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GLOBAL ASSET LIABILITY MANAGEMENT

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ABSTRACT

Dynamic financial analysis (DFA) is a technique which uses Monte Carlo simulation to investigate the evolution over time of financial models of funds, complex liabilities and entire firms. Although of increasing popularity, the drawback of DFA is the dearth of systematic methods for optimising model parameters for strategic financial planning. This paper introduces *strategic DFA* which employs the only recently mature technology of *dynamic stochastic optimisation* to fill this gap. The new approach is described in terms of an illustrative case study of a joint university/industry project to create a decision support system for strategic asset liability management involving global asset classes and defined contribution pension plans. Although the application of the system described in the paper is to fund design and risk management, the approach and techniques described here are much more broadly applicable to strategic financial planning problems; for example, to insurance and reinsurance firms, to risk capital allocation in financial institutions and trading firms and to corporate investment and business development involving real options. As well as describing the mathematical and statistical models used in the case study, the paper treats econometric estimation, asset return and liability scenario generation, model specification and optimisation, system evaluation and historical backtesting. Throughout the system visualisation plays an important role.

KEYWORDS

Dynamic Financial Analysis; Global Capital Markets; Dynamic Stochastic Optimisation; Large Scale Systems; Asset Liability Management; Risk Management; Defined Contribution Pensions; Benchmark Portfolios; Guaranteed Returns

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Dynamic optimisation is perceived to be too difficult ... It would be nice to have a generic 'sledge hammer' approach for attacking this sort of problem.

A. D. Smith, 1996, p1085

1. Introduction

1.1 *Aims*

1.1.1 Recent years have witnessed the introduction of new investment products aimed at attracting investors who are worried about the volatility of financial markets. The main feature of these products is a minimum return guarantee together with exposure to the upside movements of the markets. While such a return guarantee could be achieved simply by investing in a zero-coupon Treasury bond or similar instrument with expiration equal to the maturity date of the product, this would not allow any expectation of higher returns. Thus, there is a need to offer pension products that protect the investor from the downside while maintaining a reasonable expectation of better returns than the guaranteed one.

1.1.2 However, most such current products do not offer a high degree of flexibility; usually they accept only lump sum investments and have a predetermined maturity of only a few years. This is probably a consequence of the difficulty of reliable long-term forecasting and subsequent determination of the proper asset allocation(s) over the distant time horizon of the investment.

1.1.3 At the same time it is well known that state, and many company, run defined benefit pension plans are becoming inadequate to cover the gap between the contributions of people while working and their pensions once retired. The solution to this problem requires some form of instrument which can fill the gap to allow investors a reasonable income after retirement. A long-term minimum guarantee plan with a variable time horizon, and, in addition to the initial contribution, the possibility of making variable contributions during the lifetime of the product, is such an instrument.

1.1.4 Although societally beneficial and potentially highly profitable for the provider, the design of such instruments is not a trivial task, as it encompasses the need to do long-term forecasting for investment classes, handling a stochastic number of contributors, contributions and investment horizons, together with providing a guarantee. Stochastic optimisation methodology in the form of dynamic stochastic programming has recently made long strides, and is positioned to be the technique of choice to solve these kinds of problems.

1.1.5 This paper describes the approach and outcomes of a joint project between a university financial research centre and a leading firm operating in the European fund management industry to develop a state-of-the-art dynamic asset/liability management (ALM) system for pension fund management. The development of this system has been part of an effort undertaken by the firm for the global improvement of its ALM-related technologies and systems.

1.2 *The Pension Fund Problem*

1.2.1 Asset/liability management concerns optimal strategic planning for management of financial resources and liabilities in stochastic environments, with market, economic and actuarial risks all playing an important role. The task of a pension fund, in particular, is to guarantee benefit payments to retiring clients by investing part of their current wealth in the financial markets. The responsibility of the pension fund is to hedge the client's risks, while meeting the solvency standards in force, in such a way that all benefit payments are met.

1.2.2 Below we list some of the most important issues that a pension fund manager has to face in the determination of the optimal asset allocations over time to the product maturity:

(a) *Stochastic nature of asset returns and liabilities*
Both the future asset return and the liability streams are unknown. Liabilities, in particular, are determined by actuarial events, and have to be matched by the assets. Thus, each allocation decision will have to take into account the liabilities level, which, in turn, is directly linked to the contribution policy requested by the fund.

(b) *Long investment horizons*
The typical investment horizon is very long (30 years). This means that the fund portfolio will have to be rebalanced many times, making 'buy&hold' Markowitz-style portfolio optimisation inefficient. Various dynamic stochastic optimisation techniques are needed to take explicitly into account the on-going rebalancing of the asset mix.

(c) *Risk of under-funding*
There is a very important requirement to monitor and manage the probability of under-funding for both individual clients and the fund, that is the confidence level with which the pension fund will be able to meet its targets without resort to its parent guarantor.

(d) *Management constraints*
The management of a pension fund is also dictated by a number of solvency requirements which are put in place by the appropriate regulating authorities. These constraints greatly affect the suggested allocation, and must always be considered. Moreover, since the fund's portfolio must be actively managed, the markets' bid-ask spreads, taxes and other frictions must also be modelled.

1.2.3 The theory of *dynamic stochastic optimisation* provides the most natural framework for the effective solution of the pension fund ALM problem that will guarantee its users a competitive advantage in the market.

1.2.4 Most firms use static portfolio optimisation models, such as Markowitz mean-variance allocation, which are short-sighted, and when rolled forward lead to radical portfolio rebalancing unless severely constrained by the portfolio manager's intuition. Although such models have been extended to take account of liabilities in terms of expected solvency (surplus) levels (see e.g. Mulvey, 1989), these difficulties with static models remain. In practice, fund allocations are (thus) wealth dependent and face time-varying investment opportunities, path-dependent returns — due to cash inflows and outflows, transactions costs and time or state dependent volatilities — and conditional mean return parameter uncertainties — due to estimation or calibration errors. Hence all conditions necessary for a sequence of myopic static model allocations to be dynamically optimal are violated (see e.g. Scherer, 2002, Section 1.2).

1.2.5 By contrast, the dynamic stochastic programming models incorporated in the system described below automatically hedge current portfolio allocations against future uncertainties in asset returns and liabilities over a longer horizon, leading to more robust decisions and previews of possible future problems and benefits.

1.3 *Paper Outline*

The next section sets out the background and basic approach of practical strategic dynamic financial analysis (DFA) systems for financial planning utilising modern dynamic stochastic optimisation techniques. The remaining sections illustrate these in the context of this case study. Section 3 treats the modelling and econometric estimation of a monthly global asset return model for four major currency areas and the emerging markets, which includes macro-economic variables. In Section 4 the calibration and stochastic simulation of various versions of this statistical model for use in financial scenario generation for strategic DFA models is discussed. The basic computer-aided asset/liability management (CALM) dynamic stochastic optimisation model is treated in Section 5, including a discussion of risk management objectives, basic constraints, practical constraints and variants of the CALM model for the determination of optimal benchmark portfolios and risk managed return guarantees. Section 6 describes the generation of dynamic stochastic optimisation models for their numerical solution, together with a brief description of solution algorithms and software. Historical out-of-sample backtests of system portfolio recommendations are described in Section 7 for risk management criteria applied to both terminal fund wealth and the trajectories of the wealth accumulation process. Finally, Section 8 draws conclusions and indicates directions for future work.

2. Strategic DFA

2.1 *System Design*

Figure 2.1 depicts the processes, models, data and other inputs required to construct a strategic DFA system for dynamic asset liability management with periodic portfolio rebalancing. It should be noted that knowledge of several independent highly technical disciplines is required for strategic DFA in addition to professional domain knowledge. Corresponding to Figure 2.1, Figure 2.2 shows the system design which describes the separate — largely automated and software instantiated — tasks which must be undertaken to obtain recommended strategic decisions once statistical and optimisation models have been specified. Each of the blocks of the latter figure will be treated in detail in a subsequent section of the paper. The outer solid feedback loop recognises the iterative nature of developing any implementable strategic plan in which process visualisation of data and solutions is key. The inner solid loop will be described in Section 4. The dotted feedback loops represent possible future developments which will be mentioned in the conclusion.

2.2 *Dynamic Stochastic Optimisation*

2.2.1 As noted above, strategic ALM requires the dynamic formulation

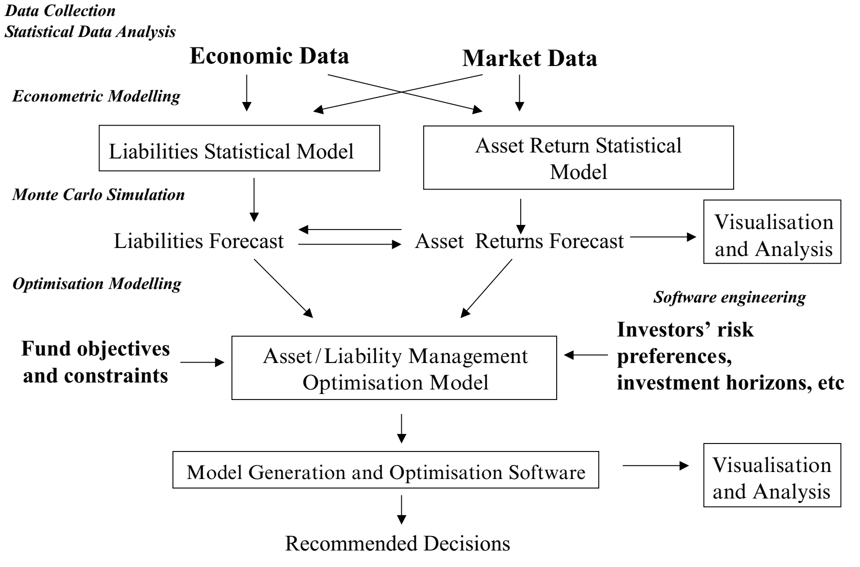


Figure 2.1. Strategic financial planning

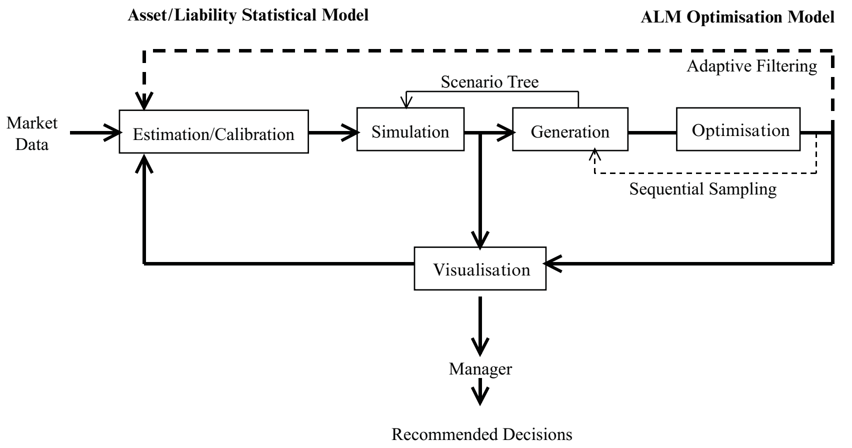


Figure 2.2. System design for strategic financial planning

of portfolio rebalancing decisions together with appropriate risk management in terms of a dynamic stochastic optimisation problem. Decisions under uncertainty require a complex process of future prediction or projection and the simultaneous consideration of a number of alternatives, some of which must be optimal with respect to a given objective. The problem is that these decisions are only known to be optimal or otherwise after the realisation of all random factors involved in the decision process. In *dynamic stochastic optimisation* (often termed *dynamic stochastic programming*, as in mathematical programming, see e.g. Dempster (1980)) the unfolding uncertain future is represented by a large number of future scenarios from the DFA simulation process (see e.g. Kaufmann *et al.* (2001) and the references therein) and contingent decisions are made in stages according to tree representations of future data and decision processes. The initial — *implementable stage* — decisions are made with respect to all possible variations of the future (in so far as it is possible to predict and generate this future), and are thus hedged within the constraints against all undesirable outcomes. This technique also allows detailed 'what-if' analysis of particular extreme future scenarios — forewarned is forearmed!

2.2.2 The methods used are computationally intensive, and have only recently become practical for real applications. Each particular optimisation problem is formulated for a specific application combining the goals and the constraints reflecting risk/return relationships. The dynamic nature of stochastic optimisation: decisions — observed output — next decisions — etc. ... allows a choice of strategy which is the best suited for the stated objectives. For example, for pension funds the objective may be a guaranteed return with a low unexpected risk and decisions reviewed every year. For a trading desk, the objective may be the maximisation of risk-adjusted cumulative trading profit, with decisions revised every minute, hour or day.

2.2.3 The basic dynamic stochastic optimisation problem treated in this paper is the following. Given a fixed planning horizon and a set of portfolio rebalance dates, find the dynamic investment strategy that maximises the expected utility of the fund's (net) wealth process subject to constraints, such as on borrowing, position limits, portfolio change and risk management tolerances, viz.:

$$\text{maximise } \mathrm{E}[\mathrm{U}(\mathbf{w}(x))]$$

$$\text{subject to } Ax \leq b.$$

Here U is a specified utility function which is used to express the *attitude to risk* adopted for a particular fund — tailored to broadly match those of its participants over the specified horizon — with regard to the wealth process **w**. (Throughout the paper we use boldface type to represent random entities.) U is used to recommend rebalance decisions which shape the state distributions of **w** over problem scenarios. Risk attitude may concern only *terminal* wealth (Hakansson, 1974; Dempster & Ireland, 1988), or be imposed at each portfolio rebalance date. The (deterministic equivalent form of the) *decision process* x represents portfolio composition at each rebalance date in each scenario subject to the data (A, b) representing the constraints. As such, it is a complete contingency plan for the events defined by the scenarios. This basic model will be detailed in Section 5 and the appendices.

2.3 *Literature Review*

2.3.1 The problem of maximising expected utility under uncertainty, subject to constraints, can be a highly non-trivial problem. From the point of view of maximising utility, the fund will naturally want its set of potential investments to be as large as possible. Thus, it will want the option to invest in global assets ranging from relatively low risk, such as cash, to relatively high risk, such as emerging markets equity. The inclusion of such assets greatly increases the complexity and the amount of uncertainty in the problem, since it necessitates the modelling to some degree of, not only the asset returns, but also of exchange rates and correlations. Further sources of complexity arise from the multi-period nature of the problem and frictions, such as market transaction costs and taxes.

2.3.2 The most well known and probably the most widely used method to solve such a problem is the *mean-variance analysis* pioneered by Markowitz (1952). This analysis can be characterised by a quadratic utility function which depends only on the mean and variance of the portfolio return parameterised by a risk aversion coefficient. Solving the utility maximisation problem for a range of values of the risk aversion parameter gives rise to the *efficient frontier*. This method is now easily implemented in a spreadsheet, and only requires an estimate of the mean and covariance of the returns, which are normally obtained from historical data and/or subjective opinion. However, as noted above, the standard implementation of the mean-variance model is static (one period), and thus fails to capture the multi-period nature of the problem. It also ignores market frictions such as transaction costs. Mean-variance analysis has been extended to incorporate multiple periods and market frictions (see e.g. Steinbach, 1999; Horniman *et al.*, 2000; and Chellathurai & Draviam, 2002), but at the cost of greatly increased complexity.

2.3.3 In this paper we apply dynamic stochastic optimisation to solve pension fund management problems with global investments. The advance of computing technology and the development of effective algorithms (see e.g. Scott, 2002) have made stochastic optimisation problems significantly more tractable. Following the early work of Bradley & Crane (1972), Lane & Hutchinson (1980), Kusy & Ziemba (1986) and Dempster & Ireland (1988), the growing body of literature concerning the application of stochastic optimisation to fund management problems includes Mulvey & Vladimirou (1992), Dantzig & Infanger (1993), Cariño *et al.* (1994), Consigli & Dempster (1998), Zenios (1998) and Geyer *et al.* (2002), and is a testament to the suitability of this method for solving such problems. A comparison of the application of mean-variance analysis, stochastic control and stochastic optimisation to fund management problems can be found in Hicks-Pedron (1998), where it is shown that dynamic stochastic optimisation performs best in terms of the appropriate Sharpe ratio.

3. Asset Return, Exchange Rate and Economic Dynamics

3.1 *Asset Return Model*

3.1.1 Our asset return model is in the econometric *estimation* tradition initiated by Wilkie (1986, 1995), and continued, for example, by Cariño *et al.* (1994), Dert (1995), Boender *et al.* (1998) and Duval *et al.* (1999). An alternative approach, in the tradition of Merton (1990), is to set up a continuous time *stochastic differential equation* (sde) model for the financial and economic dynamics of interest, discretise time to obtain the corresponding system of stochastic difference equations and calibrate the output of their simulation with history by various *ad hoc* or semi-formal methods of parameter adjustment, see, for example, Mulvey & Thorlacius (1998) and Dempster & Thorlacius (1998).

3.1.2 Several other alternative approaches have appeared in the literature which also attempt to generate scenarios known to be arbitrage free within the model. One method widely used for very specific problems in financial stochastic optimisation is sampling scenarios from arbitrage-free lattice paths for the appropriate — e.g. short rate (Zenios, 1998) — arbitrage free model. The resulting sampled scenarios, however, need not be arbitrage free unless the sampling procedure is carefully controlled (see ¶4.3). More recently, arbitrage-free methods (Cairns, 2000) and deflator techniques (Smith & Speed, 1998; Jarvis *et al.*, 2001) for designing models in more complex situations have appeared. These modelling approaches involve — at least implicitly — *risk neutral* (i.e. risk discounted) probabilities and *market price of risk premia* to allow simulation of cash flows under real world probabilities. While such approaches are appropriate — indeed necessary — for full discounting for *valuation* purposes, they are totally inappropriate for making dynamic *'what-if' forward investment decisions*, which must face an approximation of the real world risks. Even for valuation purposes, calibration of complex arbitrage-free models to current — but not necessarily past — market data is difficult, not least since the literature on estimating multivariate market prices of risk or state price densities is sparse (but see Section 3.4 for such a three-factor yield curve calculation). By contrast with the assumption of no arbitrage — when portfolio decisions are irrelevant to total return (Jarvis *et al.*, 2001) — time varying investment opportunities and potential macro-economic arbitrages occur in the real world.

3.1.3 We have therefore opted for the econometric approach, which can — if successful (c.f. the positive results of system backtests in Section 7) — model these effects, together with the fact that the estimation procedures involved have been widely employed and most pitfalls in their use documented. Although, in our experience, some further informal calibration (tuning) of parameter estimates is usually required, for the complex asset return models developed here this has been minimal.

3.1.4 Note that real world scenario generation for stochastic optimisation models by any method may still introduce spurious arbitrages due to sampling errors. Simple techniques for their suppression will be discussed in ¶4.3. In this study sampling error has been found to completely swamp statistical parameter estimation error — even assuming that the fitted econometric model actually underlies the data.

3.1.5 Figure 3.1 depicts the global structure of the asset return model involving investments in the three major asset classes — cash, bonds and equities — in the four major currency areas — the United States of America (US), the United Kingdom (UK), the European Union (EU) and Japan (JP) — together with emerging markets (EM) equities and bonds. Arrows depict possible explanatory dependence.

3.1.6 Following Dempster & Thorlacius (1998), the approach is to specify a canonical model for each currency area which is linked to the others directly via an exchange rate equation and indirectly through correlated innovations (disturbance or error terms). For capital market modelling with monthly data this approach was deemed likely to be superior to the usual

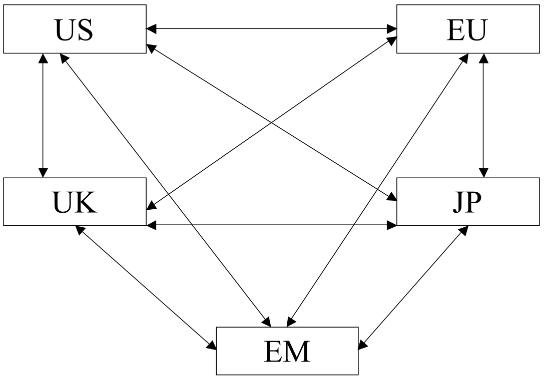


Figure 3.1. Pioneer asset return model global

macroeconomic (quarterly) trade flow linkages (see e.g. Pesaran & Shuermann, 2001) between currency areas. Figures 3.2 and 3.3 show respectively, at overall and detailed level, the structure of the canonical model of a major currency area. Potential liability models in each currency area are shown for completeness, although, of course, pension or guarantee liabilities might be needed only in fewer currencies. The next three sections

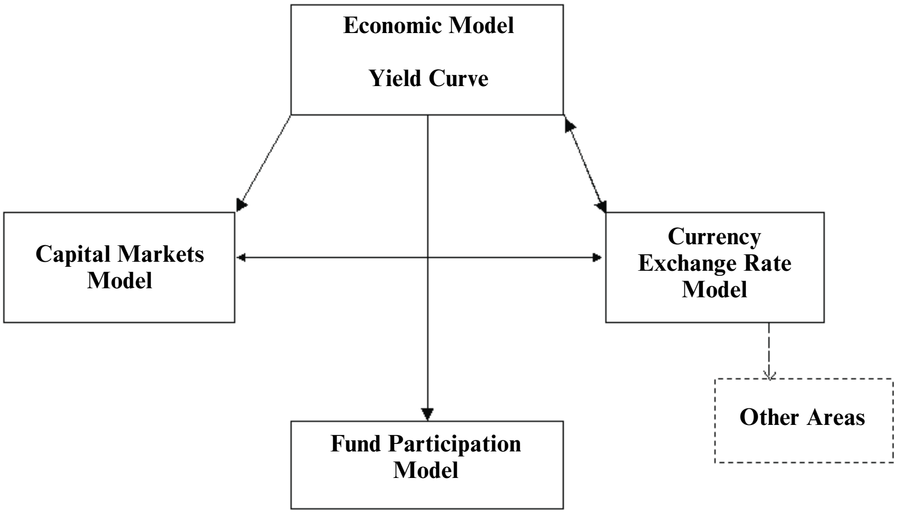


Figure 3.2. Major currency area model structure

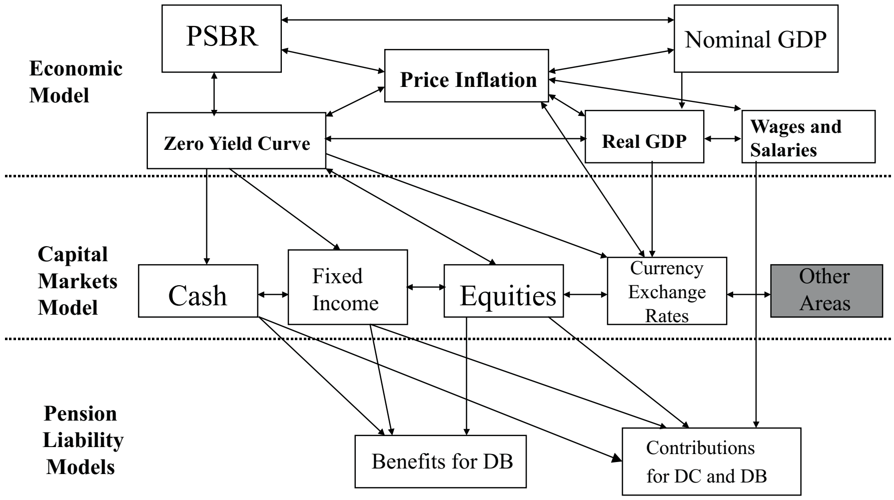


Figure 3.3. Major currency area detailed model structure

discuss respectively: the canonical model for the capital markets and exchange rate; the emerging markets model; and the canonical economic model. The home currency for these models is assumed to be the US dollar, but, of course, scenarios can be generated in any of the four major currencies, since cross rates are forecast, and any other currency (e.g. the euro) can be taken as the home currency for the statistical estimation.

3.2 *Capital Markets and Exchange Rate Model*

3.2.1 For simplicity, we specify here the evolution of the four state variables — equity (stock market) index (**S**), short-term (money market) interest rate (**r**), long-term (Treasury bond) interest rate (**l**) and exchange rate (**X**) — in continuous time form as:

$$\frac{d\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu}_s dt + \boldsymbol{\sigma}_s d\mathbf{Z}_s$$

$$d\mathbf{r} = \boldsymbol{\mu}_r dt + \boldsymbol{\sigma}_s d\mathbf{Z}_r$$

$$d\mathbf{l} = \boldsymbol{\mu}_l dt + \boldsymbol{\sigma}_l d\mathbf{Z}_l$$

$$\frac{d\mathbf{X}}{\mathbf{X}} = \boldsymbol{\mu}_X dt + \boldsymbol{\sigma}_X d\mathbf{Z}_X.$$

3.2.2 Here the drifts and volatilities for the four diffusion equations are potentially functions of the four state variables, and the $d\mathbf{Z}$ terms represent (independent) increments of correlated Wiener processes. All dependent variables in this specification are in terms of rates, while the explanatory state variables in the drift and volatility specifications are in original level (**S** and **X**) or rate (**r** and **l**) form.

3.2.3 Detailed specifications of discretised versions of this model are given in Appendix A. The resulting econometric model has been transformed to have all dependent variables in the form of returns and the disturbance structure contemporaneously correlated, but serially uncorrelated. In vector terms, the econometric discrete time model has the form:

$$\Delta \mathbf{x} = \mathrm{diag}(\mathbf{x})[\mu(\mathbf{x}) + \sqrt{\Sigma}\varepsilon]$$

where Δ denotes forward difference, diag (.) is the operator which creates a diagonal matrix from a vector, μ is a first order non-linear autoregressive filter, $\sqrt{\Sigma}$ is the Cholesky factor of the correlation matrix Σ of the disturbances, and the vector $\boldsymbol{\varepsilon}$ has uncorrelated standardised entries.

3.2.4 Although linear in the drift parameters to be estimated, this model is second order autoregressive and highly non-linear in the state variables, making its long run dynamics difficult to analyse and potentially unstable. For use in scenario generation over long horizons, the model must, therefore, be linearised so that its stability analysis becomes straightforward. Some linear variants used to date will be discussed in the sequel; we continue to experiment with appropriate forms. Due to its linearity in the parameters, this (reduced form) model may be estimated using the *seemingly unrelated regression* (SUR) technique, see e.g. Hamilton (1994) or Cochrane (1997), recursively until a parsimonious estimate is obtained in which all non-zero parameters are statistically significant.

3.3 *Emerging Markets Model*

3.3.1 After preliminary analysis of the emerging market equity and bond indices (see Table 3.1) using extreme value theory (Kyriacou, 2001) and experimentation with various ARMA/GARCH specifications, it was decided to fit the following ARMA (1,0) model with GARCH (1,1) error structure to index returns individually, viz.:

$$\begin{aligned} \mathbf{y}_t &= \alpha_0 + \alpha_1 \mathbf{y}_{t-1} - \beta_1 u_{t-1} + \mathbf{u}_t \\ H_t &= \gamma + pH_{t-1} + qu_{t-1}^2 \\ \mathbf{u}_t &:= \sqrt{H_t}\varepsilon_t \end{aligned}$$

where $\mathbf{y}$ denotes the (monthly) index return and $\boldsymbol{\varepsilon}$ is a serially uncorrelated standard normal or student t random variable. Interestingly, although in the EM index data analysed individually the equity index was less extreme than the bond index in terms of tail parameter estimate (Kyriacou, 2001), the

Table 3.1. Data proxies for model variables

Variable	Corresponding proxy
S^{US}	S&P 500 stock index
R^{US}	US 3-month T-bill rate
L^{US}	US 30-year T-yield with semi-annual compounding
S^{UK}	FTSE stock index
R^{UK}	UK 3-month T-bill rate
L^{UK}	UK 20-year GILT rate with semi-annual compounding
S^{EU}	MSCI Europe stock index
R^{EU}	German 3-month FIBOR rate
L^{EU}	German 10-year bond yield with annual compounding
S^{JP}	TOPIX stock index
R^{JP}	JP 3-month CD rate
L^{JP}	JP 10-year bond yield with annual compounding
S^{EM}	MSEMEI stock index
B^{EM}	EMBI+ bond index with 10-year average maturity
X^{UK}	UK/US exchange rate
X^{EU}	EU/US exchange rate
X^{JP}	JP/US exchange rate
C^{US}	US CPI
W^{US}	US wage index
G^{US}	US GDP
P^{US}	US public sector borrowing

above model fits both sets of index data reasonably well with Gaussian innovations. However, these innovations could be expected to be contemporaneously correlated between EM indices and with the innovations of the other variables in the model.

3.3.2 In this case the system model remains as in Section 3.2, but the enlarged contemporaneous covariance matrix Σ is no longer constant, and becomes a process Σ for the entries corresponding to the two extra EM returns. Following a general quasi-likelihood strategy (White, 1982), we may estimate a constant covariance matrix Σ, as before, using the residuals from the SUR capital market equation estimation and the normalised residuals $u_t/\sqrt{H_t}$ from the individual EM index estimations with sample variance (approximately) 1. Then we compute the Cholesky factor of the corresponding correlation matrix estimate and, for simulation of the full system equation, scale each correlated standardised innovation by the appropriate volatility estimate — constant $\hat{\sigma}$ or time and scenario dependent $\sqrt{\hat{H}_t}$.

3.4 *Economic Model*

3.4.1 In order to capture the interactions of the capital markets with the economy in each major currency area, a small model of the economy was developed with four state variables in nominal values: three financial — consumer price index (CPI), wages and salaries (WS) and public sector borrowing requirement (PSB) — and gross domestic product (GDP). For stability the specification is in terms of returns similar to the capital markets model, but with non-state-dependent volatilities, viz.:

$$\frac{CPI_{t+1} - CPI_t}{CPI_t} = \begin{pmatrix} a_{cpi_1} + a_{cpi_2} CPI_t + a_{cpi_3} WS_t + a_{cpi_4} GDP_t + a_{cpi_5} PSB_t \\ +b_{cpi_2} CPI_{t-1} + b_{cpi_3} WS_{t-1} + b_{cpi_4} GDP_{t-1} + b_{cpi_5} PSB_{t-1} \end{pmatrix} + \sigma_{cpi} \varepsilon_t^{cpi}$$

$$\frac{WS_{t+1} - WS_t}{WS_t} = \begin{pmatrix} a_{ws_1} + a_{ws_2} CPI_t + a_{ws_3} WS_t + a_{ws_4} GDP_t + a_{ws_5} PSB_t \\ +b_{ws_2} CPI_{t-1} + b_{ws_3} WS_{t-1} + b_{ws_4} GDP_{t-1} + b_{ws_5} PSB_{t-1} \end{pmatrix} + \sigma_{ws} \varepsilon_t^{ws}$$

$$\frac{PSB_{t+1} - PSB_t}{PSB_t} = \begin{pmatrix} a_{psb_1} + a_{psb_2} CPI_t + a_{psb_3} WS_t + a_{psb_4} GDP_t + a_{psb_5} PSB_t \\ +b_{psb_2} CPI_{t-1} + b_{psb_3} WS_{t-1} + b_{psb_4} GDP_{t-1} + b_{psb_5} PSB_{t-1} \end{pmatrix} + \sigma_{psb} \varepsilon_t^{psb}$$

$$\frac{GDP_{t+1} - GDP_t}{GDP_t} = \begin{pmatrix} a_{gdp_1} + a_{gdp_2} CPI_t + a_{gdp_3} WS_t + a_{gdp_4} GDP_t + a_{gdp_5} PSB_t \\ +b_{gdp_2} CPI_{t-1} + b_{gdp_3} WS_{t-1} + b_{gdp_4} GDP_{t-1} + b_{gdp_5} PSB_{t-1} \end{pmatrix} + \sigma_{gdp} \varepsilon_t^{gdp}.$$

This is again a second order autoregressive model in the state variables which, as shown, is linear in parameters and non-linear in variables. It may be estimated using the techniques mentioned in Section 3.2.

3.4.2 With a view to eventually including Treasury bond asset classes of different maturities in the system, a standard three-factor yield curve model (Campbell, 2000) was developed for fitting to spot yield curve data. The three factors in this model are a very short (one month) rate ($\boldsymbol{R}^0$) and long rate ($\boldsymbol{L}$) corresponding to the capital markets model, and a slope factor ($\boldsymbol{Y} := \boldsymbol{L} - \boldsymbol{R}$) between the short and long rates. By using a time series of monthly yield curve data, it is possible to estimate the evolution over the sample period of the market prices of risk (MPRs) for the three factors in volatility units by assuming that the model fits the yield curve exactly (commonly referred to as backing-out the MPRs).

3.4.3 In more detail, suppose that the processes for the three factors $\boldsymbol{R}_0$, $\boldsymbol{Y}$ and $\boldsymbol{L}$ under the real world probabilities satisfy:

$$d\boldsymbol{R}^0 = (k(\boldsymbol{L} + \boldsymbol{Y} - \boldsymbol{R}^0) + \alpha_{\boldsymbol{R}^0}\sigma_{\boldsymbol{R}^0})dt + \sigma_{\boldsymbol{R}^0} d\boldsymbol{W}_{\boldsymbol{R}^0} := \boldsymbol{\delta}_{\boldsymbol{R}^0} dt + \sigma_{\boldsymbol{R}^0} d\boldsymbol{W}_{\boldsymbol{R}^0}$$
$$d\boldsymbol{Y} = (\mu_y - \lambda_y \boldsymbol{Y} + \alpha_y \sigma_y)dt + \sigma_y d\boldsymbol{W}_y := \boldsymbol{\delta}_y dt + \sigma_y d\boldsymbol{W}_y$$
$$d\boldsymbol{L} = (\mu_L - \lambda_L \boldsymbol{L} + \alpha_L \sigma_L)dt + \sigma_L d\boldsymbol{W}_L := \boldsymbol{\delta}_L dt + \sigma_L d\boldsymbol{W}_L.$$

To calculate market prices of risk time series $\alpha_{R^0 t}, \alpha_{Yt}, \alpha_{Lt}$ for $t = 1, \ldots, T$, we first calibrate the model to detailed yield curve data at $t = 1$ in the usual manner, giving estimates of the model parameters $\mu_Y, \mu_L, \lambda_Y, \lambda_L, k, \sigma_{R^0}, \sigma_Y$ and σ_L. Estimates of the real world drifts $\delta_{R^0 t}, \delta_{Yt}, \delta_{Lt}, t = 1, \ldots, T$ can then be obtained from the historical (monthly) time series for the factors, using a suitable backward moving average and data prior to $t = 1$. Estimates of the market prices of risk can then be calculated using the expressions:

$$\alpha_{R^0 t} = (\delta_{R^0 t} - kL_t - kY_t + kR^0_t)/\sigma_{R^0}$$
$$\alpha_{Yt} = (\delta_{Yt} - \mu_y - \lambda_y Y_t)/\sigma_Y$$
$$\alpha_{Lt} = (\delta_{Lt} - \mu_L - \lambda_L L_t)/\sigma_L.$$

3.4.4 Figure 3.4 depicts the result of this procedure for the US over nearly a 24-year horizon. Note that, while the MPRs of the very short rate and the yield curve slope are highly positively correlated, they are both negatively correlated with the MPR of the long rate, as might be expected for a market which shifts its interest rate risk focus back and forth from short to long term.

3.4.5 As a preliminary analysis of the interactions of the US macroeconomic and capital market variables over the sample period, these

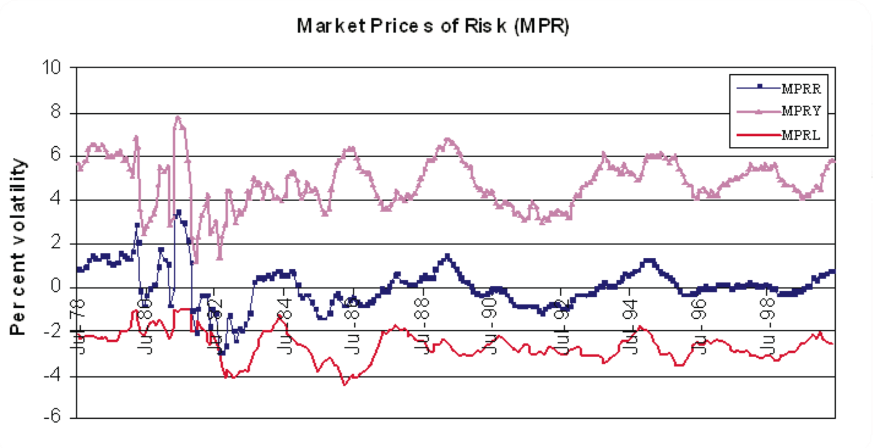


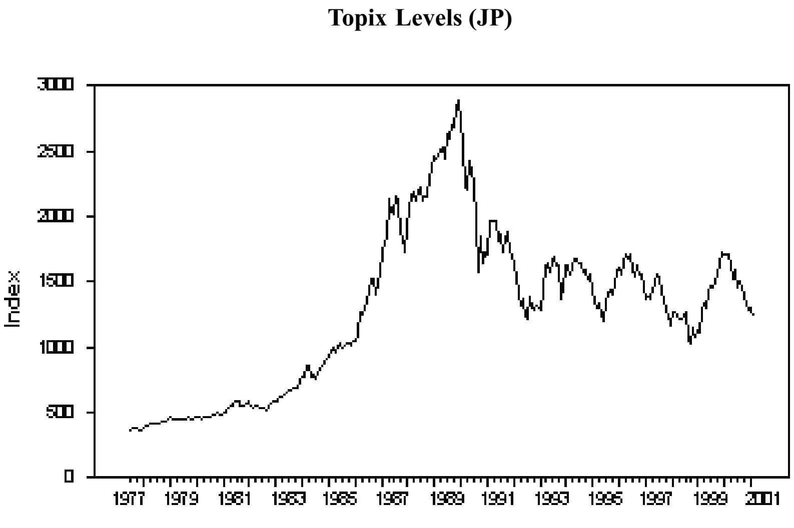
Figure 3.4. Evolution of US yield curve factor market prices of risk

MPRs were regressed on the macroeconomic variables expressed in both levels and returns and significant relationships noted. These accorded well with significant coefficients in the subsequent US system model estimation (see Section 3.6).

3.5 *Data and System Model Estimation*

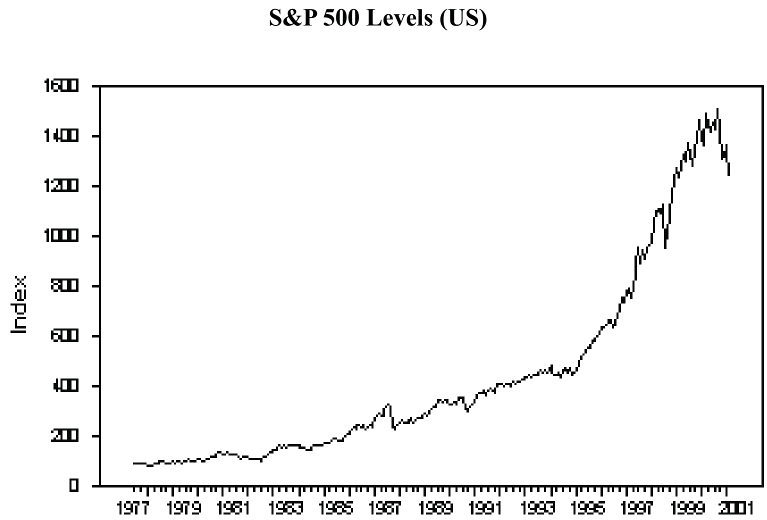
3.5.1 Table 3.1 sets out the data used as proxies for the variables of the full system so far discussed. Sources were DataStream and Bloomberg at monthly frequency from 1977, except for economic variables available only quarterly. Monthly levels were computed for the latter by taking the cube root of the actual quarterly return and finding the corresponding monthly levels between announcements. Figure 3.5 shows equity index evolution in the US and Japan over the 284-month period from July 1977 to February 2001. Dummy variable techniques were required to estimate the effects on constant terms of the bubble and crash period, thereby enabling a meaningful estimation of the Japanese currency area capital market equations. So far they have not proved necessary for recent US history! A consistent database of model data is currently being maintained and updated monthly by the fund manager.

3.5.2 Various subsystems of the full capital markets and economic model have been estimated (see Section 7) using the SURE model maximum



Source: DataStream

Figure 3.5a. Equity index evolution in JP



Source: DataStream

Figure 3.5b. Equity index evolution in the US

likelihood estimation procedures of RATS (Doan, 1996). For each model the full set of model parameters was first estimated, and insignificant (at the 5% level) variables sequentially removed to obtain a parsimonious final model with all statistically significant coefficients. This procedure has been automated in a PERL/RATS script, and (although we are well aware that, for given data, best variable selection is an NP-hard problem) the automated results agree virtually completely with the much more time consuming hand procedures. Estimation of the emerging market individual ARMA/GARCH equations to yield the AR(1)/GARCH (1,1) specification of Section 3.4 has been accomplished using S^+. The quasi-likelihood procedure for estimating full models with EM returns was described in Section 3.4.

3.6 *Results*

3.6.1 We summarise here only illustrative or highly significant findings; more detailed results are forthcoming in Arbeleche & Dempster (2003).

3.6.2 In this project we have devised a way of presenting econometric model estimation results concisely and graphically. For example, Figure 3.6 shows such an *influence diagram* for a full system model including the US economic variables. Boxes (economic variables) or circles (capital market variables) denote dependent variables (in return form with corresponding adjusted R^2 values shown in percentage terms) and arrows denote a

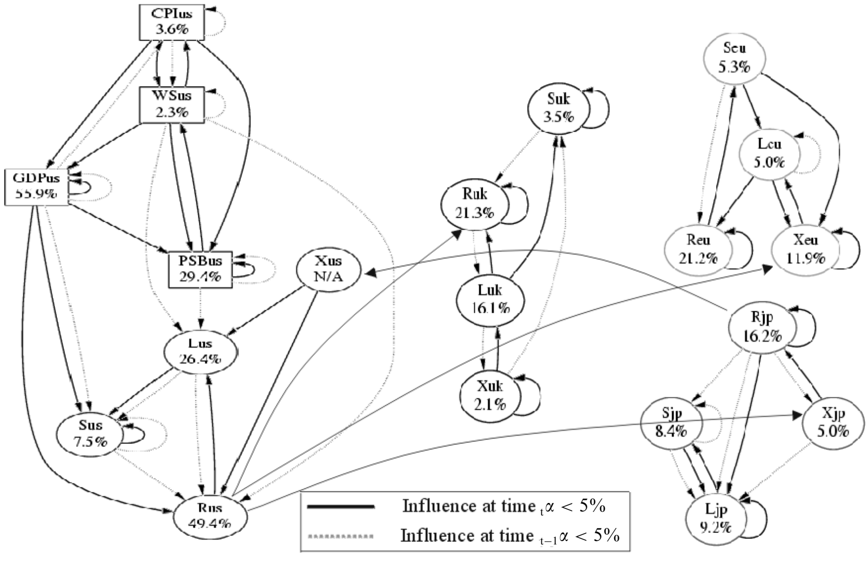


Figure 3.6. Influence diagram for CM +USE + EM 93/01-02/01 system model

significant influence (solid) or lagged influence (dotted) from a corresponding explanatory variable (tail) to a dependent variable return (tip). The seemingly unrelated regression nature of the model is obvious as each currency area is directly related only through exchange rates and indirectly related through shocks. In light of Meese & Rogoff's (1983a, b) classical view on the inefficacy of macroeconomic explanations of exchange rates even at monthly frequency, after considerable single equation and subsystem analysis we have found that interest rate parity expressed as inter-area short rate differences — together with other local capital market variables — has significant explanatory power, while purchasing power parity expressed various ways does not (c.f. Hodrick & Vassalou, 2002).

3.6.3 Figure 3.7 emphasises our main econometric finding that the world's equity and emerging bond markets and currency exchange rates are linked simultaneously through shocks. The first covariance (diagonal and below)/correlation (above diagonal) matrix is that of raw returns. The second is estimated using residuals from the fitted system model. The circled entries have high correlations, and do not change significantly — some actually increase — from the one to the other, showing that the dependent variables react mainly to current shocks (innovations) in spite of the stochastic nature of the explanatory variables.

Covariance/correlation matrix of raw returns 93:12 to 01:02

	SUS	RUS	LUS	SUK	RUK	LUK	XUK	SEU	REU	LEU	XEU	SJP	RJP	LJP	XJP	SIE	BIE	CPIUS	WSUS	GDPUS	PSBUS
SUS	0.002	-0.084	0.177	0.719	-0.008	-0.331	-0.023	0.702	-0.078	-0.274	0.202	0.371	0.023	0.075	-0.109	0.619	0.617	0.217	0.101	0.307	0.175
RUS	0.000	0.002	0.420	-0.111	0.224	0.272	-0.182	0.030	0.177	0.345	0.160	0.020	0.096	0.138	0.218	-0.155	-0.152	0.027	0.165	0.207	0.225
LUS	0.000	0.001	0.001	-0.086	-0.104	0.599	-0.192	-0.007	-0.001	0.600	0.168	0.232	-0.065	0.080	0.202	0.137	-0.160	-0.076	0.037	0.031	-0.022
SUK	0.001	0.000	0.000	0.001	-0.090	-0.373	-0.268	0.762	-0.007	-0.310	0.270	0.347	0.141	0.122	-0.073	0.583	0.566	0.139	0.006	0.208	0.075
RUK	0.000	0.000	0.000	0.000	0.001	0.064	0.277	0.131	0.186	0.021	-0.011	-0.004	0.062	-0.219	0.182	-0.097	-0.001	0.103	0.099	0.072	0.037
LUK	0.000	0.000	0.001	0.000	0.000	0.001	-0.020	-0.234	-0.062	0.654	-0.034	0.063	-0.093	-0.053	0.163	-0.035	-0.232	-0.078	0.013	-0.108	-0.045
XUK	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.225	-0.045	-0.017	-0.585	-0.058	-0.071	-0.051	-0.256	-0.142	-0.188	0.012	0.015	-0.003	0.078
SEU	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.003	-0.034	-0.220	0.437	0.414	0.115	0.174	0.131	0.554	0.529	0.204	0.103	0.323	0.194
REU	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.154	-0.038	-0.118	0.443	0.083	-0.072	-0.152	-0.070	-0.072	0.041	0.043	-0.069
LEU	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.001	-0.178	-0.018	-0.085	0.107	-0.099	-0.020	-0.225	-0.128	-0.009	-0.022	-0.035
XEU	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.271	0.063	0.097	0.369	0.169	0.256	0.082	0.091	0.138	0.067
SJP	0.001	0.000	0.000	0.001	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.003	-0.095	0.180	0.051	0.490	0.391	-0.004	-0.191	0.069	0.043
RJP	0.000	0.001	-0.001	0.002	0.001	-0.001	0.000	0.002	0.006	0.000	0.000	-0.001	0.085	0.088	-0.102	-0.108	0.037	0.097	0.025	0.068	-0.034
LJP	0.000	0.001	0.000	0.001	-0.001	0.000	0.000	0.001	0.000	0.000	0.000	0.001	0.003	0.015	-0.172	0.010	-0.120	-0.034	-0.115	0.024	-0.066
XJP	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001	-0.001	0.001	-0.050	0.441	-0.032	0.119	-0.003	0.008
SIE	0.002	0.000	0.000	0.001	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.002	-0.002	0.000	0.000	0.005	0.721	0.051	-0.086	0.038	-0.023
BIE	0.002	0.000	0.000	0.001	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.001	0.001	-0.001	0.000	0.003	0.003	0.243	0.062	0.182	0.155
CPIUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.519	0.799	0.487
WSUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.560	0.491
GDPUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.682
PSBUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Covariance/correlation matrix of residuals 93:12 to 01:02

	SUS	RUS	LUS	SUK	RUK	LUK	XUK	SEU	REU	LEU	XEU	SJP	RJP	LJP	XJP	SIE	BIE	CPIUS	WSUS	GDPUS	PSBUS
SUS	0.002	-0.138	-0.148	0.675	0.009	-0.261	0.060	0.674	0.113	-0.241	0.166	0.421	-0.017	0.197	-0.134	0.639	0.562	0.045	0.038	0.129	-0.072
RUS	0.000	0.001	0.201	0.071	-0.004	-0.014	-0.294	-0.003	0.259	-0.007	0.251	-0.027	0.278	0.094	0.319	-0.243	-0.130	-0.347	-0.090	-0.017	0.098
LUS	0.000	0.000	0.001	-0.026	-0.169	0.507	0.254	-0.009	0.010	0.449	0.206	0.144	-0.132	-0.058	0.247	0.048	-0.104	-0.062	0.171	0.141	-0.028
SUK	0.001	0.000	0.000	0.001	-0.098	-0.340	-0.236	0.732	0.002	-0.325	0.248	0.278	0.152	0.114	-0.104	0.589	0.550	0.073	-0.073	0.090	-0.105
RUK	0.000	0.000	0.000	0.000	0.001	0.103	0.292	0.171	0.138	0.018	0.067	0.175	0.031	-0.229	0.187	-0.098	-0.079	-0.034	0.100	0.031	-0.088
LUK	0.000	0.000	0.000	0.000	0.000	0.001	-0.107	-0.120	0.029	0.608	0.106	0.103	-0.170	-0.062	0.277	-0.079	-0.167	-0.140	0.233	0.110	0.061
XUK	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.184	-0.001	-0.071	-0.609	0.085	0.067	-0.061	-0.253	-0.143	-0.229	0.013	0.307	0.036	0.087
SEU	0.001	0.000	0.000	0.001	0.000	0.000	0.000	0.002	-0.049	-0.216	0.409	0.427	0.105	0.189	0.122	0.595	0.512	0.063	0.062	0.145	-0.041
REU	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.187	-0.022	-0.019	0.528	0.045	-0.059	-0.178	-0.121	-0.219	-0.043	0.286	-0.023
LEU	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	-0.086	0.052	-0.051	0.080	-0.009	-0.103	-0.150	-0.127	0.090	0.113	0.023
XEU	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.258	0.049	0.017	0.375	0.159	0.224	-0.046	-0.067	0.051	-0.009
SJP	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.002	-0.189	0.115	0.066	0.466	0.447	0.145	0.201	0.412	0.171
RJP	0.000	0.002	-0.001	0.001	0.000	-0.001	0.000	0.001	0.005	0.000	0.000	-0.002	0.069	0.110	-0.140	-0.120	-0.057	-0.036	-0.075	0.088	0.021
LJP	0.001	0.000	0.000	0.000	-0.001	0.000	0.000	0.001	0.000	0.000	0.000	0.001	0.003	0.012	-0.165	0.034	-0.045	0.067	0.010	0.053	-0.075
XJP	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001	-0.001	0.001	-0.038	0.109	-0.159	0.067	-0.154	-0.001
SIE	0.025	-0.007	0.001	0.021	-0.003	-0.002	-0.003	0.029	-0.007	-0.002	0.004	0.021	-0.031	0.004	-0.001	0.976	0.699	0.199	-0.059	0.168	0.024
BIE	0.022	-0.004	-0.003	0.020	-0.002	-0.005	-0.005	0.025	-0.005	-0.004	0.005	0.021	-0.015	-0.005	0.004	0.686	0.988	0.204	-0.150	-0.040	-0.017
CPIUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.091	0.048	-0.064
WSUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.293	0.305
GDPUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.481
PSBUS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 3.7. Covariance/correlation matrices for the model of Figure 3.6

4. Asset Return and Liability Scenario Simulation and Calibration

4.1 *Simulation and Calibration*

4.1.1 The system model discussed in the previous section is a vector non-linear second order autoregressive model with a monthly timestep. Given initial values of its state variables, it may be simulated without stochastic innovations as a discrete time, deterministic, dynamical system defining the mean paths of the state variables. The non-linear dynamics of this deterministic system may be exceedingly complex, and the system may rapidly explode or die to zero values of some variables for certain configurations of the (significant) estimated parameters. Graphical emulation of the central tendencies of the historical path by this deterministic system (see Figure 4.1) is a

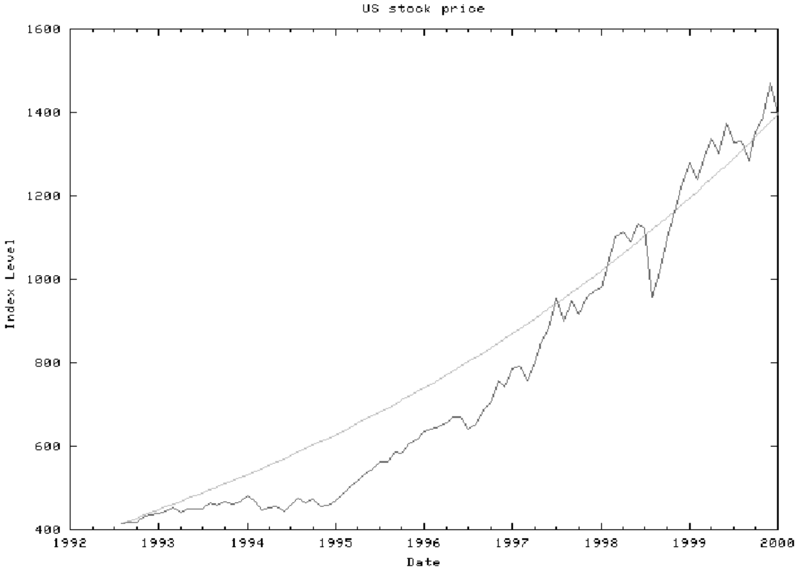


Figure 4.1a. In-sample drift with US history — stock index

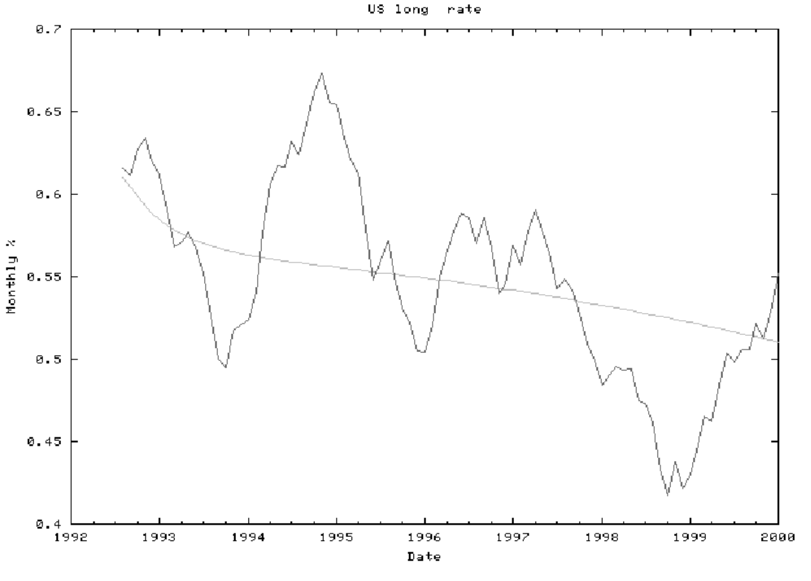


Figure 4.1b. In-sample drift with US history — long rate

necessary condition for the generation of realistic scenarios — alternative histories — by Monte Carlo simulation of the stochastic dynamical system. Monte Carlo simulation of this non-linear vector stochastic difference equation is effected by Euler (first order) stochastic simulation of the independent Gaussian or Student *t* disturbances which are correlated through the estimated Cholesky factor of the contemporaneous covariance matrix. The implication is that a limited number of estimated parameters — both coefficients and volatilities — may need adjustment to make both the deterministic and corresponding stochastic systems graphically match history (in-sample). Since the impacts of parameter changes is complex due to the non-linearity of the system, this is not an easy task. Nevertheless, intuitions can be developed to make the achievement of reasonably accurate calibrations tractable, and we have developed a prototype graphical interface tool *stochgen 3.0* (Dempster *et al.*, 2002) to aid the process graphically. Ideally, the calibration process itself should be formalised as a non-linear optimisation problem for some out-of-sample prediction error criterion, and we are currently working on limited versions of this. However, the development of appropriate prediction criteria is itself a challenge, to say nothing of the fact that the parameter optimisation problem involving an out-of-sample prediction error criterion is a non-convex optimisation problem of at least the difficulty of the dynamic stochastic optimisation problems that we wish to solve. As previously noted, we have therefore made considerable use of graphics.

4.1.2 Figure 4.1 shows a typical graphical result of a calibrated deterministic simulation of the non-linear system in the estimation period (in-sample). Figure 4.2 shows the corresponding in-sample scenario generation where one is looking for scenario paths with similar properties to the historical path. Similar scenarios may be generated out-of-sample. For calibration purposes, however, the 0%, 25%, 50%, 75% and 100% scenario values in each out-of-sample period, as shown in Figure 4.3, are more valuable. The US stock index plot in the figure shows the desirable calibration in which out-of-sample the historical path is centred in the 50% inter-quantile range of the scenario state distributions over time. The US long rate plot shows the less desirable result in which the historical path is captured by the scenario distributions, but is probabilistically over predicted. As noted above, in or out-of-sample calibration of all variables is difficult, and while the weaker criterion may always be met out-of-sample by calibration, in our experience the stronger criterion is usually only met for about 50% of the state variables in a calibration.

4.1.3 Another approach to econometric model calibration is to linearise a non-linear system to obtain a *vector autoregressive* (VAR) system which is stable in the state variable returns, so that the deterministic system converges to steady state returns and shocks to the corresponding stochastic system are non-persistent. Stability analysis for such a system is more easily conducted by appropriate eigenvalue analysis of the explanatory variable

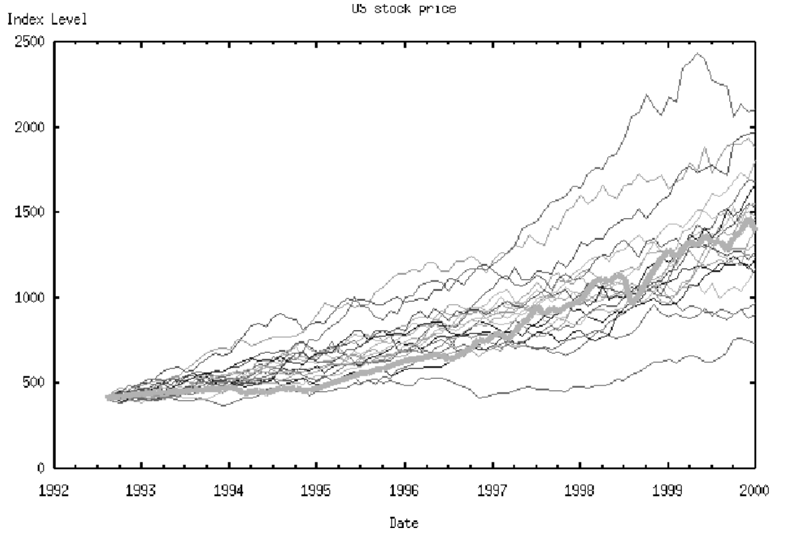


Figure 4.2a. In-sample scenarios with US history — stock index

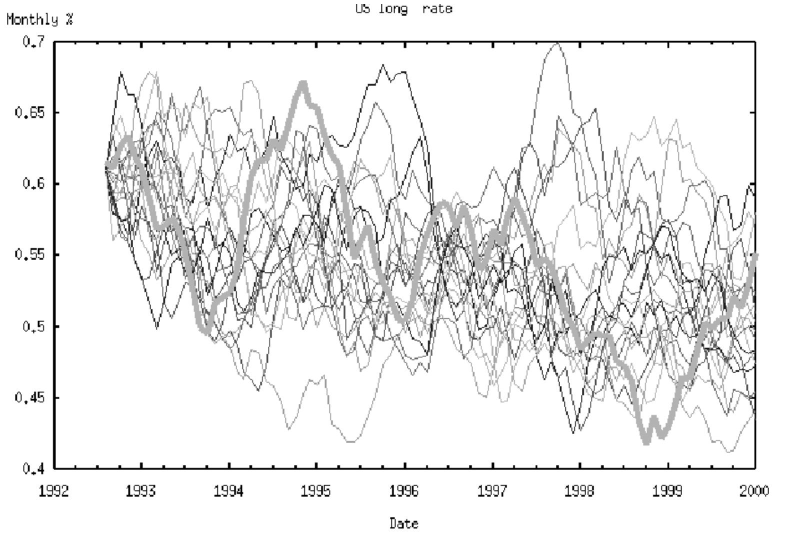


Figure 4.2b. In-sample scenarios with US history — long rate

Calibration period; 1977-1990

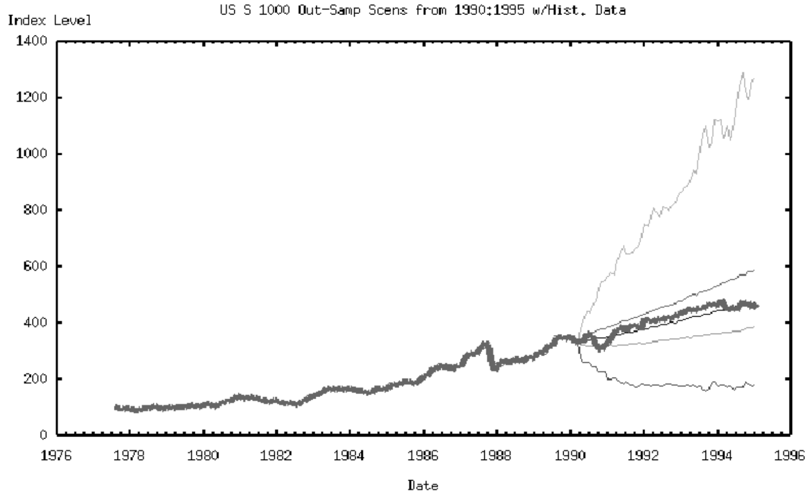


Figure 4.3a. Out-of-sample simulation quantiles with US history — stock index

Simulation period; 1990-1995

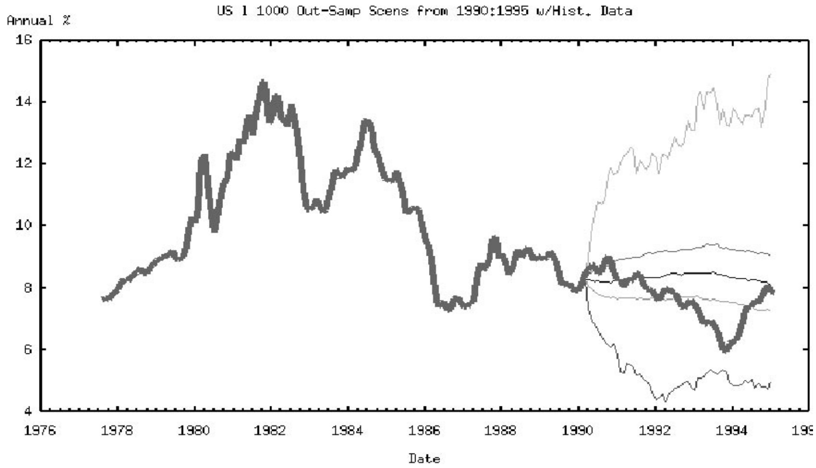


Figure 4.3b. Out-of-sample simulation quantiles with US history — long rate

coefficient matrices — the leading eigenvalue (root) must be less than one in modulus. For given data, the feasibility of fitting such a model may be checked by (autoregressive) impulse response analysis (Garratt *et al.*, 2000; Hamilton, 1994), and testing on our full model data to August 2002 has been affirmative. The VAR approach can be extended to an adaptive error-correcting VAR model (Boender *et al.,* 1998; Pesaran & Schuerman, 2001), on which we are currently engaged, and will be reported elsewhere (Arbeleche & Dempster, 2003).

4.1.4 Finally, treating the process generating the historical data as stationary with independent increments — an unrealistic assumption — we may alternatively conduct historical simulation by resampling from the empirical marginal distributions of state variable returns constructed from the historical paths over the in-sample period.

4.1.5 All these options have been evaluated, and we report dynamic stochastic optimisation backtest results for all three approaches to scenario generation for our dynamic ALM problem in Section 7.

4.2 *Comparative Scenario Return Distribution Evaluation*

Out-of-sample scenario marginal return distributions from calibrated system models were evaluated in two ways: against the empirical marginal return distribution generated by the out-of-sample historical path (Figure 4.4), and against an alternative scenario generation system (Figure 4.5). The ten-year out-of-sample annual return distributions in (the representative) Figure 4.5 were generated by the capital markets model and the market-neutral version of InQA's simulator, based on the Wilkie global model (Wilkie, 2000). These comparative results were judged to be more than acceptable.

4.3 *Suppression of Sampling Error*

Since we must always use a finite sample of scenarios, there will always be sampling error in the generation of scenario return state distributions relative to the calibrated estimated system model. This can lead to serious errors and spurious arbitrages in subsequent portfolio optimisation. These can, however, be suppressed by ensuring that the sample marginal return distributions corresponding to all generated scenarios at a specific point in time have two moments matched to those of the theoretical model underlying the simulations (Høyland & Wallace, 2001; Høyland *et al.,* 2001). This can be posed in terms of matching the moments of the sampled innovations with their theoretical — here independent standard normal or student t — distributions. The first sample moments are easily set to zero by translation, and the unit second moments can be matched by matrix calculations or in terms of a non-linear programme, which can be solved by sequential quadratic programming using the SNOPT software (Villaverde, 2003).

Calibration period; 1977-1990

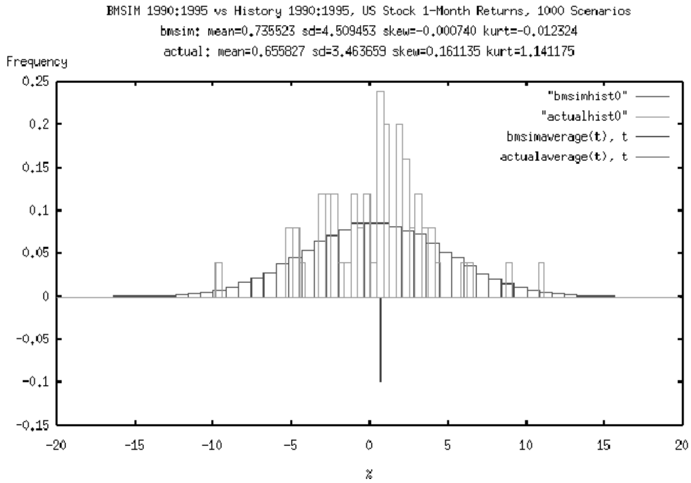


Figure 4.4a. Comparison of one-month returns with US history — stock index

Simulation period; 1990-1995

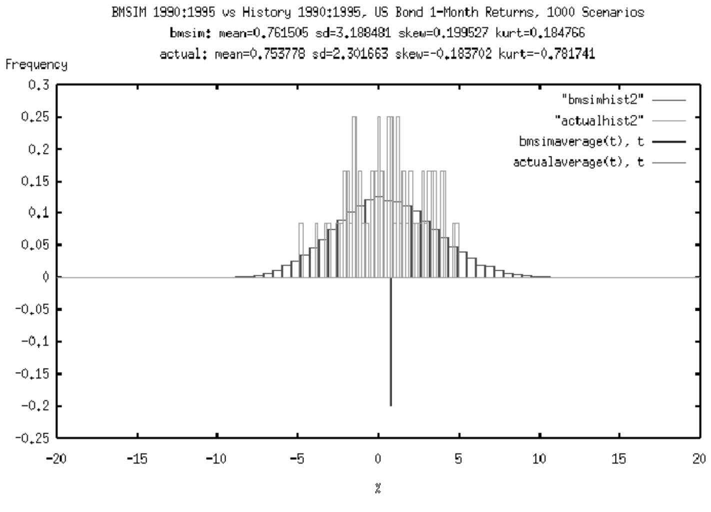


Figure 4.4b. Comparison of one-month returns with US history — long bond

Calibration period; 1992-2000

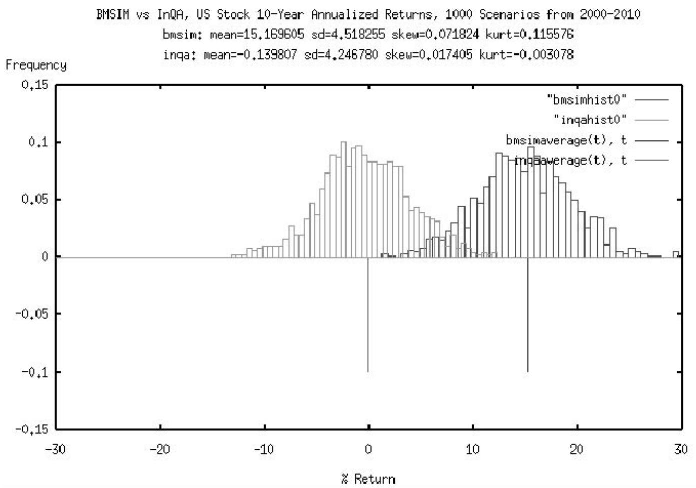


Figure 4.5a. Comparison of ten-year annualised US returns with InQA — stock index

Simulation period; 2000-2010

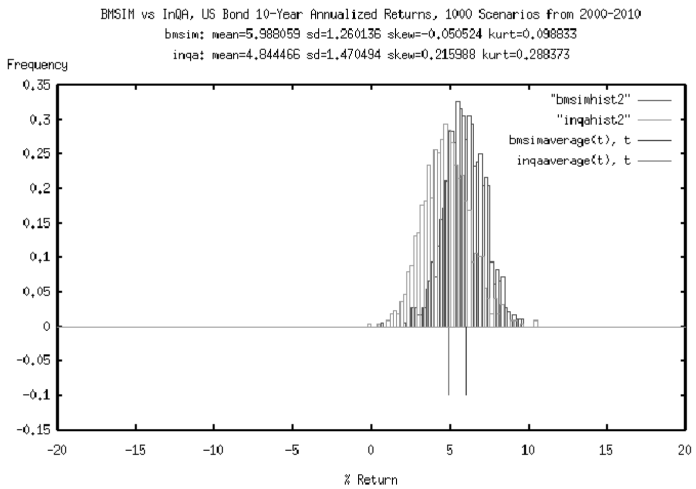


Figure 4.5b. Comparison of ten-year annualised US returns with InQA — long bond

4.4 *Liability Modelling and Simulation*

4.4.1 A proprietary stochastic Markov chain model for defined benefit pension fund liabilities has been developed, which currently assumes (unrealistically) that liabilities and fund return performance and macroeconomic variables, such as CPI and the wages and salaries index, are independent. Nevertheless, formidable calibration problems for the liability model remain due to lack of historical data.

4.4.2 For defined contribution pension funds, similar interdependence between lagged fund performance and participation rates is a reality. In principle, this can be handled (Dempster, 1988), but is again difficult to specify and calibrate.

4.4.3 Tax liabilities for funds in the jurisdiction of the fund manager are particularly simple — a 1% proportional transaction cost.

4.4.4 If complex liability models (including more complex tax liabilities) can be simulated — possibly together with asset returns and macroeconomic variables — to result in a net liability cash flow process, no difficulties arise in the optimisation model (see e.g. Consigli & Dempster, 1998). In this paper, however, we concentrate on the newer — previously unsolved — problem of incorporating the guarantee liabilities of defined contribution pension plans into scenario-based stochastic optimisation models (see Section 6.4).

4.5 *Scenario Tree Generation*

4.5.1 As mentioned in Section 2.2, in order to mirror reality, dynamic stochastic optimisation models for strategic DFA problems must face alternative scenario uncertainty at each decision point in the model — e.g. at each forward portfolio rebalance. Otherwise, the model decisions incorporate future knowledge along scenarios — hardly possible in the real world of finance! The distinction is between the so-called flat out-of-sample scenarios of Figure 4.6 and a scenario tree, an example of which is shown schematically in Figure 4.7. Each path from the root to a leaf node in the latter scheme represents a scenario, and the nodes represent decision points — the root node represents the initial implemented decision (e.g. initial portfolio balance). Subsequent nodes represent forward 'what-if' decisions facing the uncertainty represented by all scenarios emanating from that node.

4.5.2 Note that the Monte Carlo simulation of scenarios corresponding to a given scheme is a non-trivial matter requiring generic software to handle a complex simulator, such as is needed for the Pioneer model. We have used the generic *stochgen 2.3* software of the STOCHASTICS™ toolchain for dynamic stochastic optimisation (Dempster *et al.*, 2002) and its variants tailored for Pioneer.

4.5.3 In this software the input tree structure is represented for a symmetric balanced scenario tree by a product of branching factors, e.g. 3.3.3 or 3^3 for the scenario tree of Figure 4.7, or by a scenario or nodal

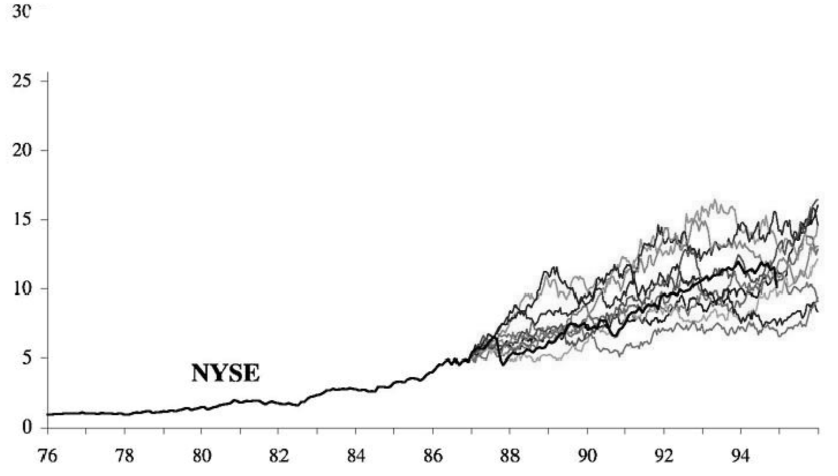


Figure 4.6. Out-of-sample flat scenario generation

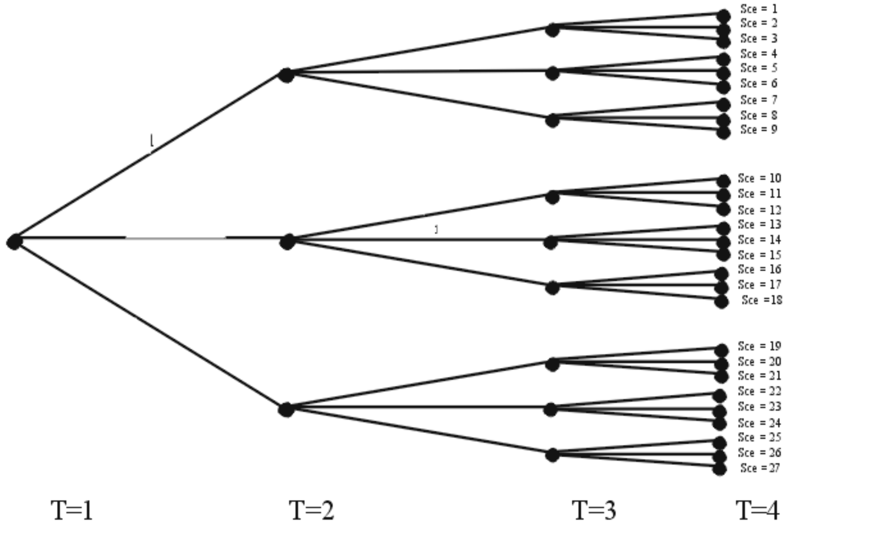
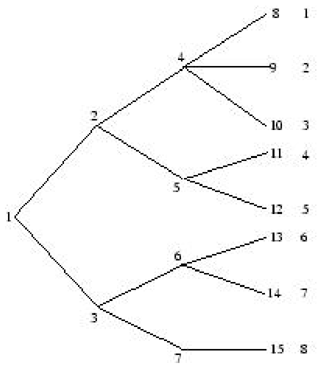


Figure 4.7. Schematic out-of-sample scenario tree branching structure with uniform branching factor three



$$L = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 5 \\ 1 & 6 & 6 & 6 \\ 1 & 6 & 6 & 7 \\ 1 & 6 & 8 & 8 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 2 & 4 & 9 \\ 1 & 2 & 4 & 10 \\ 1 & 2 & 5 & 11 \\ 1 & 2 & 5 & 12 \\ 1 & 3 & 6 & 13 \\ 1 & 3 & 6 & 14 \\ 1 & 3 & 7 & 15 \end{bmatrix}$$

Figure 4.8. Example of a scenario tree with corresponding scenario and nodal partition matrices

partition matrix for asymmetric trees as shown in Figure 4.8. The *scenario partition matrix* (Lane & Hutchinson, 1980) corresponds to the discrete scenario information partition inherent in the tree structure at each decision point, while the *nodal partition matrix* (used in *stochgen 2.3*) denotes the node through which each scenario passes at each decision point, and is useful for decomposition-based optimisers.

4.5.4 The *stochgen* software must handle, at each node, multiple conditional stochastic simulations of versions of the asset return model initialised by the data at the node and two previous timesteps (months) along the scenario path. Notice that the simulation time step (a month) is much shorter than the decision point frequency (for forward portfolio rebalancing: quarterly, semi-annually or annually), c.f. Dempster *et al.* (2000).

4.5.5 In the reported project backtests we used balanced scenario trees with high initial branching (see Section 7.2).

4.5.6 A number of variants of the BMSIM stochastic simulator for the non-linear econometric model have so far been written in C^{++}/C, but in the *stochgen 3.1* software currently under development these variants are specified as extensions or restrictions of a full model. Similarly, VARSIM and VARSIM 2 are simulators for variants of the VAR linearisation of the asset return model, and HSIM performs the historical bootstrap simulation described in Section 4.1.

4.5.7 Obtaining bond returns in a currency area is somewhat subtle, since they must be derived from bond yields. A representative derivation is given in Appendix B. Handling a complex external stochastic simulator is just one function of the variants of the *stochgen* software, and we will return to its other functions in Section 6 after describing, in the next section, the strategic ALM dynamic stochastic optimisation models used in our project.

5. Optimal Dynamic Asset Liability Management

5.1 *CALM Problem Formulation*

5.1.1 The dynamic ALM model used in the Pioneer project is a variant of the CALM model (Dempster, 1993), used previously in other projects (Consigli & Dempster, 1998; Hicks-Pedrón, 1998). Here we describe the main features of the model. A precise mathematical description is given in Appendix C.

5.1.2 We focus in this paper on what is normally called *strategic asset allocation*, which is concerned with allocation across broad asset classes such as equity and bonds of a given country. The problem is as follows:

— Given a set of assets, a fixed planning horizon and a set of rebalance dates, find the trading strategy that maximises the risk adjusted wealth accumulation process subject to the constraints.

As noted in Section 4.4, defined contribution pension plans or other complex liabilities (such as insurance or reinsurance claims) may be added to the basic model as a stochastic net cash flow stream (see e.g. Consigli & Dempster, 1998).

5.1.3 In the model description given below we begin with a discussion of alternative utility functions (fund risk tolerances) (Section 5.2) and continue on to treat the specification of risk management objectives through the problem objective function (Section 5.3) and then the constraints (Section 5.4). The last two sections discuss, respectively, the optimal setting of benchmark portfolios (Section 5.5) and the specification of probabilistic *value at risk* (VaR) constraints for the model connected with defined contribution guarantee liabilities (Section 5.6).

5.1.4 We consider a discrete time and space setting. It is assumed that the fund operates from the viewpoint of one currency which we call the *home currency*. Unless otherwise mentioned, all quantities are assumed to be in the local currency. There are $T+1$ times (the first T are decision points) indexed by $t = 1, \dots, T+1$, where $T+1$ corresponds to the planning *horizon* at which no decisions are made. Uncertainty is represented by a finite set of time evolutions of states of the world, or *scenarios*, denoted by Ω. The *probability* $p(\omega)$ of scenario ω in Ω is here always the reciprocal of the number of scenarios, since these scenarios are being generated by Monte Carlo simulation, as discussed in the previous section.

5.1.5 *Assets* take the form of equity, bonds and cash. Let I denote the set of all equity and bond assets and K denote the set of cash assets. The fund begins with an *initial endowment* of equity and bonds given by $\{x_i : i \in I\}$ and of cash in the home currency given by w_1. The fund trades in the assets at $t = 1, \dots, T$, i.e. at all times except at the planning horizon.

5.1.6 A *trading strategy* is given by $\theta_{ikt}(\omega) := (x_{it}(\omega), x^+_{it}(\omega), x^-_{it}(\omega), z^+_{kt}(\omega), z^-_{kt}(\omega))$ for i in I, k in K, $t = 1, \dots, T$, ω in Ω, where:

— $x_{it}(\omega)$ denotes the amount *held* of asset i between time t and time $t+1$ in state ω.
— $x^+_{it}(\omega)/x^-_{it}(\omega)$ denotes the amount *bought/sold* of asset i at time t in state ω. The introduction of the buy/sell variables is used to account for proportional transaction costs on buying and selling equity and bond assets. Denote by f and g respectively, the *proportional transaction cost* of buying or selling an equity or bond asset. For example, a 1% proportional transaction cost on buying and selling an equity or bond asset corresponds to $f = 1.01$ and $g = 0.99$.
— $z^+_{kt}(\omega)/z^-_{kt}(\omega)$ denotes the amount of cash *lent/borrowed* in asset k between time t and time $t+1$ in state ω. The positions in cash are split into long and short components to account for different rates of borrowing and lending. We assume that cash lent and borrowed at time t in any currency is automatically converted back to the home currency at time $t+1$.
— The *asset returns* are given by $\{v_{it}(\omega), (r^+_{kt}(\omega), r^-_{kt}(\omega))$: i in I, k in K, $t = 2, \ldots, T+1$, ω in $\Omega\}$, where:
  — $v_{it}(\omega)$ denotes the net return on asset i between time $t-1$ and time t in state ω.
  — $r^+_{kt}(\omega)/r^-_{kt}(\omega)$ denotes the net return on lending/borrowing asset k between time $t-1$ and time t in state ω.

5.1.7 The *exchange rates* are given by $\{(p_{it}(\omega), p_{kt}(\omega))$: i in I, k in K, $t = 1, \ldots, T+1$, ω in $\Omega\}$ where:
— $p_{it}(\omega)$ denotes the exchange rate of asset i at time t in state ω expressed as home currency/local currency.
— $p_{kt}(\omega)$ denotes the exchange rate of asset k at time t in state ω expressed as home currency/local currency.

5.1.8 The fund may face cash inflows and outflows given by $\{(q^+_t(\omega), q^-_t(\omega))$: $t = 2, \ldots, T$, ω in $\Omega\}$, where:
— $q^+_t(\omega)/q^-_t(\omega)$ denotes the *cash flow in/out* at time t in state ω.

5.1.9 A trading strategy θ results in a *wealth before rebalancing* of $w^\theta_t(\omega)$ for $t = 2, \ldots, T+1$ and $\omega \in \Omega$, and a *wealth after rebalancing* of $W^\theta_t(\omega)$ for $t = 1, \ldots, T$ and $\omega \in \Omega$.

5.1.10 Subject to the constraint structure, the fund acts by choosing the trading strategy which maximises the (von Neumann-Morgenstern) *expected utility* of the wealth process, which is assumed to take the form:

$$\mathrm{E}[\mathrm{U}(\mathbf{w}^\theta_2, \ldots, \mathbf{w}^\theta_{T+1})] = \sum_{\omega \in \Omega} p(\omega) \sum_{t=2}^{T+1} u_t(w^\theta_t(\omega)).$$

Alternative *period utility functions* u_t are discussed in the next section.

5.2 *Utility Functions*

5.2.1 The functional U is used to define the risk preferences of the fund over the wealth process in such a way that $E[U(\mathbf{w}_2^{\theta_1}, \ldots, \mathbf{w}_{T+1}^{\theta_1})] > E[U(\mathbf{w}_2^{\theta_2}, \ldots, \mathbf{w}_{T+1}^{\theta_2})]$ if, and only if, the wealth process generated by θ_1 is strictly preferred to the wealth process generated by θ_2. Thus, a clearly desirable property of U is that it be strictly increasing. Another desirable property of U is that it be concave. If U is concave $E[U(\mathbf{w}_2^{\theta}, \ldots, \mathbf{w}_{T+1}^{\theta})] \leq U(E[\mathbf{w}_2^{\theta}], \ldots, E[\mathbf{w}_{T+1}^{\theta}])$. The interpretation is that the utility of having the certain quantities $E[\mathbf{w}_2^{\theta}], \ldots, E[\mathbf{w}_{T+1}^{\theta}]$ is preferred to the expected utility of having the uncertain quantities $\mathbf{w}_2^{\theta}, \ldots, \mathbf{w}_{T+1}^{\theta}$. Thus, if U is strictly concave, the fund is said to be *risk averse*, and if it is linear it is said to be *risk neutral*. (If U is convex, then it is said to be risk loving or *risk seeking*.) Since U is a linear combination of the u_t, U will be strictly increasing and concave if they are.

5.2.2 As noted in ¶2.1, the utility functional is used here to represent the general attitude to risk of the fund's participants over a specified fund horizon. Short horizon funds are likely to attract more risk averse participants than very long horizon funds, whose long-term participants can afford to tolerate more risk in the short run. Even for such problems, however, the fund manager will likely wish to mitigate the long-term participants' risk tolerances in the short run in the interest of maintaining competitive participation rates. In any event, choice of a sequence of period utility functions can be used to shape the evolution of the wealth process over the scenarios in the scenario tree of the problem. Appropriate tree size and branching structure — together with variance reduction (¶4.3) — can be used to ensure that these distributional properties, resulting from the implemented decisions, continue to hold against sufficiently large samples of further flat scenarios not included in the problem scenario tree — a prerequisite for good out-of-sample performance (see Section 7.2).

5.2.3 We consider the following period utility functions:

(1) *Exponential* (CARA) $\quad u(w) = -e^{-aw} \qquad a > 0$

(2) *Power* (CRRA) $\quad u(w) = \dfrac{w^{1-a}}{1-a} \qquad a > 0$

(3) *Downside-quadratic* $\quad u(w) = (1-a)w - a(w - \tilde{w})_-^2$

$$0 \leq a \leq 1, 0 \leq \tilde{w} \leq \infty.$$

5.2.4 Note that log utility given by $u(w) = \log(w)$ is a limiting case of power utility as $a \to 1$. The $\tilde{w}$ parameter that appears in the downside-quadratic utility function denotes a target wealth. Note that this utility function reduces to linear (risk-neutral) utility, given by $u(w) = w$ for $a := 0$.

5.2.5 The exponential utility function is also referred to as the *constant absolute risk aversion* (CARA) utility function, because its Arrow-Pratt absolute measure of risk aversion, defined by $-u''(w)/u'(w)$, is equal to the

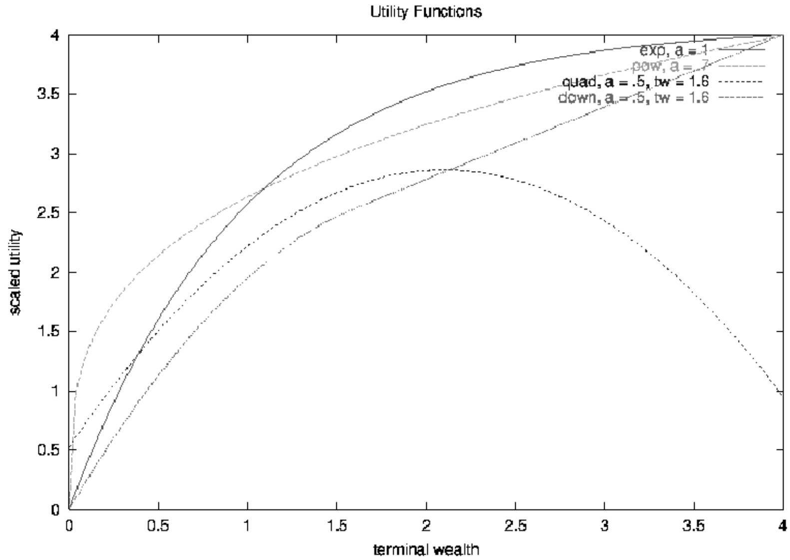


Figure 5.1. Scaled risk averse utility functions

constant a. The power utility function is also referred to as the *constant relative risk aversion* (CRRA) utility function, because the Arrow-Pratt relative measure of risk aversion, defined by $-wu''(w)/u'(w)$, is equal to the constant a. The downside-quadratic utility function, similar to the *mean-downside-variance* or *mean-semi-variance* utility function, except that it has $\tilde{w}$ in place of E[**w**], aims to maximise wealth and, at the same time, penalise downside deviations of the wealth from the target. This is illustrated in Figure 5.1, which depicts the different amounts of risk aversion implied by the curvature of the utility functions — for a fixed slope, the greater the curvature the greater the aversion to risk. Of particular interest is the curvature for wealth levels less than the initial wealth (1) or the target wealth ($\tilde{w}$).

5.2.6 Table 5.1 gives the Arrow-Pratt absolute measure of risk aversion for each utility function considered above and used in our models. Kallberg & Ziemba (1983) have shown, in the one period case, that utility functions with similar Arrow-Pratt absolute measures of risk aversion result in similar optimal portfolios.

Table 5.1. Arrow-Pratt absolute measure of risk aversion

Exponential	a
Power	a/w
Downside-quadratic	$2a/[(1-a) - 2a(w - \tilde{w})]$

5.3 *Risk Management Objectives*

As noted above, in principle different attitudes to downside risk in fund wealth may be imposed at each decision point through the additively separable utility U, which is a sum of different period utility functions u_t, $t = 2, \dots, T+1$, or may be of a common form with different period-specific values of its parameters. Adjustment of these parameter values allows the shaping of the fund wealth distribution across scenarios at a decision point, as we shall see in more detail in Section 6. In practice, however, a common specification of period utility is usually used.

5.4 *Basic, Diversification and Liquidity Constraints*

5.4.1 The basic constraints of the dynamic CALM model (c.f. Consigli & Dempster, 1998), detailed in Appendix C, are:

— *Cash balance constraints*. These are the first set of constraints of the model referring respectively to period one and the remaining periods before the horizon.

— *Inventory balance constraints*. These are the second set of constraints and involve *buy* (+), *sell* (−), and *hold* variables for each asset (and more generally liability, with buy and sell replaced by incur and discharge). This approach, due to Bradley & Crane (1972), allows (with double subscripting) all possible tax and business modelling structures to be incorporated in constraints (see e.g. Cariño *et al.*, 1994).

— *Current wealth constraints*. The third set of constraints involves the two wealth variables: beginning of period wealth before rebalancing (w) from the previous period and beginning of period wealth (W) after a possible cash infusion from borrowing, or an outflow from the costs of portfolio rebalancing and possible debt reduction, i.e. after rebalancing.

5.4.2 The remaining constraint structures required will likely differ from fund to fund. Possible constraints include:

— *Solvency constraints*. These constrain the net wealth of the fund generated by the trading strategy θ to be non-negative (or greater than a suitable regulatory constant) at each time, i.e. $w_t^\theta(\omega) \geq 0$ for $t = 2, \dots, T+1$ and ω in Ω.

— *Cash borrowing limits*. These limit the amount that the trading strategy can borrow in cash, and take the form: $p_{kt}(\omega) z_{kt}^-(\omega) \leq \bar{z}_k$ for k in K, $t = 1, \dots, T$ and ω in Ω, where, recall, p_k denotes the appropriate exchange rate.

— *Short sale constraints*. These limit the amount that the trading strategy can short the equity and bond assets, and take the form: $p_{it}(\omega) x_{it}(\omega) \geq \bar{x}_i$ for i in I, $t = 1, \dots, T$ and ω in Ω.

— *Position limits*. These limit the amount invested in an asset to be less than some proportion $\phi < 1$ of the fund wealth, and take the form:

$$p_{it}(\omega)x_{it}(\omega) \leq \phi_i W_t^0(\omega)$$

$$p_{kt}(\omega)(z_{kt}^+(\omega) - z_{kt}^-(\omega)) \leq \phi_k W_t^0(\omega)$$

for i in I, k in K, $t = 1, \ldots, T$ and ω in Ω.

— *Turnover (liquidity) constraints.* These limit the approximate change in the fraction of total wealth invested in some equity or bond asset i from one time to the next to be less than some proportion of the fund wealth $\alpha_i < 1$, and take the form:

$$|p_{it}(\omega)x_{it}(\omega) - p_{it-1}(\omega)x_{it-1}(\omega)| \leq \alpha_i W_{t-1}^0(\omega)$$

for i in I, k in K, $t = 1, \ldots, T$ and ω in Ω. They are imposed on large funds, primarily from market liquidity considerations which are not modelled.

5.4.3 All the above constraints are piecewise linear convex.

5.4.4 For backtesting purposes (see Section 7), we define the following three types of constraint structures. T1 constraints have no position limits or turnover constraints. T2 constraints have 20% position limits on all assets and no turnover constraints. T3 constraints contain both position limits and turnover constraints, as summarised in Table 5.2.

5.4.5 Short selling and borrowing are not allowed in any of these constraint structures. Assuming that the simulated price processes are non-negative, this automatically enforces the solvency constraints.

Table 5.2. Position limits and turnover restrictions by proportion of value

Asset	Position limit	Turnover constraint
US equity	0.40	0.15
US bonds	0.40	0.15
UK equity	0.80	0.15
UK bonds	0.80	0.15
EU equity	0.80	0.15
EU bonds	0.80	0.15
JP equity	0.15	0.15
JP bonds	0.15	0.15
EM equity	0.05	-
EM bonds	0.05	-
Sum of cash	0.25	-
US equity + bonds	0.50	-
JP equity + bonds	0.20	-
EM equity + bonds	0.08	-

5.5 *Benchmark Portfolio (Fixed Mix) Constraints*

5.5.1 A common problem in the management of funds of all types is the setting of realistic benchmarks. This is usually done in an *ad hoc* manner in light of experience. For a given set of asset classes, a *benchmark portfolio*, whose performance can be used to set a *return benchmark*, may be decided optimally by applying a further constraint to any variant of the dynamic strategic ALM model so far defined. The corresponding portfolio rebalance (trading) strategy is to rebalance the asset portfolio to the initial optimally determined proportions — i.e. fixed mix — at each trading date (decision point), see Mulvey (1995). Thus, assets which have appreciated since the last rebalance will be sold to finance the purchase of depreciating assets to bring their value up to the initial fixed proportion of portfolio value — buy low and sell high! — but, of course, this policy is no protection against generally falling asset values.

5.5.2 Mathematically, the *fixed mix* constraint on asset values held in each scenario ω in Ω at each time period $t = 1, \dots, T$ is given by:

$$\Sigma_{i \in I} \lambda_i = 1$$

$$p_{i1} x_{i1}^{+} = \lambda_i (w_0 - \tau) \qquad i \in I$$

$$p_{it}(\omega) x_{it}(\omega) = \lambda_i \left(\sum_{j \in I} p_{jt}(\omega) x_{jt}(\omega) \right) \qquad i \in I$$

where $\lambda_i \geq 0$, $i \in I$ are the initial portfolio proportions to be optimally determined, w_0 is initial wealth and τ is an estimate of the transaction costs of the initial portfolio balance. Obviously the imposition of these constraints reduces the terminal wealth achievable in the model relative to the full optimum without such constraints — sometimes severely in practice (Hicks-Pedrón, 1998) — and hence constitutes a benchmark to beat. Unfortunately, due to the bilinear nature of the constraints applying to the portfolio decisions subsequent to the initial one, the resulting optimisation problem becomes non-convex (Dempster *et al.*, 2003), but we shall address its practical solution in Section 6.

5.6 *Guaranteed Return Constraints*

5.6.1 Of course, the return guarantee to an individual investor in a defined contribution pension fund is absolute, given the solvency of the guarantor. In the situation of a banking group, such as the fund manager and its parent guarantor, this necessitates strategies both to implement the absolute guarantee for individuals and to manage the investment (trading) strategy of the fund, so as to ensure meeting the guarantee for all participants of the fund with a high probability.

5.6.2 Mathematically, this latter goal can be met by imposing a probabilistic constraint of the VaR type on the wealth process at specific trading dates, computing expected shortfall across scenarios which fail to meet the fund guarantee, and adding the corresponding penalty terms to period objective functions. For example, at the horizon $T+1$ or at any intermediate date t', this would take the form:

$$P\left(\mathbf{w}_{t'} \geq w_{t'}^{*}\right) \geq 1-\alpha$$

where $\alpha := 0.01$ or 0.05, corresponding to respectively 99% or 95% confidence, and $w_{t'}^{*}$ is calculated from the initial wealth and the guaranteed annualised rate r as $w_0(1+r)^{t'}$. However, such scenario-based probabilistic constraints are extremely difficult to implement, in that they again convert the convex (deterministic equivalent) large scale optimisation problem to a non-convex one. We will, nevertheless, describe a practical approximation procedure in the next section, but we leave expected shortfall penalties to future work.

6. Problem Generation and Solution Techniques

6.1 *Optimisation Problem Generation*

6.1.1 Instantiations of the CALM model and other similar strategic DFA models lead to very large deterministic equivalent non-linear optimisation problems, involving, perhaps, hundreds of thousands of scenarios and millions of variables and constraints. Moreover, in a production setting, both parameter values and the model itself are constantly changing, due to changes of view, objectives and regulations. Mathematical programming modelling languages, such as AMPL (Fourer *et al.*, 1993) and OPL (ILOG, 2000), have been developed to handle deterministic optimisation models in this regard, by specifying the variables, objective and constraints of the problem in an algebraic language in terms of entity sets, which is similar to the ordinary mathematical specification of the Pioneer CALM model given in Appendix C. Such systems take as input the model in algebraic form together with specific parameter values, and they output a structured file in a standard format such as MPS (IBM, 1972), which is readable as input by a wide range of optimisation solvers. These concepts have been extended to large scale dynamic stochastic optimisation problems with the STOCHASTICS™ software (Dempster *et al.*, 2002) and the SMPS standard solver input format (Birge *et al.*, 1986), which have been used for this project. As discussed in Section 4.4, the *stochgen* subsystem handles the scenario tree generation using routine dynamic stochastic simulation from a standardised tree structure specification — horizon and branching structure — and making use of AMPL (or a new modelling language SAMPL, currently under development for *stochgen 3.1*) outputs the optimisation problem for decomposition-based techniques — or appropriate pieces of the optimisation problem — in the SMPS or MPS formats to the solver — possibly as it runs. See Dempster & Consigli (1998) and Dempster *et al.* (2002) for more details.

6.2 *Optimal Strategic ALM Algorithms and Software*

6.2.1 A variety of large scale optimisation algorithms have been used to solve variants of the CALM model. For linear and quadratic problems — both linearly constrained — these are simplex, interior point and nested Benders decomposition methods. For general linearly constrained convex and general non-linear problems, both nested Benders decomposition and sequential quadratic programming algorithms have been used.

6.2.2 Simplex and interior point algorithms are well documented (see e.g. Vanderbei, 2002), and the basic reference to nested Benders decomposition is Gassmann (1990), see also Scott (2002). Nested Benders decomposition is a sequential cutting plane technique in which the sub-problems at each node of the scenario tree are solved independently for each major iteration, until the cuts for each sub-problem lead to the solution of the problem. Like interior point methods, the number of major iterations required for convergence by nested Benders decomposition depends more upon the size of the feasible region than on the problem dimensions (size) itself. We have used CPLEX 5.1 for linear and quadratic programming, *solgen 1.2* of the STOCHASTICS™ toolchain for nested Benders decomposition and SNOPT for general non-linear programming by sequential quadratic programming (see Gill *et al.*, 2002).

6.2.3 For the CALM model of Appendix C and its variants — which are linearly constrained convex problems generally and quadratic problems for the best performing downside quadratic utilities (see Section 7.3), we usually first solve a quadratic version of a new instantiation with a few thousand scenario trees using CPLEX interior point. For the very large scenario trees corresponding to long horizon multi-portfolio rebalance problems, however, the *solgen 1.2* implementation of nested Benders decomposition is required, since the other techniques must load the full problem into the computer's memory.

6.3 *Optimal Benchmark Portfolio Algorithms*

6.3.1 Due to the bilinear nature of the constraints — in initial portfolio proportions and subsequent portfolio asset positions — which apply to portfolio decisions subsequent to the initial one, the fixed mix problem for setting optimal benchmark portfolios is non-convex. However, these non-convexities add only finescale 'noise' to a generally well behaved, though not unimodal, problem value, considered as a function of the initial portfolio proportions (λ) to be optimised. This formulation, as a low dimensional general non-convex problem in the number of asset classes in the model, is possible, since, for fixed λs, subsequent rebalance decisions may be computed either by one iteration of nested Benders decomposition or directly by algebraic calculation (Dempster *et al.*, 2003).

6.3.2 In an attempt to reduce transaction costs, it is also possible to define a model and corresponding trading strategy which rebalances to the fixed mix proportions only when current portfolio proportions have varied by more than specified percentages. Such a *relaxed fix mix* model has dead zones in which no portfolio rebalancing is necessary, and may also be formulated as a global optimisation problem (using nested Benders decomposition) in the initial portfolio proportions.

6.3.3 To attack these non-convex problems, we have applied a variety of algorithms and software — local smooth approximate conjugate directions (Powell, 1964), the DIRECT global Lipschitz smooth partitioning algorithm (Gablonsky, 1998) and several others — to fixed mix variants of the CALM model with reasonable success (Scott, 2002). Currently we are working on improving the efficiency of these methods to make their routine operational use more robust.

6.4 *Capital Guaranteed Products Algorithm*

6.4.1 The so-called *chance-constrained* programme arising from applying one or more probabilistic VaR-type capital guarantee constraints to the CALM model would only be convex if the distribution of current wealth $w_{t'}$ satisfies certain analytic conditions (Prékopa, 1980). This is not the case, of course, for a finite scenario-based distribution, and hence the resulting problem is non-convex, and will require approximation for practical purposes. Like the benchmark portfolio problem, however, this approximation problem is not intractable. Instead of solving a problem (involving, for example, expected terminal wealth) of the form:

$$\max \mathrm{E}[\mathbf{w}_{T+1}]$$

subject to:

$$\mathbf{x} \in \mathbf{X}$$

$$\mathbf{w}_{T+1} = g(\mathbf{x})$$

$$P\left(\mathbf{w}_{T+1} \geq w^*_{T+1}\right) \geq 1 - \alpha$$

we repeatedly solve:

$$\max \mathrm{E}\left[\beta \mathbf{w}_{T+1} - (1-\beta)\left(\mathbf{w}_{T+1} - \tilde{w}_{T+1}\right)_-^2\right]$$

subject to:

$$x \in X$$

$$\mathbf{w}_{T+1} = g(\mathbf{x})$$

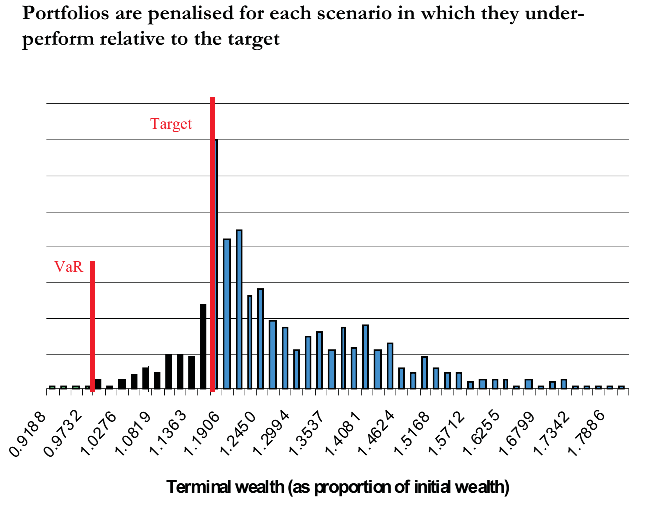


Figure 6.1. Terminal wealth distribution from a scenario tree

while searching for a value of target wealth $\tilde{w}_{T+1}$ for which the probabilistic constraint is satisfied. Alternatively, a severe downside linear penalty can be employed, and this appears to be better at shaping wealth so as to reproduce scenario-based problem confidence levels out-of-sample using simulator or historical data, see Figure 6.1. We are currently perfecting this method for operational use with long horizon problems.

6.4.2 We have also tested a 0-1 mixed integer programming formulation, in which the binary variables are used to count explicitly scenarios on which the guaranteed fund wealth is violated, but this approach currently appears intractable for anything but toy problems.

7. System Historical Backtests

7.1 *Implementation*

7.1.1 In a practical implementation of the dynamic stochastic optimisation approach to strategic DFA, a new problem is solved for each trading time $t = 1, \ldots, T$, and the initial portfolios implemented. At each time t, the asset return and exchange rate model's parameters are re-estimated and re-calibrated using historical data up to and including time t, and the initial values of the simulated scenarios are given by the actual values of the variables at that time.

7.1.2 There are several reasons for implementing our approach in this manner. The first is that the actual value of the variables at $t = 2$ are unlikely to coincide with any values of the variables in the simulated scenarios at $t = 2$. If this is the case, then the optimal investment policy will be undefined. The second, and more important, reason is that re-estimating and re-calibrating the simulator's parameters at each time t captures information in the history of the variables up to that point. Since the asset return and exchange rate models employed are only an approximation to the real dynamics, using the most recent history should improve the scenario simulation.

7.1.3 For a given problem formulation, the process of implementing the stochastic optimisation approach at each trading time t can be represented by the system diagram of Figure 7.1 (c.f. Figure 2.1). Much of this system has been automated for the purposes of this research.

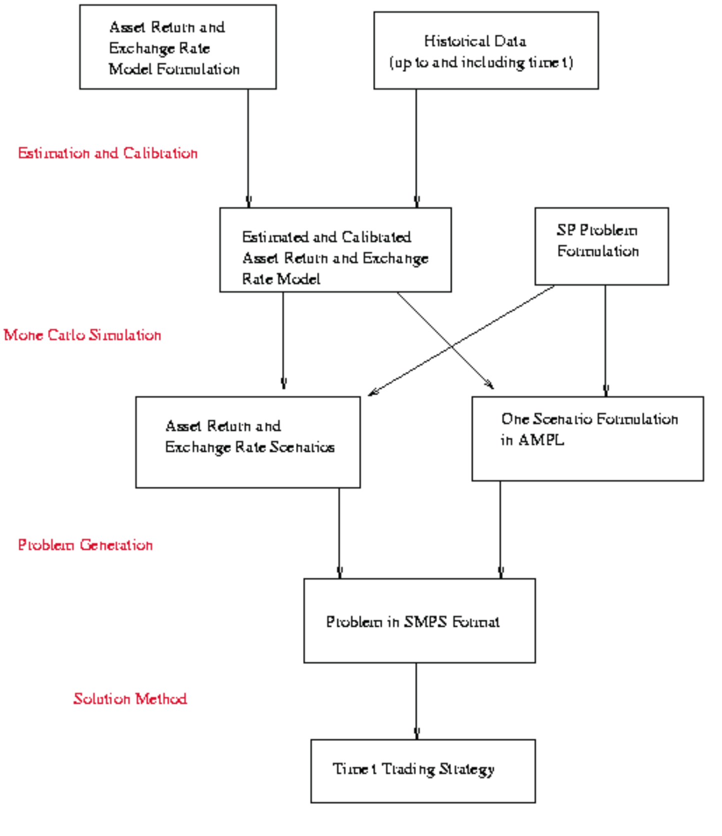


Figure 7.1. Pioneer CALM model system diagram

7.1.4 The quality of the first stage implementable and subsequent 'what-if' portfolio rebalance decisions of a dynamic stochastic optimisation ALM model clearly depend on a number of real world scenario contingent factors.

7.1.5 Obviously, the most crucial factor is the 'predictive' power of the asset return statistical model underlying the scenario (tree) simulations. (We shall return to *ad hoc* tests of this factor in the next section). Less obvious, perhaps, is the impact of the number of scenarios used in the optimisation model, and, even more importantly, the branching structure used in the scenario tree. Although there is general consensus that, in dynamic models, branching should be larger for the earliest decisions — in particular for the first implementable one — than for those later in the tree (see e.g. Dempster & Thompson, 2002), the number of scenarios required to stabilise problem value and decisions is highly model dependent. This is clearly a sampling problem for a continuous state stochastic optimisation problem — one level higher than a (discrete time) stochastic process sampling problem. Although asymptotic consistency results for both value and decisions are available (see Dempster (1998) or Shapiro (2002) and the references therein), the proofs are mathematically very difficult and the results of limited practical use. It is, however, generally agreed for a given problem that its value is stabilised by smaller scenario trees (samples) than are required to stabilise its (even implementable) decisions. Moreover, suppressing sampling error by the techniques discussed in ¶4.3 has also generally been seen to be beneficial for decision stability (although to an extent not reported in any detail in the literature). In our experiments, tree sizes (i.e. numbers of scenarios) have been reduced by a factor five by these means, with a slightly greater problem run time reduction (to several minutes on a top end PC), which is of great practical use in fund design — although much remains to be done. We define a practical decision stability criterion in the next section.

7.2 *Backtesting*

7.2.1 Backtesting strategic DFA systems out-of-sample can take two forms: experimental and historical. In the more familiar historical backtest, statistical models are fitted to data up to a trading time t, scenario trees are generated to some chosen horizon $t + T + 1$, the optimal decisions implemented at t are evaluated against historical returns at $t + 1$, and the whole procedure rolled forward for T trading times. Experimental backtests can repeat this procedure as many times as is necessary to suppress sampling error, by treating independently generated out-of-sample flat scenarios to $T + 1$ as pseudo-histories. Such tests are invaluable in exploring the stability properties of decisions in specific models, and we have termed a given model *decision stable* in scenario tree size and structure experiments when the standard deviation of the sampling error in each implementable decision portfolio proportion has been reduced to 10% of its sample mean value by a

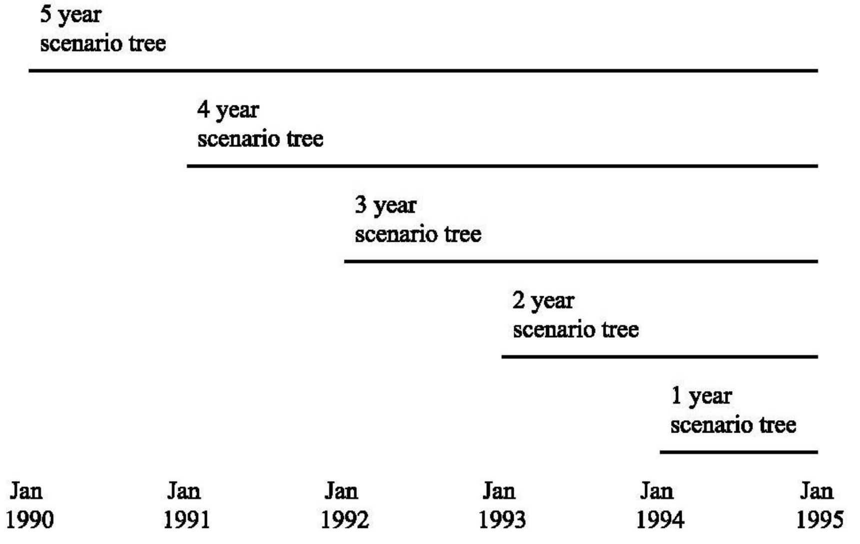


Figure 7.2. Telescoping horizon backtest schema

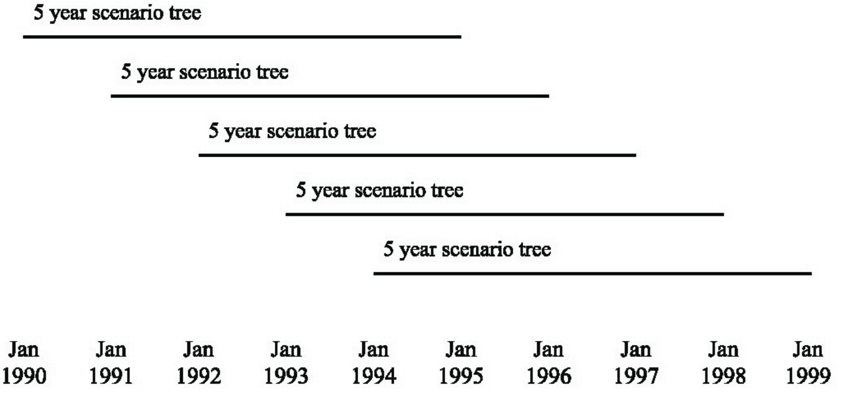


Figure 7.3. Rolling horizon backtest schema

suitable choice of scenario tree for the model. Typically, 10,000 flat scenarios are used for such experiments.

7.2.2 Either type of backtest can involve a *telescoping horizon*, as depicted in Figure 7.2, or a *rolling horizon*, as shown in Figure 7.3.

7.3 *Pioneer CALM Model Backtests*

7.3.1 A number of historical backtests have been run on variants of the CALM global model, with perhaps surprisingly uniformly good results, see Villaverde (2002) for complete details. The aims of these tests were several. First, we wished to establish the relative 'predictability' or otherwise of the alternative Pioneer tuned econometric models for short horizons. (Long horizon (20 or 30 year) experiments — where the simpler aim is merely to recapture historical statistical patterns — are currently in progress, but require significant computational resources.) Secondly, we wished to understand the impact of the alternative utility functions available to the system on optimal portfolio decisions. Thirdly, we wished to evaluate the impact of risk attitudes imposed on fund wealth trajectories period by period (in terms of additively separable utility functionals) versus their imposition only on fund terminal wealth. Fourthly, we were interested in the farsightedness or otherwise of the dynamic stochastic optimisation approach to strategic DFA relative to rolling over single period-based systems *à la* Markowitz — the *raison d'être* of dynamic models. Finally, we were interested in what effects imposing the practical diversification and liquidity (turnover) constraints (T3 in Table 4.3) would have on backtest returns. We discuss the (at least partial) evidence to date on all these topics here.

7.3.2 All historical asset allocation backtests that we report were from the viewpoint of a US dollar-based fund in Eire. The benchmark used is therefore the S&P500 Equity Index over the out-of-sample period for each test. All portfolio rebalances are subject to a 1% value tax on transactions, which, of course, does not apply to the benchmark index. Monthly data (as set out in Table 3.1) were available from July 1977 to August 2002.

7.3.3 Table 7.1 shows the results in terms of annualised returns of a typical backtest, with a two-year telescoping horizon and semi-annual rebalancing from February 1999 to February 2001, using the model of Appendix C, with 8,192 scenarios, a 128.16.2.2 branching structure and a

Table 7.1. Asset allocation backtests: annualised returns from February 1999-February 2001

	Capital markets		Capital markets + emerging markets		Capital markets + emerging markets + US economic model	
Utility function	No limits	20% limits	No limits	20% limits	No limits	20% limits
Linear	91%	9%	92%	10%	31%	11%
Quadratic	8%	9%	6%	11%	21%	6%
Downside-quadratic	54%	9%	70%	11%	29%	9%
Exponential	72%	9%	92%	10%	51%	11%
Power	91%		92%		49%	

terminal wealth criterion. During this period the S&P500 returned 0%. With no position limits, the model tends to pick the best asset(s), and so in this case a high annual historical return is an indication of predictability in the tuned econometric model used to generate the scenarios. Once more realistic constraints are imposed in this test, however portfolios become well diversified, and in the results corresponding to the various attitudes to risk there is little to choose from. However, performance is improved by the use of the emerging market asset returns, even though they were actually not used in the optimal portfolios. Corresponding results for the addition of the US economic model to the system are mixed. When this backtest was extended one period to August 2001 — when the S&P500 annualised return over the 2.5 year period was −2.3% — similar results were obtained with the best position limited result being 6.8% p.a. for the downside-quadratic utility with a:= 0.5 and target wealth a 61% increase over the period.

7.3.4 Overall, the best overall historical backtest results were obtained using the downside-quadratic utility function with appropriate parameters. A summary of the backtests performed to date for this attitude to risk is given in Table 7.2. Note here that imposing the practical liquidity (T3) constraints, which could be expected generally to reduce returns, sometimes led to significantly increased returns. Notice, also, that the imposition of an attitude to risk of wealth in each period — the ten-year five-year horizon rolling four-area backtest using the linearised VARSIM simulation — improved annual return over the position limited returns for the two constituent five-year periods (using three and four area capital market models), employing only an attitude to risk on fund terminal wealth.

7.3.5 Table 7.3 shows analysed implemented solver output for an historical backtest over the period 1996-2001, with annual rebalancing and the liquidity (T3) constraints imposed (corresponding to the bolded entry in Table 7.2). Note that the successive implemented portfolios are responding as much as possible to changing market conditions by asset allocations with varying diversification.

7.3.6 Overall, we found that the imposition of the T3 liquidity constraints in the model forced its decisions to take full advantage of the information in future scenarios and optimal forward rebalances, to result in well diversified portfolios and significant improvement in historical backtest performance over rolling myopic single period models (c.f. Hicks-Pedrón, 1998).

Table 7.2. Summary of CALM US$ fund historical backtests

Initial estimation period	Out-of-sample period	Length	Asset return model	Simulator	Number of scenarios k	Rebalance frequency	Risk management criterion	Horizon	Constraint annualised return % (see Section 5.4)			S&P 500 benchmark annualised return %
									T1	T2	T3	
1972-1990	1990-1995	5 years	3 areas (ex Japan)	BMSIM	4	annual	terminal	telescoping	10.33	9.34	-	7.41
1992-1996	1996-2001	5 years	4 areas	BMSIM	4	annual	terminal	telescoping	13.36	7.13	-	14.12
1992-1996	1996-2001	5 years	4 areas	VARSIM	4	annual	terminal	telescoping	1.51	8.30	-	14.12
1992-1999	1999-2001	2.5 years	4 areas	BMSIM	8.2	semi-annual	terminal	telescoping	27.89	6.48	2.69	−2.30
1992-1999	1999-2001	2.5 years	above + emerging markets	BMSIM	8.2	semi-annual	terminal	telescoping	16.98	5.72	3.38	−2.30
1992-1999	1999-2001	2.5 years	above + US economy	BMSIM	8.2	semi-annual	terminal	telescoping	19.16	4.64	−0.38	−2.30
1992-1999	1999-2001	2.5 years	4 areas	VARSIM	8.2	semi-annual	terminal	telescoping	−6.40	-	−3.92	−2.30
1990-1996	1996-2001	5 years	4 areas	BMSIM	8.2	annual	all periods	telescoping	8.54	-	8.37	14.12
1990-1996	1996-2001	5 years	4 areas	VARSIM	8.2	annual	all periods	telescoping	5.78	9.99	**9.37**	14.12
1990-1996	1996-2001	5 years	4 areas	HSIM	8.2	annual	all periods	telescoping	4.95	-	6.04	14.12
1972-1991	1991-2001	10 years	4 areas	VARSIM	8.2	annual	all periods	5-year rolling	3.56	-	9.98	12.72

Table 7.3. Implemented annual portfolio rebalances for an historical backtest with liquidity constraints using VARSIM

	US stock	US cash	US bond	UK stock	UK cash	UK bond	UK fx	EU stock	EU cash	EU bond	EU fx	JP stock	JP cash	JP bond	JP fx
Date: Feb-96															
First stage weights	0.19	0.25	0	0.03	0	0.52	0	0	0	0	0	0	0	0	0
Historical return (dollar)	1.23	1.05	1.00	1.22	1.13	1.24	1.07	1.21	0.92	1.01	0.9	0.78	0.88	0.98	0.87
12-month portfolio return against history	1.18														
Date: Feb-97															
First stage weights	0.23	0.25	0	0	0	0.40	0	0	0	0	0	0.12	0	0	0
Historical return (dollar)	1.33	1.05	1.19	1.28	1.08	1.26	1.01	1.36	0.96	0.99	0.92	0.87	0.96	1.04	0.95
12-month portfolio return against history	1.17														
Date: Feb-98															
First stage weights	0.40	0.06	0.06	0	0	0.28	0	0.15	0	0	0	0.04	0	0	0
Historical return (dollar)	1.18	1.05	1.13	1.03	1.04	1.24	0.98	1.11	1.06	1.15	1.02	0.93	1.06	1.04	1.06
12-month portfolio return against history	1.16														
Date: Feb-99															
First stage weights	0.40	0	0.06	0	0	0.15	0	0.29	0	0	0	0.09	0	0	0
Historical return (dollar)	1.10	1.05	0.94	1.04	1.04	1.00	0.99	1.17	0.90	0.80	0.88	1.66	1.08	1.12	1.08
12-month portfolio return against history	1.15														
Date: Feb-00															
First stage weights	0.41	0	0	0	0	0	0	0.45	0	0	0	0.14	0	0	0
Historical return (dollar)	0.91	1.06	1.19	0.88	0.97	0.98	0.92	0.86	1.01	1.06	0.96	0.68	0.94	1.00	0.94
12-month portfolio return against history	0.85														

8. Conclusions

8.1 This paper describes an innovative joint project to construct the model base for a decision support system for defined contribution pension fund design at the strategic level. Each block of the system diagram of Figure 2.2 has been described in detail (including the third party component software utilised). The methods developed are much more widely applicable to a range of strategic DFA problems in finance. Practical solutions to two new problems — optimal fund benchmark setting and value-at-risk constrained guaranteed return fund design — have been outlined. In all historical backtests, using data over roughly the past decade, the global asset allocation system equalled or outperformed the S&P500, when transaction costs are taken into account. All system returns for the non-linear statistical model were positive — even through the recent high tech crash.

8.2 A number of areas for further work have been identified throughout the paper, and much work remains to be done. However, if we have convinced the reader that the dynamic stochastic optimisation approach to strategic DFA problems is a practical reality today, the paper will have achieved its aims.

8.3 Currently, we are developing an industrial strength version of the expected value of perfect information importance sampling algorithm (Dempster, 1998), represented by the inner dotted feedback loop in Figure 2.2. Eventually, it should be possible to automate the re-estimation and updating procedure of the outer dotted loop in the figure, but this adaptive filtering approach for this application is still a long way off.

8.4 The fund manager intends to become a leader in the management of pension funds for third parties. Its collaboration with the Centre for Financial Research at Cambridge has already made possible important advances in both its long-term forecasting engines and its optimisation techniques. Such know-how is currently used in the development of its new financial products and services.

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APPENDIX A

DETAILED PIONEER ASSET RETURN MODELS

Let the home currency be USD, and for a given scenario let:

- S_t^c denote the equity price at time t for $c = US, UK, EU, JP, EM$. S_t^{EM} is assumed to be denominated in USD;
- R_t^c denote the percent return on lending cash between time $t-1$ and time t for $c = US, UK, EU, JP$. The percent return on borrowing cash is taken to be $R_t^c + \delta$ for some $\delta > 0$;
- L_t^c denote the bond yield, expressed as a monthly percent, at time t for $c = US, UK, EU, JP$. The maturity and compounding frequency depends on c, and is specified in Table 3.1;
- X_t^c denote the exchange rate at time t for $c = UK, EU, JP$ expressed as \$/local currency of c;
- B_t^{EM} denote the EM bond price at time t denominated in USD; the (average) maturity of the bond (index) is specified in Table 3.1;
- C_t^{US} denote the US Consumer Price Index (CPI) at time t;
- W_t^{US} denote US wages at time t;
- G_t^{US} denote US GDP at time t; and
- P_t^{US} denote US public sector borrowing (PSB) at time t.

The formulation of the capital markets discrete time model corresponding to the BMSIM3 simulator (see Section 4.4) is given by the following (with a monthly time step):

$$\frac{\boldsymbol{S_{t+1}^{US}} - S_t^{US}}{S_t^{US}} = \begin{pmatrix} a_S^{US} + a_{SS}^{US} S_t^{US} + a_{SR}^{US} R_t^{US} + a_{SL}^{US} L_t^{US} + a_{SX}^{US} X_t^{US} + \\ b_{SS}^{US} S_{t-1}^{US} + b_{SR}^{US} R_{t-1}^{US} + b_{SL}^{US} L_{t-1}^{US} + b_{SX}^{US} X_{t-1}^{US} \end{pmatrix} + \sigma_S^{US} \boldsymbol{\varepsilon_{St}^{US}}$$

$$\frac{\boldsymbol{R_{t+1}^{US}} - R_t^{US}}{R_t^{US}} = \begin{pmatrix} a_R^{US} + a_{RS}^{US}\left(\frac{S_t^{US}}{R_t^{US}}\right) + a_{RR}^{US}\left(\frac{1}{R_t^{US}}\right) + a_{RL}^{US}\left(\frac{L_t^{US}}{R_t^{US}}\right) + a_{RX}^{US}\left(\frac{X_t^{US}}{R_t^{US}}\right) + \\ b_{RS}^{US}\left(\frac{S_{t-1}^{US}}{R_t^{US}}\right) + b_{RR}^{US}\left(\frac{1}{R_t^{US}}\right) + b_{RL}^{US}\left(\frac{L_{t-1}^{US}}{R_t^{US}}\right) + b_{RX}^{US}\left(\frac{X_{t-1}^{US}}{R_t^{US}}\right) \end{pmatrix} + \sigma_R^{US} \boldsymbol{\varepsilon_{Rt}^{US}}$$

$$\frac{\boldsymbol{L}_{t+1}^{US} - L_t^{US}}{L_t^{US}} = \begin{pmatrix} a_L^{US} + a_{LS}^{US} S_t^{US} + a_{LR}^{US} R_t^{US} + a_{LL}^{US} L_t^{US} + a_{LX}^{US} X_t^{US} + \\ b_{LS}^{US} S_{t-1}^{US} + b_{LR}^{US} R_{t-1}^{US} + b_{LL}^{US} L_{t-1}^{US} + b_{LX}^{US} X_{t-1}^{US} \end{pmatrix} + \sigma_L^{US} \boldsymbol{\varepsilon_{Lt}^{US}}$$

$$\frac{\boldsymbol{S}_{t+1}^c - S_t^c}{S_t^c} = \begin{pmatrix} a_S^c + a_{SS}^c S_t^c + a_{SR}^c R_t^c + a_{SL}^c L_t^c + a_{SX}^c X_t^c + \\ b_{SS}^c S_{t-1}^c + b_{SR}^c R_{t-1}^c + b_{SL}^c L_{t-1}^c + b_{SX}^c X_{t-1}^c \end{pmatrix} + \sigma_S^c \boldsymbol{\varepsilon}_{St}^c$$

$$\frac{\boldsymbol{R}_{t+1}^c - R_t^c}{R_t^c} = \begin{pmatrix} a_R^c + a_{RS}^c\left(\frac{S_t^c}{R_t^c}\right) + a_{RR}^c\left(\frac{1}{R_t^c}\right) + a_{RL}^c\left(\frac{L_t^c}{R_t^c}\right) + a_{RX}^c\left(\frac{X_t^c}{R_t^c}\right) + \\ b_{RS}^c\left(\frac{S_{t-1}^c}{R_t^c}\right) + b_{RR}^c\left(\frac{1}{R_t^c}\right) + b_{RL}^c\left(\frac{L_{t-1}^c}{R_t^c}\right) + b_{RX}^c\left(\frac{X_{t-1}^c}{R_t^c}\right) \end{pmatrix} + \sigma_R^c \boldsymbol{\varepsilon}_{Rt}^c$$

$$\frac{\boldsymbol{L}_{t+1}^c - L_t^c}{L_t^c} = \begin{pmatrix} a_L^c + a_{LS}^c S_t^c + a_{LR}^c R_t^c + a_{LL}^c L_t^c + a_{LX}^c X_t^c + \\ b_{LS}^c S_{t-1}^c + b_{LR}^c R_{t-1}^c + b_{LL}^c L_{t-1}^c + b_{LX}^c X_{t-1}^c \end{pmatrix} + \sigma_L^c \boldsymbol{\varepsilon}_{Lt}^c$$

$$\frac{\boldsymbol{X}_{t+1}^c - X_t^c}{X_t^c} = \begin{pmatrix} a_X^c + a_{XS}^c\left(\frac{S_t^c}{X_t^c}\right) + a_{XR}^c\left(\frac{R_t^{US} - R_t^c}{X_t^c}\right) + a_{XL}^c\left(\frac{L_t^{US} - L_t^c}{X_t^c}\right) + a_{XX}^c\left(\frac{1}{X_t^c}\right) + \\ b_{XS}^c\left(\frac{S_{t-1}^c}{X_t^c}\right) + b_{XR}^c\left(\frac{R_{t-1}^{US} - R_{t-1}^c}{X_t^c}\right) + b_{XL}^c\left(\frac{L_{t-1}^{US} - L_{t-1}^c}{X_t^c}\right) + b_{XX}^c\left(\frac{1}{X_t^c}\right) \end{pmatrix} + \sigma_X^c \boldsymbol{\varepsilon}_{Xt}^c$$

for $c = UK, EU$ and $X_t^{US} := X_t^{UK}$. The $\boldsymbol{\varepsilon}$ terms are correlated standard normal or standardised student t random variables. The a, b and σ terms are parameters of the model. Note that, since we are assuming that the home currency is USD, modelling an exchange rate for the US is unnecessary. Salient features of the model include non-linear drifts, a lag structure and constant volatilities in this form.

The formulation of the model for the BMSIM4 simulator is identical to that for BMSIM3, with the addition of Japan, so that $c = UK, EU, JP$ and with $X_t^{US} = X_t^{JP}$. As noted in Section 3.2, additive binary dummy variables were used to remove the S_t^{JP} bubble and crash.

The formulation of the model for the BMSIM4EM simulator is identical to that for BMSIM4, with the addition of the following AR(1)/GARCH (1, 1) processes for EM equity and bonds:

$$\frac{\boldsymbol{S}_{t+1}^{EM} - S_t^{EM}}{S_t^{EM}} = a_S^{EM} + a_{S1}^{EM}\frac{S_t^{EM} - S_{t-1}^{EM}}{S_{t-1}^{EM}} - a_{S2}^{EM}\sqrt{H_{t-1}^S}\varepsilon_{t-1}^S + \sqrt{H_t^S}\boldsymbol{\varepsilon}_t^S$$

$$\boldsymbol{H}_t^S = b_S + b_{S1}H_{t-1}^S - b_{S2}H_{t-1}^S(\boldsymbol{\varepsilon}_{t-1}^S)^2$$

$$\frac{\boldsymbol{B}_{t+1}^{EM} - B_t^{EM}}{B_t^{EM}} = a_B^{EM} + a_{B1}^{EM}\frac{B_t^{EM} - B_{t-1}^{EM}}{B_{t-1}^{EM}} - a_{B2}^{EM}\sqrt{H_{t-1}^B}\varepsilon_{t-1}^B + \sqrt{H_t^B}\boldsymbol{\varepsilon}_t^B$$

$$H_t^B = b_B + b_{B1}H_{t-1}^B - b_{B2}H_{t-1}^B(\varepsilon_{t-1}^B)^2.$$

We assume that all ε terms are contemporaneously correlated, but serially uncorrelated. Because the EM variables in BMSIM4EM only influence the US, UK, EU and JP financial variables via the shocks (the contemporaneously correlated ε terms), the EM variables will normally not influence the US, UK, EU and JP financial variables significantly.

The formulation for BMSIM4EME is similar to that for BMSIM4EM, with the exception of two changes. The first is that we introduce the model for the US macroeconomic variables of Section 3.4. The second is that we the replace the US equations with:

$$\frac{\boldsymbol{S_{t+1}^{US}} - S_t^{US}}{S_t^{US}} = \begin{pmatrix} a_S^{US} + a_{SS}^{US} S_t^{US} + a_{SR}^{US} R_t^{US} + a_{SL}^{US} L_t^{US} + a_{SX}^{US} X_t^{US} + \\ b_{SS}^{US} S_{t-1}^{US} + b_{SR}^{US} R_{t-1}^{US} + b_{SL}^{US} L_{t-1}^{US} + b_{SX}^{US} X_{t-1}^{US} + \\ c_{SC}^{US} C_t^{US} + c_{SW}^{US} W_t^{US} + c_{SG}^{US} G_t^{US} + c_{SP}^{US} P_t^{US} + \\ d_{SC}^{US} C_{t-1}^{US} + d_{SW}^{US} W_{t-1}^{US} + d_{SG}^{US} G_{t-1}^{US} + d_{SP}^{US} P_{t-1}^{US} \end{pmatrix} + \sigma_S^{US} \boldsymbol{\varepsilon_{St}^{US}}$$

$$\frac{\boldsymbol{R_{t+1}^{US}} - R_t^{US}}{R_t^{US}} = \begin{pmatrix} a_R^{US} + a_{RS}^{US}\left(\frac{S_t^{US}}{R_t^{US}}\right) + a_{RR}^{US}\left(\frac{1}{R_t^{US}}\right) + a_{RL}^{US}\left(\frac{L_t^{US}}{R_t^{US}}\right) + a_{RX}^{US}\left(\frac{X_t^{US}}{R_t^{US}}\right) + \\ b_{RS}^{US}\left(\frac{S_{t-1}^{US}}{R_t^{US}}\right) + b_{RR}^{US}\left(\frac{1}{R_t^{US}}\right) + b_{RL}^{US}\left(\frac{L_{t-1}^{US}}{R_t^{US}}\right) + b_{RX}^{US}\left(\frac{X_{t-1}^{US}}{R_t^{US}}\right) + \\ c_{RC}^{US} C_t^{US} + c_{RW}^{US} W_t^{US} + c_{RG}^{US} G_t^{US} + c_{RP}^{US} P_t^{US} + \\ d_{RC}^{US} C_{t-1}^{US} + d_{RW}^{US} W_{t-1}^{US} + d_{RG}^{US} G_{t-1}^{US} + d_{RP}^{US} P_{t-1}^{US} \end{pmatrix} + \sigma_R^{US} \boldsymbol{\varepsilon_{Rt}^{US}}$$

$$\frac{\boldsymbol{L_{t+1}^{US}} - L_t^{US}}{L_t^{US}} = \begin{pmatrix} a_L^{US} + a_{LS}^{US} S_t^{US} + a_{LR}^{US} R_t^{US} + a_{LL}^{US} L_t^{US} + a_{LX}^{US} X_t^{US} + \\ b_{LS}^{US} S_{t-1}^{US} + b_{LR}^{US} R_{t-1}^{US} + b_{LL}^{US} L_{t-1}^{US} + b_{LX}^{US} X_{t-1}^{US} + \\ c_{LC}^{US} C_t^{US} + c_{LW}^{US} W_t^{US} + c_{LG}^{US} G_t^{US} + c_{LP}^{US} P_t^{US} + \\ d_{LC}^{US} C_{t-1}^{US} + d_{LW}^{US} W_{t-1}^{US} + d_{LG}^{US} G_{t-1}^{US} + d_{LP}^{US} P_{t-1}^{US} \end{pmatrix} + \sigma_L^{US} \boldsymbol{\varepsilon_{Lt}^{US}}.$$

The addition of the US macroeconomic variables is an attempt to create a more realistic model for asset returns and exchange rates. Because they influence the US financial variables through the drift terms and the shocks, they should have a significant impact on the US financial variables. Again, we assume that all ε terms are contemporaneously correlated, but serially uncorrelated.

The generation of the dynamic stochastic optimisation problems requires the asset returns and exchange rates in each scenario. Appendix B explains how bond yields are transformed into bond asset returns.

APPENDIX B

DERIVATION OF BOND RETURNS FROM BOND YIELDS

The following is a derivation of the one-month bond return for the US. The UK, EU and Japan formulae differ only in the maturity and compounding frequency of the bond yield.

The US bond has a 30-year maturity with semi-annual compounding. Let $L1_t$ denote the 30-year annualised bond yield with semi-annual compounding, i.e. $L1_t = 12L_t/100$. Let F denote the face value of the bond, and let c_t denote the annual coupon rate.

Consider holding a newly issued 30-year bond from time t to time $t+1$ which is one month later. The value of the investment at time t is the cash price of the 30-year bond, which is given by:

$$V_t = \sum_{n=1}^{60} \frac{F\frac{c_t}{2}}{\left(1+\frac{L1_t}{2}\right)^n} + \frac{F}{\left(1+\frac{L1_t}{2}\right)^{60}}$$

$$= \frac{Fc_t}{L1_t}\left(1 - \frac{1}{\left(1+\frac{L1_t}{2}\right)^{60}}\right) + \frac{F}{\left(1+\frac{L1_t}{2}\right)^{60}}.$$

At time $t+1$ or one month later there has been no coupon payment, and the value of the investment is the cash price of a bond with a $29^{11/12}$ year maturity, and which pays a coupon in five months and then every six months until maturity. The cash price of this bond is:

$$\widehat{V}_{t+1} = \sum_{n=1}^{60} \frac{F\frac{c_t}{2}}{\left(1+\frac{\widehat{L1}_{t+1}}{2}\right)^{n-1+\frac{5}{6}}} + \frac{F}{\left(1+\frac{\widehat{L1}_{t+1}}{2}\right)^{59\frac{5}{6}}}.$$

If we assume that the yield of this bond $\widehat{L1}_{t+1} = L1_{t+1}$, then we can approximate $\widehat{V}_{t+1}$ by:

$$\tilde{V}_{t+1} = \sum_{n=1}^{60} \frac{F\frac{c_t}{2}}{\left(1+\frac{L1_{t+1}}{2}\right)^{n-1+\frac{5}{6}}} + \frac{F}{\left(1+\frac{L1_{t+1}}{2}\right)^{59\frac{5}{6}}}$$

$$= \frac{Fc_t}{L1_{t+1}\left(1-\frac{L1_{t+1}}{2}\right)^{-\frac{1}{6}}} \left(1 - \frac{1}{\left(1+\frac{L1_{t+1}}{2}\right)^{60}}\right) + \frac{F}{\left(1+\frac{L1_{t+1}}{2}\right)^{59\frac{5}{6}}}.$$

Then the one-month bond return can be estimated as:

$$\frac{\tilde{V}_{t+1}}{\tilde{V}_t} - 1 = \frac{\frac{c_t}{L1_{t+1}\left(1-\frac{L1_{t+1}}{2}\right)^{-\frac{1}{6}}} \left(1 - \frac{1}{\left(1+\frac{L1_{t+1}}{2}\right)^{60}}\right) + \frac{1}{\left(1+\frac{L1_{t+1}}{2}\right)^{59\frac{5}{6}}}}{\frac{c_t}{L1_t}\left(1 - \frac{1}{\left(1+\frac{L1_t}{2}\right)^{60}}\right) + \frac{1}{\left(1+\frac{L1_t}{2}\right)^{60}}} - 1$$

and the coupon rate c_t can be approximated as some fraction of $L1_t$, i.e. $c_t = mL1_t$, with $m \leq 1$.

APPENDIX C

THE PIONEER CALM ASSET ALLOCATION MODEL

The mathematical formulation of the basic asset management problem in deterministic equivalent form for solution is given by the following version of the CALM model of Dempster (1993). We assume that u is given by one of the utility functions described in Section 5.2, that, as a consequence of Monte Carlo simulation, each scenario ω in Ω is equally likely, that there are no cash inflows or outflows, and that the only regulatory and performance constraints are cash borrowing limits, short sale constraints, position limits and turnover constraints. Liabilities are easily added in terms of cash inflows or outflows (Consigli & Dempster, 1998):

$$\max_{\theta} \sum_{\omega\in\Omega} p(\omega) \sum_{t=2}^{T+1} u_t(w_t^{\theta}(\omega))$$

such that:

$$w_1 + \sum_{i\in I} p_{i1}(\omega)(gx_{i1}^{-}(\omega) - fx_{i1}^{+}(\omega)) + \sum_{k\in K} p_{k1}(\omega)(-z_{k1}^{+}(\omega) + z_{k1}^{-}(\omega)) = 0 \qquad \omega \in \Omega$$

$$\sum_{i\in I} p_{it}(\omega)(gx_{it}^{-}(\omega) - fx_{it}^{+}(\omega)) +$$

$$\sum_{k\in K} p_{kt}(\omega)((1 + r_{kt}^{+}(\omega))z_{kt-1}^{+}(\omega) - (1 + r_{kt}^{-}(\omega))z_{kt-1}^{-}(\omega) - z_{kt}^{+}(\omega) + z_{kt}^{-}(\omega)) = 0$$

$$t = 2, \ldots, T, \omega \in \Omega$$

$$x_{i1}(\omega) = x_i + x_{i1}^{+}(\omega) - x_{i1}^{-}(\omega) \qquad i \in I, \omega \in \Omega$$

$$x_{it}(\omega) = x_{it-1}(\omega)(1 + v_{it}(\omega)) + x_{it}^{+}(\omega) - x_{it}^{-}(\omega) \qquad i \in I, t = 2, \ldots, T, \omega \in \Omega$$

$$\sum_{i\in I} p_{it}(\omega)(1 + v_{it}(\omega))x_{it-1}(\omega) +$$

$$\sum_{k\in K} p_{kt}(\omega)((1 + r_{kt}^{+}(\omega))z_{kt-1}^{+}(\omega) - (1 + r_{kt}^{-}(\omega))z_{kt-1}^{-}(\omega)) = w_t^{\theta}(\omega)$$

$$t = 2, \ldots, T + 1, \omega \in \Omega$$

$$W_t^\theta(\omega) = \sum_{j\in I} p_{jt}(\omega)x_{jt}(\omega) + \sum_{k\in K} p_{kt}(\omega)(z_{kt}^+(\omega) - z_{kt}^-(\omega)) \qquad t = 1, \ldots, T, \omega \in \Omega$$

$$p_{kt}(\omega)z_{kt}^-(\omega) \le \bar{z}_k \qquad k \in K, t = 1, \ldots, T, \omega \in \Omega$$

$$p_{it}(\omega)x_{it}(\omega) \ge \bar{x}_i \qquad i \in I, t = 1, \ldots, T, \omega \in \Omega$$

$$p_{it}(\omega)x_{it}(\omega) \le \phi_i W_t^\theta(\omega) \qquad i \in I, t = 1, \ldots, T, \omega \in \Omega$$

$$p_{kt}(\omega)(z_{kt}^+(\omega) - z_{kt}^-(\omega)) \le \phi_k W_t^\theta(\omega) \qquad k \in K, t = 1, \ldots, T, \omega \in \Omega$$

$$|p_{it}(\omega)x_{it}(\omega) - p_{it-1}(\omega)x_{it-1}(\omega)| \le \alpha_i W_t^\theta(\omega) \qquad i \in I, t = 1, \ldots, T, \omega \in \Omega$$

$$w_t^\theta(\omega) \ge 0 \qquad t = 2, \ldots, T+1, \omega \in \Omega$$

$$x_{it}^+(\omega), x_{it}^-(\omega), z_{kt}^+(\omega), z_{kt}^-(\omega) \ge 0 \qquad i \in I, k \in K, t = 1, \ldots, T, \omega \in \Omega$$

where $p(\omega) = 1/|\Omega|$ and $\theta_{ikt}(\omega) := (x_{it}(\omega), x_{it}^+(\omega), x_{it}^-(\omega), z_{kt}^+(\omega), z_{kt}^-(\omega))$ for i in I, k in K, $t = 1, \ldots, T$, ω in Ω .

The first set of constraints are known as cash balance constraints. They ensure that the net flow of cash at each time and in each state is zero. The next set of constraints are known as inventory balance constraints. They give the position in each equity and bond asset at each time and in each state. The third set of constraints define, respectively, the before and after rebalancing wealth at each time in each state. The next six constraints are the cash borrowing constraints, short sale constraints, position limit constraints, turnover constraints and solvency constraints, discussed in the previous section. This deterministic mathematical programming problem is convex, linearly constrained and (unless u is the identity) has a non-linear objective.