STOCHASTIC CLAIMS INFLATION IN IBNR

PROFESSOR DR ERHARD KREMER

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SUMMARY
This paper deals with loss reserving under inclusion of stochastic claims inflation, a topic that is of current interest. Note that recently a new paper on it was presented at the international Astin colloquium at Cairns. In the following it is basically assumed that the discounted claims increase follows an autoregressive model of ARCH-type and that the stochastic yearly interest intensity follows a classical autoregressive model. A procedure to estimate adequately the stochastic discounting factors is deduced. This is combined with the link-ratio technique and the classical forecasting procedure for autoregressive processes, giving a new stochastic loss reserving technique. The whole method is perfect and handy. Its practicability is demonstrated in an example.
1. INTRODUCTION.

During the last decades a lot was written on how to calculate adequately loss reserves in nonlife insurance. Certain survey books were published (see e.g. Taylor (1986), Institute of Actuaries (1985) and the topic was included into standard text books (see e.g. Sundt (1983), Kremer (1985)). In the present decade many attempts were made to refine and extend previous methods (see e.g. Kremer (1993), (1996) and Doray (1996)). New aspects were considered (see e.g. Kremer (1997)) and totally new approaches presented (see e.g. Verrall (1995)). Nevertheless the toolkit is not yet totally complete. For example the techniques for coping with claims inflation are not very far developed. In practice one usually adjusts in advance the claims for inflation and applies standard techniques to the adjusted data. The reason for proceeding like this is that the models underlying nearly all standard techniques do not incorporate the claims inflation effects. To the author only one standard technique, that does incorporate the claims inflation in its basic model, is known. It is the method published by Verbeek in 1972 and reconsidered by Taylor in 1977. This method is a handy recursive procedure with which one estimates mean yearly claims growth and the mean yearly claims inflation effect. Since the modelling of the claims inflation is quite simple, one called for more refined models, yielding more refined techniques. Clearly one likes to incorporate models of modern financial mathematics for adequate modelling the claims inflation. Most modern it would be to work with stochastic discounting. Just this is done in a recent article of Goovaerts and de Schepper (1997) and in the following one of the author. Whereas Goovaerts and de Schepper use a pure probability-theoretic approach, the author gives a more mathematical-statistical approach. So in the following a fairly new claims reserving technique is developed which is based on a model that incorporates stochastic claims inflation. In an example it is shown that the method works fairly well in case the claims data is not too irregular. In the author's opinion the following method is more practical than that given by Goovaerts and de Schepper.
2. MODEL.

Denote with the random variable \( Z_{ij} \) on \((\Omega, \mathcal{A}, P)\) the total claims amount of a (collective of) risk(s) in development year no. \( j \) with respect to its accident year no. \( i \). With the claims settlement period \( n \), the set of random variables:

\[
Z_{\Delta} = (Z_{ij}, j = 1, \ldots, n - i + 1, i = 1, \ldots, n)
\]

is the so-called run-off triangle. For the sequel define the increases:

\[
Y_{ij} = Z_{ij} - Z_{i,j-1}
\]

(with \( Y_{i0} := 0 \)) and with given volume measures \( V_{ii} > 0, i = 1, \ldots, n \):

\[
X_{ij} = Y_{ij} / V_{ii}
\]

Obviously the problem of forecasting the (unknown) random variables \( Z_{ij} \) out of the (known) run-off triangle \( Z_{\Delta} \) is equivalent with the problem of forecasting the (unknown)

\[
X_{ij}, j = n - i + 2, i = 2, \ldots, n
\]

out of the (known) run-off triangle \( X_{\Delta} \) (of the \( X_{ij} \)). Remember that with a forecast \( \hat{Z}_{in} \) of \( Z_{in} \) (\( i \geq 2 \)) the IBNR-reserve for accident year no. \( i \) is just

634
For the present paper we assume that there are random variables $D_i$, $i = 1,...,2n - 1$ with

$$D_i = 1, \quad D_\epsilon \sim (0,1), \ i = 2,...,2n - 1$$

such that the random variables

$$W_{ij} = X_{ij} \cdot D_{i+j-1}$$

follow the stochastic recursion:

$$(2.1) \quad W_{ij} = L_j \cdot W_{ij}^{i,j-1} + \epsilon_{ij}$$

where $L_j > 0$ is an unknown parameter and $\epsilon_{ij}$ random error terms, independent of the $W_{i,j-1}$ and with:

$$(2.2) \quad E(\epsilon_{ij}) = 0$$

$$(2.3) \quad \text{Var}(\epsilon_{ij}) = \sigma_j^2 / V_i$$

$\sigma_j^2 \approx 0$ unknown. Furthermore assume that:

$$(2.4) \quad \text{the vectors} \ (W_{i1}, e_{i2},...,e_{in}), \ i = 1,...,n .$$
are stochastically independent,

\[ E(W_{ij}) = E(W_{j1}) \quad \text{for all } i,j, \]

\[ \text{Var}(W_{i1}) = s^2 / V_i \quad \text{for all } i, \]

with an unknown \( s^2 \approx 0 \)

Note that (2.1), (2.2), (2.3), (2.5) imply that:

\[ E(W_{ik}) = E(W_{jk}) = \mu_k \quad \text{for all } i,j, \]

\[ V_i \cdot \text{Var}(W_{ik}) = V_{j} \cdot \text{Var}(W_{jk}) = \eta_k \quad \text{for all } i,j. \]

for all \( k = 1, \ldots, n \). The \( D_i \) have to be interpreted as stochastic deflation factors. The inverses \( D_i^{-1} = A_i \) describe the stochastic claims inflation, more concretely the random variable:

\[ I_i = ((X_{1i}) - 1) \cdot 100 \]

is the percentage of claims size increase in year no. \( i \) due purely to claims inflation. Note that the assumption \( D_1 = 1 \) means that the deflation is done up to the end of the first period. Lateron the \( D_2, \ldots, D_n \) will be calculated (estimated) from the run-off triangle \( X_\Delta \), resulting in \( \hat{D}_2, \ldots, \hat{D}_n \). For forecasting the \( X_{ij}, j = n - i + 2 \) one needs
forecasts $\hat{A}_{n+1}, \ldots, \hat{A}_{2n-1}$ of the future inflation factors $A_{n+1}, \ldots, A_{2n-1}$.

For giving a reasonable forecast we assume in addition that:

$$D_i = \exp(-\sum_{j=1}^{i} S_j)$$  \hspace{2cm} (2.7)

with random variables $S_j$ following an autoregressive time series model of order one (=AR(1)):

$$S_j = a \cdot S_{j-1} + (1-a) \cdot b + \epsilon_j$$  \hspace{2cm} (2.8)

with $S_1 = 0$,

where $a, b$ are unknown parameters and the $\epsilon_j$ are uncorrelated error terms with:

$$E(\epsilon_j) = 0, \quad \text{Var}(\epsilon_j) = \sigma^2$$

with an unknown $\sigma^2 \geq 0$.

Note that (2.7) is equivalent with

$$S_i = \ln(D_{i-1}) - \ln(D_i), \quad i = 2, \ldots, n$$  \hspace{2cm} (2.9)

with $S_1 = 0$ ,

and that for $i > j$:
or equivalently:

\begin{equation}
A_{i+1} = A_i \cdot \exp(S_{i+1})
\end{equation}

The model (2.7), (2.8) is a special case of those used in Dhaene (1989).

3. ESTIMATION.

For carrying through the forecasts of the $X_{ij}$, $j \geq n - i + 2$ from the run-off triangle $X_\Delta$ (of the $X_{ij}$, $j \leq n - i + 1$) one needs estimators of the

a) $D_2, \ldots, D_n$

b) $L_2, \ldots, L_n$

c) $a, b$

Based on the data of the run-off triangle $X_\Delta$. From statistics one knows that a good estimator of $\mu_k$ is just:

\begin{equation}
\hat{\mu}_k = \left( \frac{1}{V_i^{(k)}} \right) \cdot \sum_{i=1}^{n-k+1} V_i \cdot W_{ik}
\end{equation}

with:

\begin{equation}
V_i^{(k)} = \sum_{i=1}^{n-k+1} V_i
\end{equation}

According to consequence (2.6) a reasonable criterion for giving estimators $\hat{D}_i$ of $D_i$ is the minimisation of the sum of squares:
in the \( D_i \), \( i = 2, \ldots, n \) (note that \( D_1 = 1 \)). This sum of squares can be rewritten as:

\[
\sum_{i=1}^{n-i+1} \sum_{j=1}^{n} V_i \cdot \left( W_{ij} - \hat{\mu}_j \right)^2
\]

By differentiating with respect to \( D_k \) and putting the result equal to zero, one arrives after routine manipulations at the equation:

\[
(3.1) \quad \frac{D}{\kappa} \cdot \frac{\hat{\kappa}}{\kappa} = \Lambda_{1\kappa} (D_1, \ldots, D_n) + \Lambda_{2\kappa} (D_1, \ldots, D_n)
\]

with the sums:

\[
\Lambda_{1\kappa} (D_1, \ldots, D_n) = \sum_{j=1}^{\kappa} \left( 1 - \frac{V_j}{V} \right) \cdot V_j \cdot X_j^2 \cdot \hat{\mu}_j
\]

\[
\Lambda_{2\kappa} (D_1, \ldots, D_n) = \sum_{k=1}^{n} \sum_{\min(k, \kappa)} \left( 1 - \frac{V_j}{V_j(j)} \right) \cdot V_j \cdot X_j^2 \cdot \hat{\mu}_j
\]

That stochastic equation system (3.1), with \( \kappa = 2, \ldots, n \) and \( D_j = 1 \), can
not be solved analytically. But one can take an iterative numerical procedure. According to ideas of numerical mathematics one guesses the recursive procedure:

\[
D_{\kappa}^{(p+1)} = \Lambda_{\kappa}^{-1} \cdot [\Lambda_{\kappa} \cdot (D_{1}^{(p)}, \ldots, D_{n}^{(p)}) + \Lambda_{\kappa} \cdot (D_{1}^{(p)}, \ldots, D_{n}^{(p)})]
\]

with \(D_{1}^{(p)} = 1\) for all \(p\), and \(\kappa = 2, \ldots, n\).

where \(D_{1}^{(1)}, 1 = 2, \ldots, n\) are chosen suitable start values in \((0, \infty)\). When the \(D_{\kappa}^{(p)}\) have stabilized sufficiently good, one takes them as estimates \(\hat{D}_{\kappa}\) of \(D_{\kappa}\) (\(\kappa = 2, \ldots, n\)).

For estimating \(L_{1}, \ldots, L_{n}\) one can look into the author's paper Kremer (1984). That paper gives under the model (2.1.) - (2.3) the estimator:

\[
\hat{L}_{j} = \frac{\sum_{i=1}^{n-j+1} V_{i} \cdot W_{ij} \cdot W_{i,j-1}}{\sum_{i=1}^{n-j+1} V_{i} \cdot W_{i,j-1}^{2}}
\]

The \(W_{ij}\) are not exactly known. Clearly one replaces them in (3.2) by their approximations:

\[
\hat{W}_{ij} = X_{ij} \cdot \hat{D}_{i+j-1}
\]

the discounted claims(-ratio) increments. For estimating the parameters
a, b, one computes according to (2.9) estimators $\hat{S}_i$ of $S_i$ as:

\begin{equation}
(3.3) \quad \hat{S}_i = 1n(\hat{D}_{i-1}) - 1n(\hat{D}_i)
\end{equation}

with $\hat{S}_i = 0$.

From time series analysis one knows (see e.g. Fuller (1976)) that for the model (2.8) reasonable estimators of $a$ and $b$ are:

\[
\hat{b} = \frac{1}{n} \cdot \sum_{i=1}^{n} \hat{S}_i
\]

\[
\hat{a} = \frac{\sum_{i=2}^{n} (\hat{S}_i - \hat{b}) \cdot (\hat{S}_{i-1} - \hat{b})}{\sum_{i=2}^{n} (\hat{S}_{i-1} - \hat{b})^2}
\]

Again the $S_i$ are unknown. One clearly replaces them by their approximations $\hat{S}_i$.

4. METHOD.

First one estimates the $\hat{D}_2, ..., \hat{D}_n$ like described in the previous section, resulting in the $\hat{D}_2, ..., \hat{D}_n$. With these estimated deflation factors one deflates the $X_{ij}$, giving the $\hat{W}_{ij}$. From (3.2) one calculates the
lagfactors $\hat{L}_j$ (with $\hat{W}_{ij}$ instead of $W_{ij}$) and completes the run-off triangle $\hat{W}_\Delta$ to a rectangle according to the prediction advice:

$$\hat{W}_{ij} = \hat{L}_j \cdot \hat{W}_{i,j-1}$$

for $j = n - i + 2, \ldots, n$, $i = 2, \ldots, n$

For having the predictions $\hat{X}_{ij}$ of the $X_{ij}$ for $j = n - i + 2$ one needs an estimator $\hat{A}_j$ for $A_j$ with $j = n + 1$, since one would take:

$$(4.1) \quad \hat{X}_{ij} = \hat{W}_{ij} \cdot \hat{A}_{i+j-1}$$

For computing the $\hat{A}_i$ for $i = n + 1$ one clearly applies the recursion (2.10):

$$(4.2) \quad \hat{A}_{i+1} = \hat{A}_i \cdot \exp(\hat{S}_{i+1})$$

with start:

$$\hat{A}_n = \hat{D}^{-1}$$

and predictions $\hat{S}_i$, $i = n + 1$ of the $S_i$, $i = n + 1$

One knows from time series analysis that an optimal prediction $\hat{S}_i$ of $S_i$, $i = n + 1$ (in the model (2.8)), based on $S_1, \ldots, S_n$, can be calculated as:
\[ \hat{S}_{n+j} = a^j \cdot S_n + (1-a^j) \cdot b \]

(see e.g. Fuller (1976)). For the unknown \( a, b, S \) one inserts \( \hat{a}, \hat{b} \) (see previous section) and the \( \hat{S} \) according to the definition (3.3). Having those predictions \( \hat{S}_i, \ i \geq n + 1 \), one applies (4.1) with (4.2).

5. EXAMPLE.
For \( n = 7 \) consider the following run-off triangle \( Z_\Delta \):

\begin{align*}
23.20 & 32.33 & 35.40 & 37.69 & 39.13 & 39.76 & 40.16 \\
25.08 & 35.06 & 38.61 & 41.61 & 43.37 & 44.56 \\
29.29 & 39.34 & 43.85 & 47.30 & 49.45 \\
31.14 & 43.99 & 49.52 & 53.33 \\
36.37 & 49.39 & 55.47 \\
37.32 & 51.25 \\
44.55
\end{align*}

giving for choice \( V_i^1 = 1 \) for all \( i \), the run-off triangle \( X_\Delta \):

\begin{align*}
23.20 & 9.13 & 3.07 & 2.29 & 1.44 & 0.63 & 0.40 \\
25.08 & 9.98 & 3.55 & 3.00 & 1.76 & 1.19 \\
29.29 & 10.05 & 4.51 & 3.45 & 2.15 \\
31.14 & 12.85 & 5.53 & 3.81 \\
36.37 & 13.02 & 6.08 \\
37.32 & 13.93 \\
44.55
\end{align*}
With 100 iterations and start values:

\[
D^{(1)}_2 = 0.9, \quad D^{(1)}_3 = 0.8, \quad D^{(1)}_4 = 0.7
\]

\[
D^{(1)}_5 = 0.6, \quad D^{(1)}_6 = 0.5, \quad D^{(1)}_7 = 0.4
\]

one gets:

\[
\hat{D}_2 = 0.91107, \quad \hat{D}_3 = 0.78763, \quad \hat{D}_4 = 0.74467
\]

\[
\hat{D}_5 = 0.63088, \quad \hat{D}_6 = 0.61321, \quad \hat{D}_7 = 0.52406
\]

what corresponds to the yearly inflation percentages (in %):

\[
\hat{I}_2 = 9.76, \quad \hat{I}_3 = 15.67, \quad \hat{I}_4 = 5.77
\]

\[
\hat{I}_5 = 18.04, \quad \hat{I}_6 = 2.75, \quad \hat{I}_7 = 17.01
\]

The deflates run-off triangle \( \hat{W}_\Delta \) is:

\[
\begin{array}{ccccccc}
23.20 & 8.32 & 2.42 & 1.71 & 0.91 & 0.39 & 0.21 \\
22.85 & 7.86 & 2.64 & 1.89 & 1.08 & 0.62 &          \\
23.07 & 7.48 & 2.85 & 2.12 & 1.16 &          &          \\
23.19 & 8.11 & 3.39 & 2.00 &          &          &          \\
22.95 & 7.98 & 3.19 &          &          &          &          \\
22.89 & 7.30 &          &          &          &          &          \\
23.55 &          &          &          &          &          &          \\
\end{array}
\]
with what one gets the lag factors:

\[
\hat{L}_2 = 0.341 \quad \hat{L}_3 = 0.364 \quad \hat{L}_4 = 0.676 \\
\hat{L}_5 = 0.545 \quad \hat{L}_6 = 0.515 \quad \hat{L}_7 = 0.543
\]

with the estimates:

\[
\hat{a} = -0.6075 \quad \hat{b} = 0.0923
\]

one computes the predictions:

\[
\hat{A}_8 = 2.213 \quad \hat{A}_9 = 2.346 \quad \hat{A}_{10} = 2.627 \\
\hat{A}_{11} = 2.845 \quad \hat{A}_{12} = 3.144 \quad \hat{A}_{13} = 3.432
\]

giving with \(g(4.1)\) the forecasts \(\hat{X}_{ij}, \ j > n - i + 1:\)

\[
\begin{array}{cccc}
0.75 & & & \\
1.28 & 0.74 & & \\
2.41 & 1.31 & 0.80 & \\
4.77 & 2.75 & 1.59 & 0.93 \\
5.88 & 4.21 & 2.57 & 1.43 & 0.86 \\
17.60 & 6.79 & 5.14 & 3.03 & 1.72 & 1.06 \\
\end{array}
\]

and the predicted cumulative values \(\hat{Z}_{ij}, \ j > n - i + 1:\)
what looks quite reasonable.

6. REMARKS.
Clearly one can try a more complicate ARIMA (p,d,q)-model instead of the simple AR(1)-model (2.8), like discussed in Dhaene (1989), e.g. an AR(2)-model. Then parameter-estimation of the \( S_j \)-process becomes more elaborate and forecasting of the \( S_j \), \( j > n + 1 \) more complicated. But note that for smaller \( n \) the parameter-estimation of these models becomes quite unreliable since one has a too short past time series \( D_2, \ldots, D_n \) and consequently it is mostly better to use the AR(1)-model.

REFERENCES:
Goovaerts, M. and De Schepper, A. (1997): IBNR reserves under stochastic interest rates. ASTIN colloquium at Cairns,


Author:
E. Kremer
Institut für Mathematische Stochastik
Bundesstrasse 55
20146 Hamburg
Germany