Chain Ladder and Bornhuetter/Ferguson – Some Practical Aspects

by Thomas Mack, Munich Re
• Consider one single accident year
• Paid claims after k years of development
• Expected cumulative payout pattern
  \( p_1, p_2, \ldots, p_k, \ldots, p_n = 1 \)  e.g.
  10\%, 30\%, 50\%, 70\%, 85\%, 95\%, 100\%
• \( U_0 = \text{prior est. of ultimate claims amount} \)
  \( R_{BF} = q_k U_0 \) with \( q_k = 1 - p_k \)  Bornhuetter/Ferguson.
• $C_k =$ claims amount paid up to now  
  (completely ignored by BF)
• $U_{BF} = C_k + R_{BF}$ posterior estimate (≠ $U_0$)
• $U = C_k + R$ (axiomatic relationship)
• $U_{CL} = C_k / p_k$ Chain Ladder ult. claims
• $R_{CL} = U_{CL} - C_k = q_k U_{CL}$ CL reserve  
  (ignores $U_0$ completely)
Comparison:

- With CL, different actuaries usually come to similar results.
- With BF, there is no clear way to $U_0$.
- $U_0$ can be manipulated:
  If you want to have reserve $X$, simply put $U_0 = X / q_k$. 
• Gunnar Benktander's proposal (1976):

\[ R_{GB} = p_k R_{CL} + (1-p_k)R_{BF} \]

\[ = p_k q_k C_k / p_k + q_k R_{BF} \]

\[ = q_k ( C_k + R_{BF} ) = q_k U_{BF} \]

• **Iterated** Bornhuetter/Ferguson

• The more the claims develop, the higher the weight \( p_k \) of \( R_{CL} \).
Ultimate U(R) Connection Reserve R(U)

\[ U_0 \]

\[ U_1 = U_{BF} = C_k + q_k U_0 \]
\[ = (1 - q_k) U_{CL} + q_k U_0 \]

\[ U_2 = U_{GB} \]
\[ = (1 - q_k^2) U_{CL} + q_k^2 U_0 \]

\[ R_{BF} = q_k U_0 \]

\[ R_1 = q_k U_1 = q_k U_{BF} = R_{GB} \]
\[ = (1 - q_k) R_{CL} + q_k R_{BF} \]
<table>
<thead>
<tr>
<th>Ultimate $U(R)$</th>
<th>Connection</th>
<th>Reserve $R(U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_n = (1-q_k^n)U_{CL} + q_k^n U_0$</td>
<td>$*q_k$</td>
<td>$R_n = (1-q_k^n)R_{CL} + q_k^n R_{BF}$</td>
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<tr>
<td>$U_{n+1} = (1-q_k^{n+1})U_{CL} + q_k^{n+1} U_0$</td>
<td></td>
<td>$C_k+$</td>
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<tr>
<td>$U_\infty = U_{CL}$</td>
<td></td>
<td>$R_\infty = R_{CL}$</td>
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Chain Ladder and Bornhuetter/Ferguson

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Chain Ladder and Bornhuetter/Ferguson

$R_{GB}$ is a **credibility mixture** of $R_{CL}$ and $R_{BF}$:

$$R_c = c \ R_{CL} + (1-c) \ R_{BF} \quad \text{with} \quad c = p_k \in [0; 1]$$

It gives $R_{BF}$ for $c = 0$ and $R_{CL}$ for $c = 1$.

**Best mixture**

if mean squared error is minimized:

$$\text{mse}(R_c) = E(R_c - R)^2 = \min \quad (\Rightarrow c^*)$$
• $R_{c^*}$ is always better than $R_0 = R_{BF}$, $R_1 = R_{CL}$

• $R_{GB}$ is not always better but mostly

• How to determine $c^*$ ?

• How to decide
  which of $R_{GB}$, $R_{CL}$, $R_{BF}$ is best
  at a given data set?
\[ R_c = c R_{CL} + (1-c) R_{BF} = c(R_{CL} - R_{BF}) + R_{BF} \]

\[ E(R_c - R)^2 = E[c(R_{CL} - R_{BF}) + (R_{BF} - R)]^2 \]

\[ = c^2 E(R_{CL} - R_{BF})^2 + 2c E[(R_{CL} - R_{BF})(R_{BF} - R)] + E(R_{BF} - R)^2 \]

\[ C^* = \frac{E((R_{CL} - R_{BF})(R - R_{BF}))}{E(R_{CL} - R_{BF})^2} \]

\[ = p_k \cdot \frac{\text{Cov}(C_k, R) + p_k q_k \text{Var}(U_0)}{q_k \text{Var}(C_k) + p_k^2 \text{Var}(U_0)} \]
So far, we have not used any assumptions. But for $\text{Var}(C_k)$, $\text{Cov}(C_k, R)$ we need a model.

Model A: (with $U = C_n$)

$$E(C_k|U) = p_k U, \quad \text{Var}(C_k|U) = p_k q_k \alpha^2(U)$$

$$\Rightarrow \quad \text{Var}(C_k) = p_k q_k E(\alpha^2(U)) + p_k^2 \text{Var}(U)$$

$$\text{Cov}(C_k, R) = p_k q_k \left( \text{Var}(U) - E(\alpha^2(U)) \right)$$

But $E(\alpha^2(U))$ is difficult to estimate.
\(B:\) Increments \(S_j = C_j - C_{j-1}, \ m_j = p_j - p_{j-1}\)

\(E(S_j/m_j | \Theta) = \mu(\Theta), \ S_j | \Theta \) independent,

\(\text{Var}(S_j/m_j | \Theta) = \sigma^2(\Theta)/m_j , \ (\text{Bühlmann/S.})\)

\(\Theta\) indicates the "quality" of the acc. year

\(\Rightarrow \) \(\text{Var}(C_k) = p_k q_k E(\sigma^2(\Theta)) + p_k^2 \text{Var}(U)\)

\(\text{Cov}(C_k, R) = p_k q_k \left( \frac{\text{Var}(U) - E(\sigma^2(\Theta))}{\text{Var}(\mu(\Theta))} \right)\)
Both models are math. equivalent and lead to

\[ c^* = \frac{p_k}{p_k + t} \quad \text{and} \quad t = \frac{\mathbb{E}(\sigma^2(\Theta))}{\text{Var} (\mu(\Theta)) + \text{Var} (U_0)} \]

\[ \mathbb{E}(\sigma^2(\Theta)) = \text{inner variабл. random error} \]
\[ \text{Var}(\mu(\Theta)) = \text{level variабл. Var}(U) \]
\[ \text{Var}(U_0) = \text{estimation error} \]

\[ \text{to be est. by actuary} \]
An actuary who presumes to establish a point estimate $U_0$ should also be able to estimate its uncertainty $\text{Var}(U_0)$ and the variability $\text{Var}(U)$ of the underlying claims process.
For $E(\sigma^2(\Theta))$, we have an unbiased estimate based on the data observed:

$$\frac{1}{k-1} \sum_{j=1}^{k} m_j \left( \frac{S_j}{m_j} - \frac{C_k}{p_k} \right)^2 = \frac{p_k}{k-1} \sum_{j=1}^{k} m_j \left( \frac{S_j}{m_j} - U_{CL} \right)^2$$

Note that $\sum_{j=1}^{k} m_j = p_k$ and $\sum_{j=1}^{k} \frac{S_j}{m_j} = U_{CL}$
Having estimated \[ t = \frac{E(\sigma^2(\Theta))}{\text{Var}(\mu(\Theta)) + \text{Var}(U_0)} \]

we can compare the precisions:

\[
\begin{align*}
\text{mse}(R_{BF}) &= E(\sigma^2(\Theta)) \left( q_k + \frac{q_k^2}{t} \right) \\
\text{mse}(R_{CL}) &= E(\sigma^2(\Theta)) \frac{q_k}{p_k} \\
\text{mse}(R_c) &= c^2 \text{mse}(R_{CL}) + (1-c)^2 \text{mse}(R_{BF}) + 2c(1-c)q_k E(\sigma^2(\Theta))
\end{align*}
\]
and obtain the following results:

\[ \text{mse}(R_{BF}) < \text{mse}(R_{CL}) \iff p_k < t \]

i.e. use BF for green years

use CL for rather mature years

\[ \text{mse}(R_{GB}) < \text{mse}(R_{BF}) \iff t < 2-p_k \]

\[ \text{mse}(R_{GB}) < \text{mse}(R_{CL}) \iff t > p_k q_k/(1+p_k) \]
**Example:** \[ U_0 = 90\%, \quad k = 3 \]

\[ \{p_j\} = 10\%, 30\%, 50\%, (70, 85, 95, 100 \%) \]

\[ \{C_j\} = 15\%, 27\%, 55\% \text{ (of the premium)} \]

\[ \Rightarrow \]

\[ R_{BF} = 45\%, \quad U_{CL} = 110\%, \quad R_{CL} = 55\% \]

\[ \{m_j\} = 10\%, 20\%, 20\%, (20\%, 15\%, 10\%, 5\%) \]

\[ \{S_j\} = 15\%, 12\%, 28\% \]
Inner variability

\[ S_1/m_1 = 1.5, \quad S_2/m_2 = 0.6, \quad S_3/m_3 = 1.4 \]

\[ E(\sigma^2 (\Theta)) = \]

\[ \frac{0.50}{3-1} \left( \frac{10}{50} (1.5-1.1)^2 + \frac{20}{50} (0.6-1.1)^2 + \frac{20}{50} (1.4-1.1)^2 \right) \]

\[ = 0.042 = (20.5\%)^2 \]
Actuary's estimates:

\[ \text{Var}(U) = (35\%)^2 \]

(e.g. lognormal with 5% above 150%)

\[ \text{Var}(U_0) = (15\%)^2 \]

\[ \Rightarrow \]

\[ \text{Var}(\mu(\Theta)) = (35\%)^2 - (20.5\%)^2 = (28.4\%)^2 \]

\[ t = \frac{(20.5\%)^2}{(28.4\%)^2 + (15\%)^2} = 0.408 \]
Results:

\[ R_{BF} = 45.0\% \pm 21.6\% \]
\[ R_{CL} = 55.0\% \pm 20.5\% \]
\[ R_{GB} = 50.0\% \pm 18.1\% \]
\[ R_{c*} = 50.5\% \pm 18.0\% \quad \text{with } c^* = 0.55 \]

Note the high standard errors!
Check by **distributional assumptions**:

**U ~ Lognormal(μ, σ^2)** with

\[ E(U) = 90\%, \ Var(U) = (35\%)^2 \]

\[ \Rightarrow \mu = -0.176, \ \sigma^2 = 0.141 \]

C_k|U ~ Lognormal(ν, τ^2) with

\[ E(C_k|U) = p_kU, \ Var(C_k|U) = p_kq_k\alpha^2U^2 \]

where \( \alpha^2 \) is such that \( Var(C_k) \) is as before

\[ \Rightarrow \alpha^2 = 0.045, \ \tau^2 = 0.044 \]
Then, according to Bayes' theorem:

\[ U|C_k \sim \text{Lognormal}(\mu_1, \sigma_1^2) \]

with \( \mu_1 = z(\tau^2 + \ln(C_k/p_k)) + (1-z)\mu = 0.0643 \)

\[ \sigma_1^2 = z\tau^2 = 0.0335 \]

\[ z = \sigma^2 / (\sigma^2 + \tau^2) = 0.762 \]

\[ \Rightarrow E(U|C_k) = \exp(\mu_1 + \sigma_1^2/2) = 108.4\% \]

\[ \Rightarrow E(R|C_k) = 53.4\% \]

\[ \text{Var}(U|C_k) = (20.0\%)^2 = \text{Var}(R|C_k) \]
• No estimation error
• Standard error still very high

Results:

\[ R_{BF} = 45.0\% \pm 21.6\% \]
\[ R_{CL} = 55.0\% \pm 20.5\% \]
\[ E(R|C_k) = 53.4\% \pm 20.0\% \]
\[ R_{GB} = 50.0\% \pm 18.1\% \]
\[ R_{c^*} = 50.5\% \pm 18.0\% \quad \text{with } c^* = 0.55 \]
Conclusions:

- Use of a priori knowledge \((U_0)\) may be better than distributional assumptions.
- A way is shown how to assess the variability of the Bornhuetter/Ferguson reserve, too.
Conclusions (ctd.):

- Benktander's credibility mixture of BF and CL is simple to apply and gives almost always a more precise estimate.

- The volatility measure t is not too difficult to estimate and improves the precision even more or helps to decide on BF, CL, GB.