Optimisation in a capital scarce world
Edinburgh, October 2009
Gareth Haslip
Colin Kerley

Outline

- Plan to talk about ways to investigate the structure of an “optimal” portfolio
- Start with risk / return definitions
- Look at investment portfolios
- Extend the idea to portfolios of liabilities
- And possibly mix the two
Introduction

- The problem: we want to maximise the return from our investment of risk capital subject to a defined level of risk
- We’d like to talk about some possible approaches to this problem

Applications

- Selecting the optimal investment strategy for a given risk budget
- Optimising reinsurance portfolios or insurance linked security fund allocation
- Minimising regulatory capital requirements for a target required level of asset return
Definitions : Risk and Return

- **Return**:
  - Assets: income + capital gains or losses
  - Liabilities: premium - expenses - losses

- **Risk**:
  - Many definitions of risk
  - StDev / PML / VaR / TVaR
  - We will focus on TVaR as it has some attractive properties as a risk measure

---

Why TVaR?

- Let $f(x)$ be the distribution of possible returns from the proposed portfolio
  - VaR looks at a single point on the distribution, say the 99%
  - TVaR is the average of all losses for $f(x)$ given they are greater than a certain point
- **Coherent risk measure**
  - Lots of good properties including sub-additivity
- **CoTVaR gradient**
  - For hill climbers can calculate “risk gradient” from co-TVaR
General Problem

- Optimisation problem
- Maximise return function subject to constraints
  - or minimise risk subject to constraints
- Lots of algorithms to do optimisation
  - Analytic solution
  - Random search
  - Hill Climbers
  - Linear programming
  - Genetic Algorithm

Simple Example: Investments
Simple example : investments

- Case study from a paper
- Over a one month time horizon we want to choose the amount to invest in each class to minimise the risk – given a certain minimum target return
- Underlying assumptions
  - Returns normally distributed
  - Gaussian copula defines the dependencies

Simple example : investments

- Definitions
  - $r_1, r_2, r_3$ returns from each asset class
  - $w_1, w_2, w_3$ chosen weights for our portfolio
  - Portfolio return $R = w_1 * r_1 + w_2 * r_2 + w_3 * r_3$
- For a given return we want to find weights to minimise the risk of the portfolio subject to some constraints
  - $w_1 + w_2 + w_3 = 1$
  - for this problem all weight must be $> 0$
- We will assume we can generate return distributions via monte-carlo simulation so have access to vectors $r(i)$ with $n$ samples
Random Search

- Start with the simplest numeric algorithm: random weight selection
  - “brute force and ignorance”

- Method
  - Choose random weights subject to constraints
  - Generate the return distribution for the portfolio
  - If the return exceeds the target threshold then look at the risk
  - If the risk is the smallest so far, remember the results
  - Repeat until bored

Results

- Or just look at all random outputs

<table>
<thead>
<tr>
<th>S &amp; P 500</th>
<th>Mean</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0%</td>
<td>0.7%</td>
<td>5.7%</td>
</tr>
<tr>
<td>US Bonds</td>
<td>0.4%</td>
<td>2.2%</td>
</tr>
<tr>
<td>Small Cap</td>
<td>7.4%</td>
<td>8.7%</td>
</tr>
</tbody>
</table>

Return vs. Risk / TVaR(90%)
Linear programming

- Constrained maximisation where risk is measured using TVaR can be expressed as a linear programming problem
- Using Monte Carlo simulation output with n samples the problem is translated into a linear system with n + q + 1 variables (q = # asset classes)
  - E.g. for 50K simulations we have to maximise a linear constrained system with over 50K variables…

---

Linear programming

- …luckily modern computing power can handle large linear systems with ease
  - Optimal solution can be found within minutes and is guaranteed to be the true global maxima
  - Additional constraints can easily be added with virtually no additional overhead, e.g.
    - Restrictions on the movement in book value
    - Min / max allocations to each asset class
    - Rating agency capital requirement
Genetic Algorithms

- Have been used successfully for a wide variety of optimisation problems

**Basic recipe**
- Create a population of individuals – all candidates for a solution
- Define a “gene” that specifies how fit an individual is for solving the solution
- Repeatedly create new generations of individuals where those with the highest fitness are more likely to have their genes passed on to the next generation
- Each generation genes are altered via mutation & crossover

**Some initial results**

Used the three methods to solve the problem
- We wanted to find the portfolio with the minimum risk subject to a minimum return threshold
- Both random & GA can be run forever but we set them to run until they came to a solution within x% of the “true” solution

<table>
<thead>
<tr>
<th>TVaR%</th>
<th>Method</th>
<th>VaR</th>
<th>TVaR</th>
<th>Time (mn)</th>
<th>S&amp;P 500</th>
<th>Bonds</th>
<th>Small Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>Random</td>
<td>13.26%</td>
<td>15.279%</td>
<td>1.89</td>
<td>0.3891</td>
<td>0.1374</td>
<td>0.4735</td>
</tr>
<tr>
<td>1%</td>
<td>Linear</td>
<td>13.23%</td>
<td>15.264%</td>
<td>0.62</td>
<td>0.3927</td>
<td>0.1367</td>
<td>0.4706</td>
</tr>
<tr>
<td>1%</td>
<td>GA</td>
<td>13.24%</td>
<td>15.268%</td>
<td>1.84</td>
<td>0.3359</td>
<td>0.1555</td>
<td>0.5085</td>
</tr>
<tr>
<td>5%</td>
<td>Random</td>
<td>9.06%</td>
<td>11.614%</td>
<td>1.89</td>
<td>0.4256</td>
<td>0.1231</td>
<td>0.4513</td>
</tr>
<tr>
<td>5%</td>
<td>Linear</td>
<td>9.06%</td>
<td>11.599%</td>
<td>0.67</td>
<td>0.4364</td>
<td>0.1199</td>
<td>0.4437</td>
</tr>
<tr>
<td>5%</td>
<td>GA</td>
<td>9.06%</td>
<td>11.601%</td>
<td>1.83</td>
<td>0.4013</td>
<td>0.1333</td>
<td>0.4654</td>
</tr>
<tr>
<td>10%</td>
<td>Random</td>
<td>6.77%</td>
<td>9.706%</td>
<td>1.89</td>
<td>0.4221</td>
<td>0.1251</td>
<td>0.4528</td>
</tr>
<tr>
<td>10%</td>
<td>Linear</td>
<td>6.76%</td>
<td>9.701%</td>
<td>0.68</td>
<td>0.4529</td>
<td>0.1136</td>
<td>0.4335</td>
</tr>
<tr>
<td>10%</td>
<td>GA</td>
<td>6.76%</td>
<td>9.704%</td>
<td>1.82</td>
<td>0.4312</td>
<td>0.1254</td>
<td>0.4434</td>
</tr>
</tbody>
</table>
Some initial results (2)

<table>
<thead>
<tr>
<th>TVaR%</th>
<th>Method</th>
<th>VaR</th>
<th>TVaR%</th>
<th>Time (min)</th>
<th>S&amp;P 500</th>
<th>Bonds</th>
<th>Small Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>Random</td>
<td>13.19%</td>
<td>15.301%</td>
<td>4.90</td>
<td>0.4444</td>
<td>0.1177</td>
<td>0.4379</td>
</tr>
<tr>
<td>1%</td>
<td>Linear</td>
<td>13.18%</td>
<td>15.298%</td>
<td>9.50</td>
<td>0.4447</td>
<td>0.1177</td>
<td>0.4376</td>
</tr>
<tr>
<td>1%</td>
<td>GA</td>
<td>13.18%</td>
<td>15.302%</td>
<td>4.63</td>
<td>0.4183</td>
<td>0.1409</td>
<td>0.4408</td>
</tr>
<tr>
<td>5%</td>
<td>Random</td>
<td>11.57%</td>
<td>11.571%</td>
<td>4.91</td>
<td>0.4746</td>
<td>0.1059</td>
<td>0.4195</td>
</tr>
<tr>
<td>5%</td>
<td>Linear</td>
<td>8.97%</td>
<td>11.564%</td>
<td>9.50</td>
<td>0.4418</td>
<td>0.1188</td>
<td>0.4394</td>
</tr>
<tr>
<td>5%</td>
<td>GA</td>
<td>8.98%</td>
<td>11.566%</td>
<td>4.63</td>
<td>0.4203</td>
<td>0.1270</td>
<td>0.4327</td>
</tr>
<tr>
<td>10%</td>
<td>Random</td>
<td>6.80%</td>
<td>9.685%</td>
<td>4.94</td>
<td>0.4372</td>
<td>0.1204</td>
<td>0.4423</td>
</tr>
<tr>
<td>10%</td>
<td>Linear</td>
<td>6.80%</td>
<td>9.684%</td>
<td>9.50</td>
<td>0.4492</td>
<td>0.1160</td>
<td>0.4349</td>
</tr>
<tr>
<td>10%</td>
<td>GA</td>
<td>6.80%</td>
<td>9.685%</td>
<td>4.62</td>
<td>0.4311</td>
<td>0.1229</td>
<td>0.4460</td>
</tr>
</tbody>
</table>

- Models run against 50k simulations of output

Optimisation Speed

- Random and GA scale linearly with volume of simulation data
- Linear scales $o(n^2)$ – stratified sampling is recommended
Observations

- Linear method will find an optimal solution
- Random and GA can come arbitrarily close to a good solution but time is a problem
- Disappointing the GA does not perform much better

Investments – six assets

- We ran the same optimisation exercise with six assets
- Again, the linear approach found the best solution
- However clear difference between the Random and GA
  - Random finding it hard to get close to a good answer
Good algorithm

- Linear programming works well for these sort of problems
- Some issues though
  - The constraints need to be linear
  - Careful in definition of TVaR if the underlying risk distributions are not continuous
  - Algorithm scales $o(n^2)$ with respect to sample size

Problem 2: Insurance world
Insurance world

- No linearity in risk
  - Risk not 100% correlated
  - The payoff distribution from a class of business is a function of how much you have invested and where
- Complex payoff functions
  - Excess of loss
  - Multi year structured deals
- Dependencies odd
  - Primary vs xl
  - Cat models

Possible solutions

- Linear approach works well
  - Only restriction is linear constraints – but probably not a major problem for many standard applications
- Use GA
  - Slower
  - Allows non-linear constraints
  - Not restricted to TVaR as risk measure
- Random should be a last resort
A Reinsurance case study

- We have a pool of risk capital and six property cat reinsurance treaties to participate in
- We can participate up to 100% in each risk
- Must find a portfolio mix that maximises our expected profit given our risk capital limit
  - All the treaties are exposed to US Hurricane risk
  - Mixture of ILW’s and Cat XL
  - RI Premiums consistent with the market

The portfolio

- Mixture of binary / high xs payoff – not very smooth
Cat Portfolio returns

<table>
<thead>
<tr>
<th>Risk</th>
<th>Type</th>
<th>Premium</th>
<th>Losses</th>
<th>Profit</th>
<th>StDev</th>
<th>CoV</th>
<th>Prob (Loss)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>ILW</td>
<td>10.00</td>
<td>4.52</td>
<td>5.62</td>
<td>20.59</td>
<td>3.7</td>
<td>4.8%</td>
</tr>
<tr>
<td>R2</td>
<td>14.99</td>
<td>4.11</td>
<td>10.33</td>
<td>37.51</td>
<td>3.6</td>
<td>2.5%</td>
<td></td>
</tr>
<tr>
<td>R3</td>
<td>ILW</td>
<td>4.50</td>
<td>1.89</td>
<td>2.64</td>
<td>5.85</td>
<td>2.2</td>
<td>9.3%</td>
</tr>
<tr>
<td>R4</td>
<td>ILW</td>
<td>6.25</td>
<td>3.06</td>
<td>3.23</td>
<td>7.80</td>
<td>2.4</td>
<td>13.6%</td>
</tr>
<tr>
<td>R5</td>
<td>ILW</td>
<td>10.50</td>
<td>8.72</td>
<td>1.77</td>
<td>13.62</td>
<td>7.7</td>
<td>29.1%</td>
</tr>
<tr>
<td>R6</td>
<td>10.00</td>
<td>4.52</td>
<td>5.62</td>
<td>20.59</td>
<td>3.7</td>
<td>4.6%</td>
<td></td>
</tr>
</tbody>
</table>

Random - scatterplot

Risk / Reward - Cat portfolio

The Actuarial Profession
making financial sense of the future
Results

- Target max return on portfolio subject to TVaR < 100 (risk capital target)
- Time ran for Random set to be the same as the GA
- GA run time set so answer about 1% near optimal value

<table>
<thead>
<tr>
<th>TVaR% Method</th>
<th>TVaR</th>
<th>Return</th>
<th>Time (mn)</th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>w5</th>
<th>w6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5% Random</td>
<td>98.51</td>
<td>9.63</td>
<td>20.10</td>
<td>43.4%</td>
<td>2.3%</td>
<td>96.0%</td>
<td>97.9%</td>
<td>61.5%</td>
<td>6.4%</td>
</tr>
<tr>
<td>0.5% Linear</td>
<td>100.00</td>
<td>10.53</td>
<td>11.10</td>
<td>51.4%</td>
<td>0.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.5% GA</td>
<td>99.64</td>
<td>10.12</td>
<td>21.51</td>
<td>45.1%</td>
<td>2.2%</td>
<td>79.9%</td>
<td>98.6%</td>
<td>97.3%</td>
<td>0.3%</td>
</tr>
<tr>
<td>1.0% Random</td>
<td>92.37</td>
<td>9.63</td>
<td>20.20</td>
<td>43.4%</td>
<td>2.3%</td>
<td>96.0%</td>
<td>97.9%</td>
<td>61.5%</td>
<td>6.4%</td>
</tr>
<tr>
<td>1.0% Linear</td>
<td>100.00</td>
<td>10.53</td>
<td>11.12</td>
<td>51.4%</td>
<td>0.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>1.0% GA</td>
<td>99.54</td>
<td>10.38</td>
<td>21.48</td>
<td>45.7%</td>
<td>0.4%</td>
<td>99.2%</td>
<td>98.6%</td>
<td>97.3%</td>
<td>8.7%</td>
</tr>
<tr>
<td>2.5% Random</td>
<td>99.01</td>
<td>11.28</td>
<td>20.10</td>
<td>43.0%</td>
<td>3.2%</td>
<td>94.2%</td>
<td>99.7%</td>
<td>66.6%</td>
<td>59.5%</td>
</tr>
<tr>
<td>2.5% Linear</td>
<td>100.00</td>
<td>11.73</td>
<td>11.14</td>
<td>36.0%</td>
<td>20.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>2.5% GA</td>
<td>99.86</td>
<td>11.70</td>
<td>21.20</td>
<td>31.9%</td>
<td>6.1%</td>
<td>100.0%</td>
<td>99.1%</td>
<td>100.0%</td>
<td>60.1%</td>
</tr>
</tbody>
</table>

Convergence

- We can see the GA outperforming the Random approach very clearly now
- Random does not get close to an “optimal” solution

<table>
<thead>
<tr>
<th>TVaR 0.5%</th>
<th>TVaR 1.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence speed</td>
<td>Convergence speed</td>
</tr>
<tr>
<td>Difference from last iteration</td>
<td>Difference from last iteration</td>
</tr>
<tr>
<td>Time (seconds)</td>
<td>Time (seconds)</td>
</tr>
</tbody>
</table>
Portfolio Composition

- Portfolio mix relatively stable with change of risk measure
- Optimisation – algorithms often find problems in the question
- Need human judgement – we can keep our jobs

Problem 3: Asset Liability Management
Understanding Market Risk

- Market risk is often misunderstood for insurers
- It is not about how the value of assets change due to movements in the financial markets
  - It is about how the surplus (net asset value) changes in response to market movements
- Worse case scenario is where market movements can decrease assets and increase liabilities simultaneously
- Managing market risk is about managing the sensitivity of the surplus process to movements in market variables

![Diagram showing assets, liabilities, surplus, and deficit]

Asset Liability Optimisation

- Aim of asset allocation for insurers is to maximise expected outperformance of assets over liabilities
  - Subject to constraints on the potential downward movement in the surplus process
- We can use the same optimisation framework to solve the asset allocation problem in this setting
- Procedure is very similar to before except that an additional Monte Carlo output vector is required
  - The % change in the discounted value of the liabilities at the end of the time period under consideration (e.g. end of year for Solvency II / SST)
  - Important that the interest rate scenarios applied to generate asset returns are ordered consistently with the liability simulation
  - Only works for non-life insurance where liabilities are independent of asset allocation
Asset Liability Optimisation

- Optimisation method is applied to surplus process:

\[ S = \text{Initial Assets} \times (w_1r_1 + w_2r_2 + w_3r_3) - 100\% \times \text{Initial Liability} \times r_l \]

- Idea is that there is a fixed -100% holding in the liabilities and then the optimisation algorithm is applied as before

- This allows asset allocation strategies to be developed in the context of Solvency 2 and the Swiss Solvency Test definitions of market risk

- For example, develop an asset strategy that minimises the regulatory market risk capital requirement subject to achieving a target level of return
  - Allows insurers to control market risk budget and concentrate on applying capital to insurance risk

Conclusions

- Linear Programming approach seems to work best for these optimisation problems
  - But only works for TVaR as a risk measure, not VaR
  - And again, constraints need to be linear

- Powerful tool for risk management…for both assets and liabilities
Questions ?