FINANCIAL MATHEMATICS

DEFINITIONS – GENERAL

Version 10/1/2002

Prepared by David Forfar, MA, FFA with the assistance of David Raymont, Librarian, Institute of Actuaries, London

Interest:-the reward paid by the borrower to the lender for use of the lender's money. The borrowed money is referred to as capital.

Compound interest:- where interest is paid on the original money (capital) and on interest arising from the original capital i.e. you get interest on interest.

Simple interest:- where interest is paid only on the original money (capital) but not on interest arising from that capital i.e. you do not get interest on interest.

Time period:- the time period selected in order to solve the problem at hand- it could be one year, a half-year, a month etc. It could even be a period of two years.

Effective rate of interest i over a time period (which must be stated):- the rate of interest i which operates over that selected time period and such that £1 invested at the beginning of that time period accumulates to £(1+i) at the end of the time period, where i is the effective rate of interest e.g. if the rate of interest is 6% effective per annum then £1 invested at the start of the year will accrue £0.06 interest at the end of the time period which, together with the original £1, gives £1.06. If the effective rate of interest is 3% effective per half-year then after one year an investment of £1 will have grown to £1.03*1.03 = £1.0609. This means that an effective rate of 3% per half-year is equivalent to an effective rate of 6.09% per annum (year).

Effective discount rate d over a time period (which must be stated) :- the rate such that the present/discounted value of £1 payable at the END of the stated time period is £(1-d) at the beginning of the stated time period. Thus if i is the effective rate of interest over the time period 1/(1+i) = v = (1-d). If the effective discount rate per annum is 6.5% then the present value at the start of the year of £1 payable at the end of the year is £(1-.065) = £0.935. The equivalent effective rate of interest per annum is such that (1+i) =1/.935 which gives i equal to 6.95187%

Instantaneous force of interest of δ(t) per annum :- the instantaneous rate of growth of a fund, per unit of the fund, reflecting the operation of a force of interest of δ(t) per year. If the fund at time t is *F(t)* then $\frac{1}{F(t)}\frac{dF(t)}{dt} = \delta(t)$. This gives as a solution $F(t) = F(0)e^{\int_{s=0}^{s=t}\delta(s)ds}$. If δ(t) is a constant equal to δ then $F(1) = F(0)e^{\delta}$ so that (1+i)=e^{δ} or δ=$\log_e$(1+i).

Nominal rates of interest:- rates of interest in name only which have to be converted into effective rates of interest over the appropriate time period.

$i^{(2)}$:- a nominal rate of interest of **$i^{(2)}$** per annum convertible half-yearly means an rate of $i^{(2)}/2$ effective per half-year, which is equivalent to an effective rate of i per annum where $(1+i) = (1+i^{(2)}/2)^2$.

$i^{(4)}$:- a nominal rate of interest of **$i^{(4)}$** per annum convertible quarterly means an rate of $i^{(4)}/4$ effective per quarter, which is equivalent to an effective rate of i per annum where $(1+i) = (1+i^{(4)}/4)^4$.

$i^{(12)}$:- a nominal rate of interest of **$i^{(12)}$** per annum convertible monthly means an rate of $i^{(12)}/12$ effective per month, which is equivalent to an effective rate of i per annum where $(1+i) = (1+i^{(12)}/12)^{12}$.

Nominal rates of discount :- rates of discount in name only which have to be converted into effective rates of discount over the appropriate time period,

$d^{(2)}$:- a nominal rate of discount of **$d^{(2)}$** per annum convertible half-yearly means an discount rate of $d^{(2)}/2$ effective per half-year, which is equivalent to an effective rate discount of d per annum where $v = 1/(1+i) = (1-d) = (1-d^{(2)}/2)^2$.

$d^{(4)}$:- a nominal rate of discount of **$d^{(4)}$** per annum convertible quarterly means an discount rate of $d^{(4)}/4$ effective per quarter, which is equivalent to an effective rate discount of d per annum where $v = 1/(1+i) = (1-d) = (1-d^{(4)}/4)^4$.

$d^{(12)}$:- a nominal rate of discount of **$d^{(12)}$** per annum convertible monthly means an discount rate of $d^{(12)}/12$ effective per month, which is equivalent to an effective rate discount of d per annum where $v = 1/(1+i) = (1-d) = (1-d^{(12)}/12)^{12}$.

v :- the discount factor $1/(1+i)$

Time value of money:- takes account of the fact that £1 to-day is not the same as £1 tomorrow. If the effective rate of interest is 6% pa. effective, you are equally happy with £1 to-day or £1.06 in one year's time or £$1.06^{1/2}$ in six-months time etc. This follows because £1 invested to-day at 6% pa. effective will be worth £1.06 in a year's time etc. Correspondingly you are equally happy with £1 in one year's time or £1/1.06 =£0.9434 now. You are equally happy with £1 in six-months time or £$1/1.06^{1/2}$ now.

Equivalent payments :- means payments at different times but which, allowing for the **time value of money,** are identical i.e. the payments accumulate or discount to the exactly same thing at any point of time e.g. $i^{(2)}/2$ payable at the end of the first six months and again at the end of the second six months is equivalent to i at the end of the year. For example the following payments are all equivalent at a rate of interest of i p.a. effective:-

i at the end of the year
$i^{(2)}/2$ payable at the end of each half-year
$i^{(4)}/4$ payable at the end of each quarter-year
$i^{(12)}/12$ payable at the end of each month
d payable at the start of each year
$d^{(2)}/2$ payable at the start of each half-year
$d^{(4)}/4$ payable at the start of each quarter-year
$d^{(12)}/12$ payable at the start of each month

Hence:-

Two payments of ½ at the end of each half-year are equivalent to one payment of $i/i^{(2)}$ at the end of the year
Four payments of 1/4 at the end of each quarter-year are equivalent to one payment of $i/i^{(4)}$ at the end of the year
Twelve payments of 1/12 at the end of each month are equivalent to one payment of $i/i^{(12)}$ at the end of the year
A payment of 1 at the start of each year is equivalent to one payment of i/d at the end of the year
Two payments of ½ at the start of each half-year are equivalent to one payment of $i/d^{(2)}$ at the end of the year
Four payments of 1/4 at the start of each quarter-year are equivalent to one payment of $i/d^{(4)}$ at the end of the year
Twelve payments of 1/12 at the start of each month are equivalent to one payment of $i/d^{(12)}$ at the end of the year

Accumulation factor:- if the effective rate of interest per time period (N.B. not necessarily a year) is i and there are n time periods plus a further fraction f of a time period, then the accumulation factor is $(1+i)^{n+f}$ .This is the amount that would be in your bank account if you invested £1 at an effective rate of i per time period and closed your account after (n+f) time periods.

Discount factor:- The reciprocal of the **accumulation factor**. If the effective rate of interest per time period (N.B. not necessarily a year) is i and there are n time periods plus a further fraction f of one time period, then the discount factor is$1/(1+i)^{n+f}$ or equivalently v^{n+f} . This is the amount that would need to be in your bank account now if you wished to have £1 to remove on closing your account after (n+f) time periods. It represents the **present value** or **discounted value** of £1 payable at the end of (n+f) time periods.

Discounted cash flow:- cash payments whose present value (discounted value) is being calculated by applying the appropriate **discount factor** to each payment.

Accumulated cash flow:- cash payments whose accumulated value is being calculated by applying the appropriate **accumulation factors** to each payment.

Geometric series :- A series of the form $\mathbf{1+x+x^2+x^3\ldots..+x^{n-1}}$. It is seen that because $\mathbf{(1-x^n) =(1-x)*(1+x+x^2+x^3\ldots..x^{n-1})}$, the sum of the above geometric series is $\mathbf{(1-x^n)/(1-x)}$.

Annuity yearly/half-yearly/quarterly/monthly in arrear:- payments made at the **end** of each year/half-year/quarter/month.

Annuity yearly/half-yearly/quarterly/monthly in advance (also called annuities-due):- payments made at the **start** of each year/half-year/quarter/month.

Term of the annuity:- the total of the time periods for which the annuity runs.

$a_{\overline{n|}}$ **:-**represents the present/discounted value of payments (an annuity) of £1 at the end of each time period repeated for n time periods. Where the effective rate of interest is i per time period, the formula for $a_{\overline{n|}}$ is $\mathbf{v+v^2+v^3+\ldots..+v^n}$ which equals $\mathbf{(1-v^n)/i}$ **.**

$\ddot{a}_{\overline{n|}}$ **:-**represents the present/discounted value of payments (an annuity) of £1 at the beginning of each time period repeated for n time periods. Where the effective rate of interest is i per time period, the formula for $\ddot{a}_{\overline{n|}}$ is $\mathbf{1+v+v^2+v^3+\ldots\ldots.+v^{n-1}}$ which equals $\mathbf{(1-v^n)/d}$ which equals $\frac{i}{d}a_{\overline{n|}}$**.**

$a_{\overline{n}|}^{(2)}$ **:-** represents the present/discounted value of payments of ½ after half a time period and ½ after a whole time period repeated for n time periods. If i is the effective rate of interest per time period then $a_{\overline{n}|}^{(2)}$ is **½{$v^{0.5}+v^{1}+v^{1.5}+v^{2}+v^{2.5}....+v^{n}$}** which is equal to **$(1-v^{n})/i^{(2)}$** which equals $\frac{i}{i^{(2)}}a_{\overline{n}|}$ where **$i^{(2)}$** is the nominal rate of interest per time period convertible per half-time period.

$\ddot{a}_{\overline{n}|}^{(2)}$ **:-** represents the present/discounted value of payments of ½ at the beginning of the first half time period and ½ at the beginning of the second half time period repeated for n time periods. If i is the effective rate of interest per time period then $\ddot{a}_{\overline{n}|}^{(2)}$ is **½{$1+v^{0.5}+v^{1}+v^{1.5}+v^{2}+.....+v^{n-1/2}$}** which is equal to **$(1-v^{n})/d^{(2)}$** which equals $\frac{i}{d^{(2)}}a_{\overline{n}|}$ where $d^{(2)}$ is the nominal rate of discount per time period convertible per half-time period.

$a_{\overline{n}|}^{(4)}$ **:-** represents the present/discounted value of payments of 1/4 at the end of the first, second, third and fourth quarter time periods repeated for n time periods. If i is the effective rate of interest per time period then $a_{\overline{n}|}^{(4)}$ is **1/4{$v^{0.25}+v^{0.5}+v^{0.75}+v^{1}+.....+v^{n}$}** which is equal to **$(1-v^{n})/i^{(4)}$** which equals $\frac{i}{i^{(4)}}a_{\overline{n}|}$ where **$i^{(4)}$** is the nominal rate of interest per time period convertible per quarter period.

$\ddot{a}_{\overline{n}|}^{(4)}$ **:-** represents the present/discounted value of payments of 1/4 at the beginning of the first, second, third and fourth quarter time periods repeated for n time periods. If i is the effective rate of interest per time period then $\ddot{a}_{\overline{n}|}^{(4)}$ is **1/4{$1+v^{0.25}+v^{0.5}+v^{0.75}+v^{1}+.....+v^{n-1/4}$}** which is equal to **$(1-v^{n})/d^{(4)}$** which equals $\frac{i}{d^{(4)}}a_{\overline{n}|}$ where $d^{(4)}$ is the nominal rate of discount per time period convertible per quarter period.

$a_{\overline{n}|}^{(12)}$ **:-** represents the present/discounted value of payments of 1/12 at the end of the first, second, third, fourth, fifth ....eleventh and twelfth *sub-time-periods* where each *sub-time-period* is one-twelfth of a time period and where these payments are repeated for n time periods. If i is the effective rate of interest per time period then $a_{\overline{n}|}^{(12)}$ is **1/12{$v^{1/12}+v^{2/12}+v^{3/12}+v^{4/12}+.....+v^{n}$}** which is equal to **$(1-v^{n})/i^{(12)}$** which equals $\frac{i}{i^{(12)}}a_{\overline{n}|}$ where **$i^{(12)}$** is the nominal rate of interest per time period convertible per one-twelfth time period.

$\ddot{a}_{\overline{n}|}^{(12)}$ **:-** represents the present/discounted value of payments of 1/12 at the beginning of the first, second, third, fourth, fifth ....eleventh, twelfth *sub-time-periods* where each *sub-time-period* is one-twelfth of a time period time and where these payments are repeated for n time periods. If i is the effective rate of interest per time period then $\ddot{a}_{\overline{n}|}^{(12)}$ is **1/12{$1+v^{1/12}+v^{2/12}+v^{3/12}+v^{4/12}+.....+v^{n-1/12}$}** which is equal to **$(1-v^{n})/d^{12)}$** which equals $\frac{i}{d^{(12)}}a_{\overline{n}|}$ where $d^{(12)}$ is the nominal rate of discount per time period convertible per one-twelfth time period.

$\bar{a}_{\overline{n}|}$:- represents the present/discounted value of a payment of £1 per annum payable continuously throughout n years. If i is the effective rate of interest then $\bar{a}_{\overline{n}|}$ is $(1-v^n)/\delta$ which equals $\frac{i}{\delta}a_{\overline{n}|}$ where δ is the force of interest per annum.

Deferred annuity:- an annuity where there is a deferred period i.e. a period of time when nothing is payable i.e. the period before the annuity comes into effect.

Immediate annuity :- means an annuity where there is no deferred period.

$_{m|}a_{\overline{n}|}$ **:-** represents the present/discounted value of the payments represented by the symbol to the right of the bar but in the case where these payments are deferred m years. The symbols $_{m|}\ddot{a}_{\overline{n}|}$ **,** $_{m|}a^{(2)}_{\overline{n}|}$, $_{m|}\ddot{a}^{(2)}_{\overline{n}|}$, $_{m|}a^{(4)}_{\overline{n}|}$, $_{m|}\ddot{a}^{(4)}_{\overline{n}|}$, $_{m|}a^{(12)}_{\overline{n}|}$, $_{m|}\ddot{a}^{(12)}_{\overline{n}|}$, $_{m|}\bar{a}_{\overline{n}|}$ are defined similarly. For example, $_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$

$s_{\overline{n}|}$ **:-**represents the accumulated value of payments of £1 at the <u>end</u> of each time period repeated for n time periods. Where the effective rate of interest is i per time period, the formula for $s_{\overline{n}|}$ is $\mathbf{1+(1+i)+(1+i)^2+(1+i)^3+\ldots\ldots+(1+i)^{n-1}}$ which equals $\mathbf{((1+i)^n-1)/i}$ **.**

$\ddot{s}_{\overline{n}|}$ (i.e. with double dot over the s) **:-**represents the accumulated value of payments of £1 at the <u>beginning</u> of each time period repeated for n time periods. Where the effective rate of interest is i per time period, the formula for $\ddot{s}_{\overline{n}|}$ is $\mathbf{(1+i)+(1+i)^2+(1+i)^3+\ldots\ldots+(1+i)^n}$ which equals $\mathbf{((1+i)^n-1)/d}$ which equals $\frac{i}{d}s_{\overline{n}|}$ **.**

$s^{(2)}_{\overline{n}|}$ **:-** represents the accumulated value of payments of ½ after half a time period and ½ after a whole time period repeated for n time periods. If i is the effective rate of interest per time period then $s^{(2)}_{\overline{n}|}$ is $\mathbf{\frac{1}{2}\{1+(1+i)^{0.5}+(1+i)^1+(1+i)^{1.5}+(1+i)^2+\ldots..+(1+i)^{n-1/2}\}}$ which is equal to $\mathbf{((1+i)^n-1)/i^{(2)}}$ which equals $\frac{i}{i^{(2)}}s_{\overline{n}|}$ where $\mathbf{i^{(2)}}$ is the nominal rate of interest per time period convertible per half-time period.

$\ddot{s}^{(2)}_{\overline{n}|}$ (i.e. with double dot over the s) :-represents the accumulated value of payments of ½ immediately and ½ after half a time period repeated for n time periods. If i is the effective rate of interest per time period then $\ddot{s}^{(2)}_{\overline{n}|}$ (i.e. with double dot over the s) is $\mathbf{\frac{1}{2}\{(1+i)^{0.5}+(1+i)^1+(1+i)^{1.5}+(1+i)^2+\ldots..+(1+i)^n\}}$ which is equal to $\mathbf{((1+i)^n-1)/d^{(2)}}$ which equals $\frac{i}{d^{(2)}}s_{\overline{n}|}$ where $d^{(2)}$ is the nominal rate of discount per time period convertible per half-time period.

$s^{(4)}_{\overline{n}|}$ **:-** represents the accumulated value of payments of 1/4 at the end of the first, second, third, fourth quarter time periods repeated for n time periods. If i is the effective rate of interest per time period then $s^{(4)}_{\overline{n}|}$ is $\mathbf{1/4\{1+(1+i)^{0.25}+(1+i)^{0.5}+(1+i)^{0.75}+(1+i)^1+\ldots..+(1+i)^{n-1/4}\}}$ which is equal to $\mathbf{((1+i)^n-1)/i^{(4)}}$

which equals $\frac{i}{i^{(4)}} s_{\overline{n|}}$ where $\mathbf{i^{(4)}}$ is the nominal rate of interest per time period convertible per quarter time period.

$\ddot{s}^{(4)}_{\overline{n|}}$ (i.e. with double dot over the s) :- represents the accumulated value of payments of 1/4 at the beginning of the first, second, third, fourth quarter time periods repeated for n time periods. If i is the effective rate of interest per time period then

$\ddot{s}^{(4)}_{\overline{n|}}$ is 1/4$\mathbf{\{(1+i)^{0.25}+(1+i)^{0.5}+(1+i)^{0.75}+(1+i)^{1}+\ldots..+(1+i)^{n}\}}$ which is equal to $\mathbf{((1+i)^{n}-1)/d^{(4)}}$ which equals $\frac{i}{d^{(4)}} s_{\overline{n|}}$ where $d^{(4)}$ is the nominal rate of discount per time period convertible per quarter time period.

$s^{(12)}_{\overline{n|}}$ **:-** represents the accumulated value of payments of 1/12 at the end of the first, second, third, fourth, fifth ....eleventh and twelfth *sub-time-periods* where each *sub-time-period* is one-twelfth of a time period and where these payments are repeated for n time periods. If i is the effective rate of interest per time period then $s^{(12)}_{\overline{n|}}$ is $\mathbf{1/12\{1+(1+i)^{1/12}+(1+i)^{2/12}+(1+i)^{3/12}+(1+i)^{4/12}+\ldots..+(1+i)^{n-1/12}\}}$ which is equal to $\mathbf{((1+i)^{n}-1)/i^{(12)}}$ which equals $\frac{i}{i^{(12)}} s_{\overline{n|}}$ where $\mathbf{i^{(12)}}$ is the nominal rate of interest per time period convertible per one twelfth time period.

$\ddot{s}^{(12)}_{\overline{n|}}$ (i.e. with double dot over the s)**:-** represents the accumulated value of payments of 1/12 at the beginning of the first, second, third, fourth, fifth ....eleventh and twelfth *sub-time-periods* where each *sub-time-period* is one-twelfth of a time period and where these payments are repeated for n time periods. If i is the effective rate of interest per time period then $\ddot{s}^{(12)}_{\overline{n|}}$ (with double dot over the s) is $\mathbf{1/12\{(1+i)^{1/12}+(1+i)^{2/12}+(1+i)^{3/12}+(1+i)^{4/12}+\ldots..+(1+i)^{n}\}}$ which is equal to $\mathbf{((1+i)^{n}-1)/d^{(12)}}$ which equals $\frac{i}{d^{(12)}} s_{\overline{n|}}$ where $\mathbf{d^{(12)}}$ is the nominal rate of discount per time period convertible per one twelfth time period.

$\bar{s}_{\overline{n|}}$ **:-** represents the accumulated value of payments of £1 pa. payable continuously for n years and is equals to which equals $\frac{i}{\delta} s_{\overline{n|}}$

$Ia_{\overline{n|}}$ **:-**represents the present/discounted value of payments of £1 at the <u>end</u> of the first time period, £2 at the end of the second time period .... and £n at the end of the nth time period. Where the effective rate of interest is i per time period, the formula for $Ia_{\overline{n|}}$ is $\mathbf{v+2v^{2}+3v^{3}+\ldots..+nv^{n}}$ which equals $\frac{(\ddot{a}_{\overline{n|}} - nv^n)}{i}$**.** Where the payments are twice in every time period, four times or twelve times then the corresponding symbols are $Ia^{(2)}_{\overline{n|}}$**,** $Ia^{(4)}_{\overline{n|}}$**,** $Ia^{(12)}_{\overline{n|}}$ and $Ia^{(2)}_{\overline{n|}} = \frac{i}{i^{(2)}} Ia_{\overline{n|}}$ etc.

$I\ddot{a}_{\overline{n}|}$:-represents the present/discounted value of payments of £1 at the <u>beginning</u> of the first time period, £2 at the beginning of the second time period .... and £n at the beginning of the nth time period. Where the effective rate of interest is i per time period, the formula for $I\ddot{a}_{\overline{n}|}$ is **$1+2v+3v^2+4v^3+.......+nv^{n-1}$** which equals $\frac{(\ddot{a}_{\overline{n}|}-nv^n)}{d}$ or $\frac{i}{d}Ia_{\overline{n}|}$**.** Where the payments are twice in every time period, four times, twelve times then the corresponding symbols are $I\ddot{a}^{(2)}_{\overline{n}|}, I\ddot{a}^{(4)}_{\overline{n}|}, I\ddot{a}^{(12)}_{\overline{n}|}$ and $I\ddot{a}^{(2)}_{\overline{n}|}=\frac{i}{d^{(2)}}Ia_{\overline{n}|}$ etc.

$I\bar{a}_{\overline{n}|}$:- represents the present/discounted value of payments of £1pa. payable continuously during the first time period, £2pa. payable continuously during the second time period .... and £n pa. payable continuously during the nth time period i.e. the rate of payments are step functions.

$\bar{I}\,\bar{a}_{\overline{n}|}$:- represents the present/discounted value of payments at a rate of t pa. at time t i.e. the rate of payments increases continuously for n years.

$Is_{\overline{n}|}$ **:-**represents the accumulated value of payments of £1 at the <u>end</u> of the first time period, £2 at the end of the second time period .... and £n at the end of the nth time period. Where the effective rate of interest is i per time period, the formula for $Is_{\overline{n}|}$ is **$(1+i)^{n-1}+2(1+i)^{n-2}+3(1+i)^{n-3}+.....+n$.** Where the payments are twice in every time period, four times or twelve times then the corresponding symbols are $Is^{(2)}_{\overline{n}|}$, $Is^{(4)}_{\overline{n}|}, Is^{(12)}_{\overline{n}|}$.

$I\ddot{s}_{\overline{n}|}$ (with double dot over the s) :-represents the accumulated value of payments of £1 at the beginning of the first time period, £2 at the beginning of the second time period .... and £n at the beginning of the nth time period. Where the effective rate of interest is i per time period, the formula for $I\ddot{s}_{\overline{n}|}$ (double dot) is **$(1+i)^n+2(1+i)^{n-1}+3(1+i)^{n-2}+.......+n(1+i)$**. Where the payments are twice in every time period, four times or twelve times then the corresponding symbols are $I\ddot{s}^{(2)}_{\overline{n}|}, I\ddot{s}^{(4)}_{\overline{n}|}, I\ddot{s}^{(12)}_{\overline{n}|}$ (with a double dot over the a in each case).

$I\,\bar{s}_{\overline{n}|}$ **:-** represents the accumulated value of payments of £1pa. payable continuously during the first time period, £2pa. payable continuously during the second time period .... and £n pa. payable continuously during the nth time period i.e. the rate of payments are step functions.

$\bar{I}\,\bar{s}_{\overline{n}|}$:- represents the accumulated value of payments at a rate of t pa. at time t i.e. the rate of payments increases continuously for n years.

$Da_{\overline{n}|}$:- represents the present/discounted value of payments of £n at the <u>end</u> of the first time period, £(n-1) at the end of the second time period .... and £1 at the end of the nth time period. Where the effective rate of interest is i per time period, the formula for $Da_{\overline{n}|}$ is **$nv+(n-1)v^2+(n-2)v^3+.....+v^n$.**

Perpetuity:- an annuity of infinite term, it just goes on being paid for ever! The value is obtained by letting n go to ∞ so that, for example, $a_{\overline{\infty}|} = \frac{1}{i}$

Equation of value:- An equation formed by taking the **present value** (**discounted value**) of the cash inputs and equating them algebraically to the present value of the cash outputs and solving for the effective rate of interest which gives equality between the two sides of the algebraic equation. Alternatively take the present value of all payments after allowing inputs to have a positive sign and outputs a negative sign (or vice versa) and the equating to zero. A corresponding equation can also be formed by equating the accumulated value of the inputs and outputs and this equation is equivalent to the first one and gives the same solution. Example:- you invest £1000 in a project and get back £560 at the end of the first year and £627.20 at the end of the second year. The equation of value is $1000=560v + 627.20v^2$ and the positive solution to this quadratic equation is i=.12 or 12%.

Discounted payback period:- you wish to invest in a project and borrow money (a loan) from the bank in order to do so. From the monies you are paid as a result of the project being successful you gradually pay back to your banker both interest on the loan and the loan capital itself thus gradually paying off the loan. The time it takes you to pay off the loan is the **discounted payback period** and any further monies you are paid from the project thereafter are pure profit since you have no outstanding loan interest or loan capital to pay off after the **discounted payback period**. [note:- the **payback period** is the **discounted payback period** calculated assuming you are able to borrow from the bank at 0% i.e. zero percent interest, but in reality is an almost meaningless number because you will almost always be charged interest on loans].

Repayment schedule for a loan:- a table showing in full detail how a loan is repaid. The schedule will show (1) the loan outstanding just before a repayment is made (2) the amount of the repayment that is used to pay interest on the outstanding loan (3) the balance of the repayment which pays off part of the outstanding loan itself and (4) the amount of the outstanding loan after the repayment has been made. For example if a loan of £$a_{\overline{n}|}$ is made, repayable by instalments of £1p.a. yearly in arrear, then the interest content of the first repayment (of £1) is $ia_{\overline{n}|}$ or $(1\text{-}v^n)$ and the amount going to repay the loan (the capital content) is v^n leaving the outstanding loan as $a_{\overline{n-1}|}$**. T**he next year the interest content of the second repayment of £1 is $ia_{\overline{n-1}|}$ or $(1\text{-}v^{n-1})$ and the amount going to repay the loan (the capital content) is v^{n-1} leaving the outstanding loan as $a_{\overline{n-2}|}$ etc.

Negative rates of interest:- normally when you invest say £1 you expect it to grow say to £1.06 at the end of a year thus giving you an effective rate of interest of 6% pa.effective. However if your investment has not been successful and is now worth only £0.95 pence then your effective rate of interest is negative and is -5%pa. effective.

Linked-internal rate of return :- this return is produced by calculating the internal rates of return for successive periods and compounding them to give an average annual rate of return over the total period. If the total time period is n years and the rates of investment return <u>effective over the first, second, third ….. time periods</u> (not necessarily exact years) are i_1, i_2, i_3, …i_m then $(1+i)^n = (1+i_1)*(1+i_2)*(1+i_3)\ldots(1+i_m)$ where i is the internal linked rate of return per annum.

Money weighted rate of return (MWRR):- the effective rate of interest which is the solution to the **equation of value** of the investment or project. The **MWRR** depends both on the amounts of money invested, when they are invested and the rate of investment return achieved.

Internal rate of return:- the **money weighted rate of return** achieved by investing in a project.

Time weighted rate of return (TWRR) :- a rate of interest which eliminates the effect of the amounts of money invested and which only depends on the underlying investment return achieved. The total time period considered is divided into time periods determined by the time periods between the dates when money is invested and which are not necessarily whole years. The effective rate of interest (investment return) is calculated in each of these time periods and the equivalent average effective interest rate per annum calculated . If the total time period is n years and the rates of investment return <u>effective over the time periods</u> (not necessarily exact years) are i_1, i_2, i_3, …i_m then $(1+TWRR)^n = (1+i_1)*(1+i_2)*(1+i_3)\ldots(1+i_m)$.

APR (Annual Percentage Rate of Charge) :- is the effective rate of interest per annum on a financial transaction. It is calculated as the least positive root of the equation of value and is rounded to the higher 0.1%. Under the Consumer Credit Act 1974, the APR has to be disclosed in advertisements and quotations. Example:- A loan of £4,000 is repaid by monthly instalments of £200 over 2 years. The APR is the solution to $4000=(200*12)\ a_{\overline{2}|}^{(12)}$ which is i=.19727 so the APR is 19.8%.