Agenda

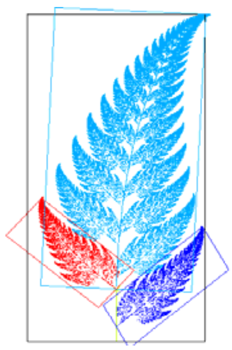
- Fractals and self-similarity
- Fractal behaviour in financial markets
- Fitting fractals – Hausdorff Dimension
- Fractional Brownian motion
- Lévy stable processes
- Conclusions

What is a fractal?

- A shape or pattern...
- ...that can be broken down into components...
- ...each of which resemble the whole
- Key property is *self-similarity*

Self-similarity

- Self similarity can be near-exact
  - e.g. for a fern...
- ...or more approximate
  - e.g. for a coastline...
  - ...or for



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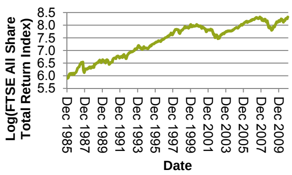
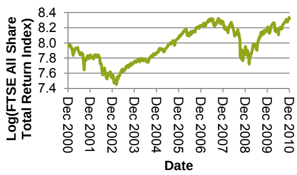
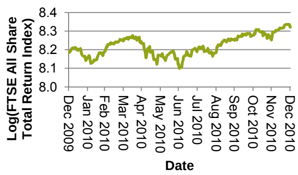
How are financial returns self-similar?







- Consider the log of equity
- By considering charts using different timescales...
- ...we see that they look interchangeable...
- ...but how are they affected by scale?

Power Laws for Self-Similarity

- Consider the risk (of a change in equities or interest rates) measured over a time interval t
- Traditionally assume the dispersion of the change is proportional to $t^{½}$, based on Brownian motion
- More generally, can consider power law processes where the dispersion grows like t^{2-d} for some d.

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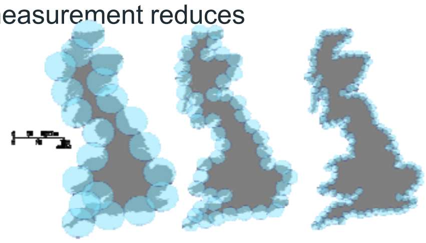
Measuring the fractional dimension

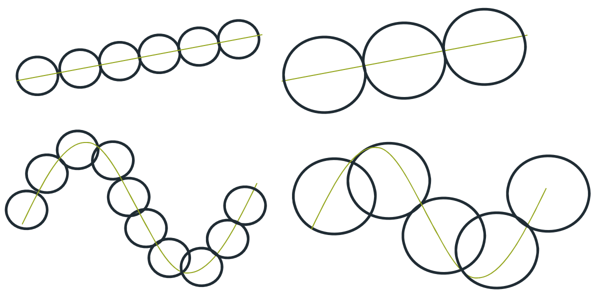
- The Hausdorff dimension measures the rate at which a measurement increases...
- ...as the scale of measurement reduces



- Given by d in $N = c/r^d$

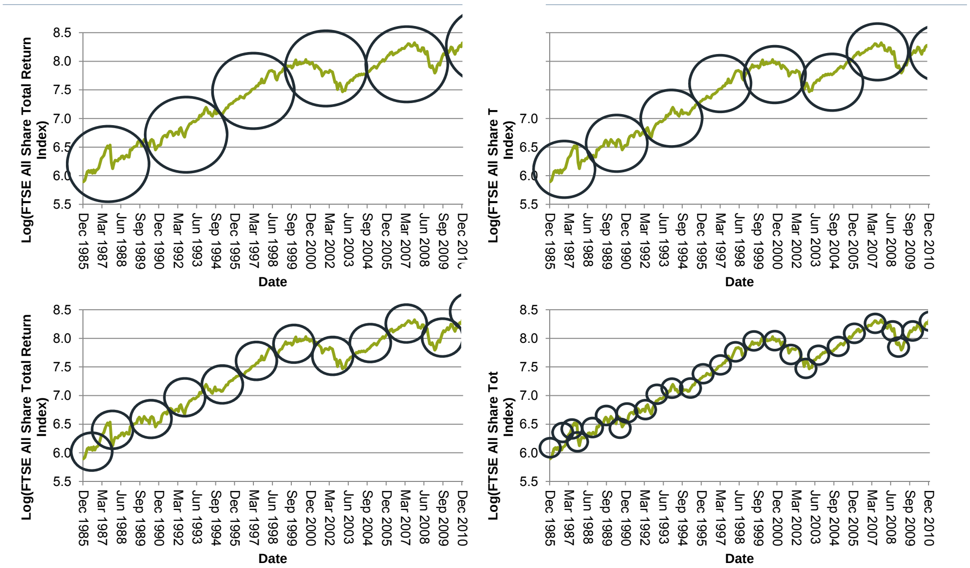
Hausdorff dimension for some common shapes

- Straight line: 1
- Curve: >1



- West coast of England: 1.25
- Financial time series?

Calculating the Hausdorff Dimension for UK Equities



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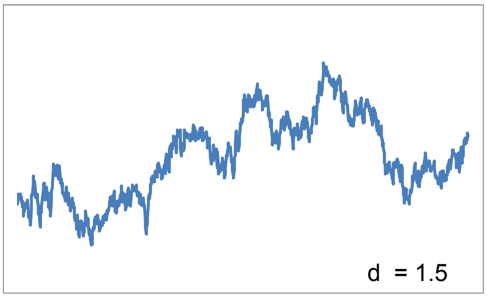
Fractional Brownian Motion
Mandelbrot & van Ness (1968)

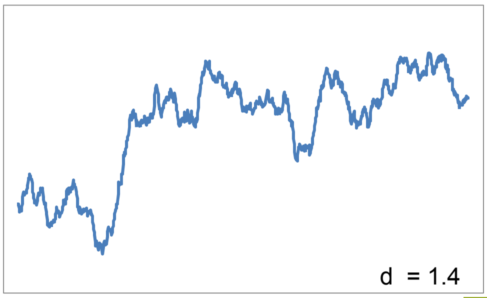
Standard Brownian Motion

- Zero mean Gaussian process
- $Var(X_t-X_s) = |t-s|$
- Independent increments

Fractional Brownian Motion

- Zero mean Gaussian process
- $Var(X_t-X_s) = |t-s|^{4-2d}$
- Long-range positive autocorrelation





Fractional Brownian Motion
Practical Issues

- Permits arbitrage (if you know the dimension d)
- Not a Markov process
  - Need to know the entire history of the process to project it
- Retains Gaussian assumption – no fat tails or jumps
- Has not seen much application in serious financial models

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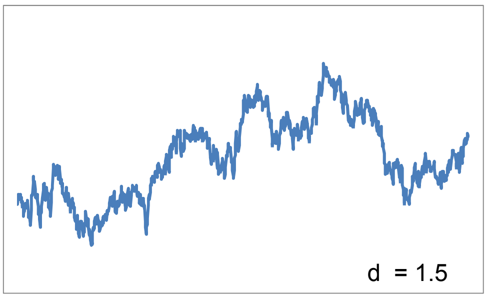
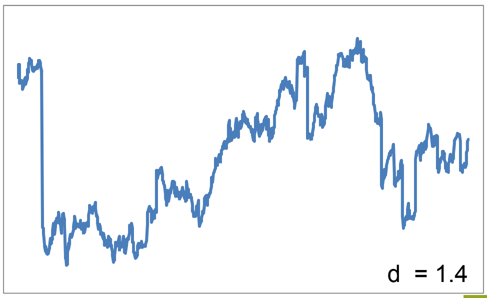
Lévy Stable Processes (Mandelbrot, 1964) Infinite Variance Central Limit Theorem

Standard Brownian Motion
- Independent Gaussian Increments
- Continuous paths

Lévy Stable Process
- Independent non-Gaussian increments
- Has jumps





Lévy Stable Processes
Practical Issues

- Four parameters: location, scale, asymmetry and tail exponent
- Parameter estimation is difficult
  - Likelihood function not known on closed form
  - Second and higher moments do not exist (apart from Normal)
  - Methods using characteristic functions are notoriously unstable
- Infinitely many jumps on any finite interval
- Generally imply the depressing conclusion that extreme events are more probable than previously thought

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Hausdorff Dimension Measures and Value-at-Risk Time Scaling

Hausdorff Dimension	Brownian motion d = 1.5	Empirical in range 1.3 to 15. In this table we use d = 1.4
Gross-up for annual VaR from monthly VaR	Sqrt(12) = 3.46	4.44
Gross-up for annual VaR from weekly VaR	Sqrt(52) = 7.21	10.71

Mandelbrot's Solutions to Dimension d < 1.5 A Summary

	Independent Increments	Correlated increments
Continuous paths	Brownian motion	Fractional Brownian motion
Jump processes	Lévy Stable processes	

Since the 1960's, many other possible explanations have emerged to account for scaling properties not being exactly 1.5. These include alternative time series models (time varying drift or volatility) as well as sampling biases in the estimation of the Hausdorff dimension.

Recent Developments
Tempering the Tails of Lévy Processes

- Recent surge in empirical work
- Invention of versions without the fat tails
  - KoBoL processes (Koponen, 1995, extended by Boyarchenko & Levendorskiĭ, 2000)
  - Independently constructed by Carr, Madan, Géman & Yor, (2002) and also known as the CGMY model.
- Retains fractional dimension but a change of measure is needed to construct the scaling property.
- Extension to stochastic volatility models (Barndorff-Nielsen & Shephard, 2006)

Using fractals for risk management

- Applying fractals to one financial series is straightforward...
- ...but most investment risk relates to portfolios of assets
- Two approaches could be used
  - Fit a fractal to the historical performance of the portfolio
  - Fit a fractal to independent factors of the returns using factor analysis

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and its staff are encouraged.

The views expressed in this presentation are those of the presenter.