Travels into Several Remote Nations of the World. In Four Parts. 
By Lemuel Gulliver, First a Surgeon, and then a Captain of Several Ships

International Mortality and Longevity Symposium
15-17 September 2014

Xu Shi and Bridget Browne

Adding stochastics to a deterministic forecasting method (such as the CMI method) – calibrated and exemplified with Australian data, or
A "simple" stochastic model for longevity risk revisited through bootstrap

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Or…
How to make a deterministic mortality forecast stochastic.

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Why does the issue arise?

• CMI method is deterministic
• How to determine percentiles for
  – Pricing
  – Reserving
  – Internal capital modelling
  – Etc
• How to incorporate prudence or conservatism with quantification?
The Original Model (Koller 2011)

\[ q_{x,t} = \bar{q}_{x,t} \times C_t + \varepsilon_{x,t} \]

\[ C_t = e^{X_t} \times C_{t-1} \]

\[ C_0 = 1 \]

\( (X_t)_{t \in \mathbb{N}_0}, \text{iid } \mathcal{N}(\mu, \sigma^2) \)

\( (e^{X_t})_{t \in \mathbb{N}_0}, \text{iid lognormal} \)

\[ \mathbb{E}[e^{X_t}] = 1, \text{ is imposed, so that best estimate is followed} \]

\[ \mathbb{E}[e^{X_t}] = e^{\mu + 0.5\sigma^2}, \text{ follows from above } * \]
Modifications

• What if $X_t$ is not normal?
• Transformation - rejected
• Non-parametric approach – bootstrap
• Improved model selection
• Consider X rather than $X_t$

A numerical example

• Australian Females
• Human Mortality Database
• Ages 55 to 89 inclusive
• 1954 to 2008
• Surface of 35 ages and 55 cal. years
Crude Mortality Improvement Rates

Histogram of Crude MI Rates
Model assessment

- Use smoothing models only to help determine our best estimate of future variability
- M1-M8 from LifeMetrics (JP Morgan)
  - Standardised residuals and BIC
  - Forecasting properties

QQ Plots
# Summary statistics for epsilon and BIC

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.000606</td>
<td>1.489723</td>
<td>-0.087587</td>
<td>-10 152</td>
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<tr>
<td>2</td>
<td>0.007536</td>
<td>1.088392</td>
<td>0.134849</td>
<td>-10 192</td>
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<td>3</td>
<td>-0.003997</td>
<td>2.028962</td>
<td>-0.085894</td>
<td>-10 917</td>
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<td>4</td>
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<td>2.078517</td>
<td>0.172425</td>
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<td>5</td>
<td>0.166069</td>
<td>3.876925</td>
<td>0.139640</td>
<td>-12 399</td>
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<tr>
<td>6</td>
<td>0.034808</td>
<td>1.842268</td>
<td>2.350169</td>
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<tr>
<td>7</td>
<td>-0.005507</td>
<td>1.496240</td>
<td>-1.015056</td>
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</tr>
<tr>
<td>8</td>
<td>0.020308</td>
<td>1.582045</td>
<td>1.035231</td>
<td>-10 297</td>
</tr>
</tbody>
</table>
Forecasting properties

- Now choosing between M2 and M3
- M2 unstable
- Backtesting for M3

**Prediction Intervals I**
Prediction Intervals II

QQ Plot of $X_t$ for M3 Australian Females
Age dependence or independence?

- P-values of two-variable regression of X against age (x) and year (t)

<table>
<thead>
<tr>
<th>Model Type</th>
<th>Variable</th>
<th>ANOVA p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Case</td>
<td>Year</td>
<td>0.1079</td>
</tr>
<tr>
<td></td>
<td>Age</td>
<td>0.8350</td>
</tr>
<tr>
<td>Discrete Case</td>
<td>Year</td>
<td>0.9965</td>
</tr>
<tr>
<td></td>
<td>Age</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

Paths of Ct, q65,t for Australian Females
Fan plots of Ct, q65,t for Australian Females

Aust Females (Age Independent Case, X)

Histogram (LHS) of a_{65|31} (at 4% pa) and cohort survival curves (RHS)
Aust Females (Age Dependent Case, $X_t$)

Histogram (LHS) of $a_{65\{31\}}$ (at 4% pa) and cohort survival curves (RHS)

Expected Present Value of $a_{65\{31\}}$

Paths of cohort survival curves for $a_{65\{31\}}$ for Australian Females (Two Cases)
Age independence => coherence?

• Example of crossover in Crude Mortality Rates for Australian Females (red cell indicates mortality in year $t$ is greater for age $x$ than for age $x+1$)

Age independence => coherence?

• Example of crossover in one simulation for Australian Females (red cell indicates mortality in year $t$ is greater for age $x$ than for age $x+1$)
Comparison with other methods

- Tickle and Booth (2014) evaluated and updated a forecast of mortality for Australian seniors using the Booth-Maindonald-Smith variant of Lee-Carter (generously shared underlying work)
  - Central estimate and 80% PI for qx,t
- LPS 115 Section 38 (APRA, 2013, p7)
  - 20% drop in qx,t
  - < 0.5% probability actual claims exceed
Paths of cohort survival curves (ie cashflows) for Australian Females, central estimate, APRA basis and 99.5\textsuperscript{th} percentiles for both cases

![Graph showing survival curves]

Comparison of Cohort Life Expectancy and Annuity Values (at 4\% pa) for Australian Females aged 65 in 2010, central estimate and selected percentiles

<table>
<thead>
<tr>
<th></th>
<th>$e_{65}[31], 2010$</th>
<th>% diff</th>
<th>$a_{65}[31], 2010$</th>
<th>% diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central estimate (here, Tickle &amp; Booth)</td>
<td>22.668</td>
<td></td>
<td>14.166</td>
<td></td>
</tr>
<tr>
<td>APRA 20% drop (99.5% percentile)</td>
<td>23.655</td>
<td>4.35%</td>
<td>14.610</td>
<td>3.14%</td>
</tr>
<tr>
<td>Bootstrap, Age independent, X (10% percentile)</td>
<td>22.287</td>
<td>-1.68%</td>
<td>14.019</td>
<td>-1.03%</td>
</tr>
<tr>
<td>Bootstrap, Age independent, X (90% percentile)</td>
<td>22.845</td>
<td>0.78%</td>
<td>14.254</td>
<td>0.62%</td>
</tr>
<tr>
<td>Bootstrap, Age independent, X (99.5% percentile)</td>
<td>23.102</td>
<td>1.91%</td>
<td>14.325</td>
<td>1.12%</td>
</tr>
<tr>
<td>Bootstrap, Age dependent, X (10% percentile)</td>
<td>21.415</td>
<td>-5.53%</td>
<td>13.636</td>
<td>-3.74%</td>
</tr>
<tr>
<td>Bootstrap, Age dependent, X (90% percentile)</td>
<td>23.817</td>
<td>5.07%</td>
<td>14.645</td>
<td>3.39%</td>
</tr>
<tr>
<td>Bootstrap, Age dependent, X (99.5% percentile)</td>
<td>24.912</td>
<td>9.90%</td>
<td>15.038</td>
<td>6.16%</td>
</tr>
</tbody>
</table>
Conclusion

- CMI method more widely used - UK, USA, Canada, Australia and China at least
- Some national statistics bodies also deterministic (ABS, 2013, ONS, 2013)
- This method allows the user to add stochastic variation to a deterministic model
- Bootstrapping improves tail modelling
- Help to inform understanding of longevity risk

References