Robustness, Model Ambiguity and Pricing

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Pricing contracts in incomplete markets

Examples:
- Pricing very long-dated cash flows $T \sim 30 - 100$ years
- Pricing long-dated equity options $T > 5$ years
- Pricing pension & insurance liabilities

Actuarial premium principles typically “ignore” financial markets
- Actuarial pricing is “static”: price at $t = 0$ only

Financial pricing considers “dynamic” pricing problem:
- How does price evolve over time until time $T$?

Financial pricing typically “ignores” unhedgeable risks
Outline of This Talk

1. Pricing in Complete Market
2. Robustness & Model Ambiguity
3. Applications
4. Summary & Conclusion
Suppose we have a stock price $S$ with return process $x = \ln S$:

$$dx = m \, dt + \sigma \, dW_x,$$

Discretisation in binomial tree:

$$x(t + \Delta t) = x(t) + \begin{cases} +\sigma \sqrt{\Delta t} & \text{with prob. } \frac{1}{2} \left(1 + \frac{m}{\sigma} \sqrt{\Delta t}\right) \\ -\sigma \sqrt{\Delta t} & \text{with prob. } \frac{1}{2} \left(1 - \frac{m}{\sigma} \sqrt{\Delta t}\right). \end{cases}$$

Model ambiguity as $m \in [m_L, m_H]$. 
Valuation with Model Ambiguity

Suppose we have a derivative contract with value \( f(t + \Delta t, x(t + \Delta t)) \) at time \( t + \Delta t \).

Given uncertainty about drift \( m \), “ambiguity averse” rational agent will consider “worst case” expectation:

\[
\min_{m \in [m_L, m_H]} e^{-r\Delta t} \mathbb{E}_t^m [f(t + \Delta t, x(t + \Delta t))]
\]

Explicit solution for binomial tree:

\[
\begin{cases}
    e^{-r\Delta t} \left( f_1 + (f_x m_L + \frac{1}{2} f_{xx}\sigma^2)\Delta t \right) & \text{if } f_x > 0 \\
    e^{-r\Delta t} \left( f_1 + \left( \frac{1}{2} f_{xx}\sigma^2 \right)\Delta t \right) & \text{if } f_x = 0 \\
    e^{-r\Delta t} \left( f_1 + (f_x m_H + \frac{1}{2} f_{xx}\sigma^2)\Delta t \right) & \text{if } f_x < 0.
\end{cases}
\]
Interpretation of Valuation Equation

Take limit for $\Delta t \downarrow 0$.
Leads to semi-linear pde: $f_t + f_x m^* + \frac{1}{2} f_{xx} \sigma^2 - rf = 0$ with $m^* = m_L$ if $f_x > 0$ and $m^* = m_H$ if $f_x < 0$.

- Actuarial notion of *prudence* (not “risk-neutral”)
- Time-consistent *coherent risk-measure* with “$Q \in [m_L, m_H]$”
- *Good Deal Bound pricing* with upper bound on pricing kernel volatility
- GDB pricing with upper bound on Radon-Nikodym volatility
Suppose that rational agent can trade in the share price $S$.

Buy $\theta/S(t)$ shares at $t$, financed by borrowing an amount $\theta$ from the bank account $B$.

At time $t + \Delta t$, net position has value $(e^{x(t+\Delta t) - x(t)} - e^{r\Delta t})\theta$.

Find optimal amount $\theta$ that maximises worst-case expectation:

$$\max_{\theta} \min_{m \in [m_L, m_H]} e^{-r\Delta t} \left( f_1 + (f_x m + \frac{1}{2} f_{xx} \sigma^2 + (m + \frac{1}{2} \sigma^2 - r)\theta)\Delta t \right)$$

Two-player game: “mother nature” vs. agent.
Optimum \((m, \theta)\) depends on sign of partial deriv’s:

\[
\frac{\partial}{\partial \theta} : e^{-r\Delta t}(m + \frac{1}{2}\sigma^2 - r)\Delta t \quad \frac{\partial}{\partial m} : e^{-r\Delta t}(f_x + \theta)\sigma\Delta t
\]

Optimal choice for \(m\) depends on sign of \(\frac{\partial}{\partial m}\):

- Suppose agent chooses \(\theta\) such that \(f_x + \theta > 0\),
- then “mother nature” chooses \(m = m_L\).
- If \(m_L < r - \frac{1}{2}\sigma^2\), then agent can improve by lowering \(\theta\),
- until \(\theta = -f_x\).

- Similar argument for \(f_x + \theta < 0\), if \(m_H > r - \frac{1}{2}\sigma^2\).
Model Ambiguity & Hedging (3)

Conclusion: optimal choice for agent is $\theta^* = -f_x$.
- But this is delta-hedge for derivative $f$
- Leads to risk-neutral valuation!

How severe is restriction $m_L < r - \frac{1}{2}\sigma^2$? (Equivalent to $\mu_L < r$)
Good Deal Bound should be higher than Market Price of Risk

Thought-experiment:
- Suppose 25 years of data
- $\hat{\mu} = 8\%$, $\sigma = 15\%$
- Then std.err. of estimate for $\hat{\mu}$ is $\sigma/\sqrt{25} = 15\%/5 = 3\%$
- So, 95%-conf.intv. for $\hat{\mu}$ is $8\% \pm 6\%$.
- Need about $(2 \cdot 15/(8 - 4))^2 \approx 50$ years of data to distinguish between $8\%$ and $4\%$ if $\sigma = 15\%$!
Tree Setup for Incomplete Market

Remember we have a stock price $S$ with return process $x = \ln S$:

$$dx = m\, dt + \sigma\, dW_x,$$

Discretisation in binomial tree:

$$x(t + \Delta t) = x(t) + \begin{cases} \sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma}\sqrt{\Delta t}) \\ -\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma}\sqrt{\Delta t}) \end{cases}.$$

Model ambiguity as $m \in [m_L, m_H]$.

- Change mean $\iff$ change probability $\iff$ stoch. discount factor
- “Local Volatility” of stoch. discount factor: $m/\sigma\sqrt{\Delta t}$
- Conf.Intv. on mean $\iff$ Good Deal Bounds on discount factor vola
- Indistinguishable models $\iff$ Likelihood ratio test
Introduce additional non-traded process $y$:

$$dy = a \, dt + b \, dW_y,$$

with $dW_x \, dW_y = \rho \, dt$.

“Quadrinominal” discretisation:

<table>
<thead>
<tr>
<th>State: $x + \sigma \sqrt{\Delta t}$</th>
<th>$y + b\sqrt{\Delta t}$</th>
<th>$y - b\sqrt{\Delta t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + \sigma \sqrt{\Delta t}$</td>
<td>$p_{++} = \left(\frac{1+\rho + \left(\frac{m}{\sigma} + \frac{a}{b}\right)}{4}\right)\sqrt{\Delta t}$</td>
<td>$p_{+-} = \left(\frac{1-\rho + \left(\frac{m}{\sigma} - \frac{a}{b}\right)}{4}\right)\sqrt{\Delta t}$</td>
</tr>
<tr>
<td>$x - \sigma \sqrt{\Delta t}$</td>
<td>$p_{-+} = \left(\frac{1-\rho - \left(\frac{m}{\sigma} + \frac{a}{b}\right)}{4}\right)\sqrt{\Delta t}$</td>
<td>$p_{--} = \left(\frac{1+\rho - \left(\frac{m}{\sigma} - \frac{a}{b}\right)}{4}\right)\sqrt{\Delta t}$</td>
</tr>
</tbody>
</table>
Model Ambiguity

Uncertainty in both parameters $m$ and $a$.

Additional notation:

$$\mu := \begin{pmatrix} m \\ a \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma^2 & \rho \sigma b \\ \rho \sigma b & b^2 \end{pmatrix}.$$ 

Describe ambiguity set as ellipsoid:

$$\mathcal{K} := \{ \mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2 \}.$$ 

- Motivated by shape of confidence interval of estimator $\hat{\mu}$
- Motivated by Good Deal Bound
- Motivated by Likelihood Ratio Testing
Ellipsoid Ambiguity Set

![Graph showing ellipsoid ambiguity set with axes for drift of insurance process and return on financial market. The graph includes symbols for confidence intervals, mean, and ambiguity.]
Robust Optimisation Problem

Consider derivative $f$ with payoff $f(t + \Delta t, x(), y())$ at time $t + \Delta t$.
Consider hedged position: $f(t + \Delta t) + \theta(e^{x(t+\Delta t)-x(t)} - e^{r\Delta t})$

Ambiguity averse rational agent solves the following optimisation problem for a time-step $\Delta t$:

$$
\max \min_{\theta, \mu \in K} e^{-r\Delta t} \left( f_1 + (\nabla f' \mu + \theta(e_1' \mu - r + \frac{1}{2} \sigma^2) + \frac{1}{2} \text{tr}(f_{xx} \Sigma)) \Delta t \right),
$$

where $\nabla f$ denotes gradient $(f_x, f_y)'$ and $e_1$ denotes the vector $(1, 0)'$.

Reformulate & simplify problem

$$
\max \min_{\theta} \theta q + \varepsilon'(\nabla f + \theta e_1)
$$

s.t. \hspace{1cm} \varepsilon' \Sigma^{-1} \varepsilon \leq k^2.

with $q = (e_1' \mu_0 - r + \frac{1}{2} \sigma^2)$ is excess return
Optimal Response for Mother Nature

Two-player game: agent vs. "Mother Nature"

Worst-case choice for Mother Nature given any $\theta$ is "opposite direction" of vector $(\nabla f + \theta e_1)$:

$$\varepsilon^* := -\left(\frac{k}{\sqrt{\left(\nabla f + \theta e_1\right)'\Sigma\left(\nabla f + \theta e_1\right)}}\right)\Sigma\left(\nabla f + \theta e_1\right).$$

If we use this value for $\varepsilon^*$ we obtain the reduced optimisation problem for the agent:

$$\max_{\theta} \theta q - k \sqrt{\left(\nabla f + \theta e_1\right)'\Sigma\left(\nabla f + \theta e_1\right)}.$$ 

Maximise expected excess return $\theta q$ minus $k$ times st.dev. of portfolio. Similar to maximise w.r.t. Cost-of-Capital “penalty”.
Optimal Response for Agent

Solution to reduced optimisation problem for agent:

\[ \theta^* := - \left( f_x + \frac{b \rho}{\sigma} f_y \right) + \frac{q/\sigma}{\sqrt{k^2 - (q/\sigma)^2}} \frac{b \sqrt{1 - \rho^2}}{\sigma} |f_y|. \]

Nice economic interpretation:

- Left term is best possible hedge
  - Perfect hedge for “pure financial” risks
  - Induces market-consistent pricing
- Right term is “speculative” position, which is product of:
  - Function of Sharpe ratio \( q/\sigma \)
  - Residual unhedgeable risk
  - Absolute value of \( f_y \)
Agent’s Valuation of Contract

If we substitute optimal $\varepsilon^*$ and $\theta^*$ into original expectation, we obtain semi-linear pde

$$f_t + f_x(r - \frac{1}{2}\sigma^2) + f_y a^* + \frac{1}{2}\sigma^2 f_{xx} + \rho\sigma b f_{xy} + \frac{1}{2}b^2 f_{yy} - rf = 0,$$

where the drift term $a^*$ for the insurance process is given by

$$a^* = \left( a_0 - \frac{\rho b}{\sigma} q \right) \mp \left( \sqrt{k^2 - \left( \frac{q}{\sigma} \right)^2} \right) b \sqrt{1 - \rho^2},$$

where $\mp$ depends on sign of $f_y$.

Again, nice economic interpretation for $a^*$. Same result as *Good Deal Bound* pricing. Interpretation as *Cost-of-Capital* pricing from insurance industry.
Agent’s Valuation of Contract – Graphical
Different Interpretations for $k$

- Equivalence between Good Deal Bound & Model Ambiguity
- Parameter $k$ is:
  - Width of confidence interval for trend
  - Volatility of pricing kernel; stoch.disc.factor
  - Sharpe-ratio of risks
  - Cost-of-Capital times # of st.dev's for unhedgeable risk

Calculation of $k$:
- Sharpe-ratio for equity: $k > (8\% - 4\%)/16\% = 0.25$
- Conf.intv.: $k \approx 2/\sqrt{25} = 0.4$
- Cost-of-Cap: $k \approx 0.06 \times 2.5 = 0.15$: too low?

It seems reasonable to set $k \approx 0.3$. 
Applications

- Pricing Very Long-Dated Cash Flows
- Pricing Longevity Risk
Pricing Very Long-Dated Cash Flows

- Life Insurance and Pension cash flows extend to 70 years
- Market for Government Bonds extends to only 30 years
- Market for Discount Bonds incomplete beyond 30 years
- Use term-structure for interest rate up to 30 years
- After 30 years use “robust pricing”
- Example:
  - Assume 1-factor Hull-White model
  - $\sigma_{HW} = 0.01$, $a_{HW} = 0.05$, long term rate: $z_\infty = 4\%$
  - Take $k = 0.3$, then price at LT rate of $z_\infty - k/a \sigma_{HW} = -2\%$
  - Mean-rev determines transition between $z_{30}$ and $z_\infty - k/a \sigma_{HW}$
Extrapolation of Zero-curve

Extrapolated zero-curve

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Robust Pricing
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Life Expectancy (at birth) for Dutch Males 1950 until 2006
Conf.intv. for trend: [0.9, 2.8] months per annum.
Pricing Longevity Risk

- Best Estimate trend for increase in life expectancy is 1.8 months per annum
- Standard Deviation of process is 3.7 months per annum
- Robust Approach:
  - Price long life risk at trend of 3.7 months p/a
  - Price short life risk at trend of 0.0 months p/a
  - Combined portfolio: price at “net exposure”
Summary & Conclusion

- Robust agent holds hedge portfolio + speculative position
- Price contracts in incomplete markets in a “market-consistent” way
- Robust agent prices unhedgeable risks using a “worst case” drift

Connections to:

- Actuarial notion of *prudence*
- *Good Deal Bound* pricing (see Cochrane & Saa-Requejo)
- *Confidence Interval* for drift parameters
- *Likelihood Ratio Testing* of models (see Hansen & Sargent)
- *Cost-of-Capital* pricing used by industry (see QIS5)