Stochastic Made “Simple”

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General Insurance Reserving Seminar
Institute of Actuaries
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Aims and Objectives

Aims

- To explain core concepts
- To take away some of the mystique

Agenda

- Conceptual framework
- A basic example
- The over-dispersed Poisson model
- Mack’s model
- The 1 year view of reserve risk
- Conclusions
Stochastic claims reserving

- This has become a new academic discipline
- It has spawned several PhDs
- Numerous papers appearing in academic journals
- Presentations at every actuarial conference
- A book has appeared
- There is a Wikipedia page
Reserving Risk

- Reserving is concerned with forecasting outstanding liabilities
- There is uncertainty associated with any forecast
- Reserving risk attempts to capture that uncertainty
- We are interested in the predictive distribution of ultimate losses AND the associated cash flows
  - Don’t just focus on “Ultimates” or “Reserves”
  - We need distributions of cash flows for discounting and for capital models
- We need methods that can provide those distributions
- The methods are still evolving
Conceptual Framework

- **Reserve Estimate** (Measure of Location): Traditional deterministic methods.
- **Predictive Distribution**: Usually cannot be obtained analytically. Simulation methods required.
Approaches to stochastic reserving

“We can do this the easy way, or the hard way…”

- A lot of work in the academic literature has focused on specifying a model, then devising analytic formulae for the standard deviation of the forecast. This is the hard way.
  - It doesn’t get us very far. A standard deviation is useful, but the formulae are specific to the model. What if we want other models, other risk measures, or a full distribution?
- More recent work has focused on using simulation techniques (bootstrap or MCMC) to provide a full distribution of cash-flows (hence reserves). This is the “easy” way.
  - We still need to specify the model, and the analytic methods are useful for checking the results
  - There are still many practical difficulties and limitations
The “ultimo” view and the one-year view
An added complication

- A standard actuarial reserving analysis tries to find the expected outstanding liabilities, giving the expected ultimate cost of claims over the lifetime of the liabilities.

- The traditional actuarial approach to reserving risk is to look at the uncertainty in the outcomes over the lifetime of the liabilities (the “ultimo” perspective).

- Under Solvency II, a 1 year view is taken. We need a distribution of the expected outstanding liabilities after 1 year. This is a different view of reserving risk.

- Can the two views be reconciled in some way?
Basic Concepts

Uncertainty when Forecasting: Prediction errors and Predictive distributions
A Basic Example

- Suppose you are an established Private Motor insurer and have written the same number of policies for the last 11 years.
- You have had the following number of large claims:

![Bar Chart]

- How many large claims do you expect next year?
- What is the uncertainty in your estimate?
- What is the uncertainty in the outcome?
A Basic Example – A Solution

- Suppose that the large claims come from a Poisson distribution with mean $\lambda$.
- Can estimate the mean $\lambda$ from the observed large claims.
- What is the variance of a Poisson (6) distribution? $\text{Process Variance} = 6$
- How can we measure the uncertainty of using the sample mean?
  
  \[
  \text{Sample mean} = \frac{\sum x_i}{n}
  \]
  
  If $x_i$ are independent and identically distributed, then variance of the sample mean $\text{sample variance} / n$.

  \[
  \text{Estimation Variance} = 0.314
  \]

\[
\text{Prediction Variance} = \text{Process Variance} + \text{Estimation Variance}
\]
The Bootstrap

An alternative way of calculating the estimation variance is the **bootstrap**

Original Sample

Data must be independent and identically distributed

Produce a pseudo data sample by re-sampling with replacement

Calculate the parameters of interest

The SD gives a bootstrap estimate of the standard error of the parameters

Repeat many times, giving a distribution of the parameters
A Basic Example - Bootstrap

Observed data: 3 7 6 7 5 10 8 4 5 6 5 6

Pseudo data 1: 4 10 8 7 5 3 5

Pseudo data 2: 6 8 3 10 6 6 5 5 7 5 8

Pseudo data 10k: 8 5 7 8 4 5 6 3 10 5 5

Sample mean = 6.00
Sample variance = 0.313
Estimation Variance

If we sample from Poisson distributions with these means, we can derive the predictive distribution including process variance.
A Basic Example - Bootstrap

Observed data:

<table>
<thead>
<tr>
<th>3</th>
<th>7</th>
<th>6</th>
<th>7</th>
<th>5</th>
<th>10</th>
<th>8</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>5</th>
</tr>
</thead>
</table>

Mean = 6

Pseudo data 1:

<table>
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<th>7</th>
<th>7</th>
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<th>3</th>
<th>5</th>
<th>5</th>
<th>4</th>
<th>3</th>
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</table>

Mean = 5.5

Pseudo data 2:

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<th>5</th>
<th>5</th>
<th>7</th>
<th>5</th>
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</table>

Mean = 6.3

Pseudo data 10k:

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<th>3</th>
<th>10</th>
<th>5</th>
<th>5</th>
</tr>
</thead>
</table>

Mean = 6.0

Sample mean = 6.00

Sample variance = 6.313

Prediction Variance
Important Lessons

- We could calculate the SD of the forecast (“prediction error”) analytically, taking account of parameter uncertainty.
  - This is the HARD way.

- Bootstrapping gives a distribution of parameters, hence an estimate of the estimation error, without the hard maths

- When supplemented by a second simulation step incorporating the process error, a distribution of the forecast is generated
  - This is the EASY way
A More Complicated Example

Suppose now that the number of large claims had been:

i.e. The same number of large claims but in a different order

How many large claims do you expect next year?

What is the uncertainty in your estimate?

Residual for linear regression = Actual - Fitted
Regression-type problems and Bootstrapping

- Define and fit statistical model
- Obtain residuals and pseudo data
- Re-fit statistical model to pseudo data
- Obtain forecast, including process error

Any model that can be clearly defined can be bootstrapped
Bootstrapping a Linear Regression Problem

- The bootstrap process for the estimation variance is then:
  - Specify a model (e.g. linear regression)
  - Define the residuals
  - Re-sample the residuals with replacement
  - Rearrange the residual definition to create new ‘pseudo’ data
  - Refit the model on the ‘pseudo’ data
  - Project forward to get a mean claim amount for the next time period
  - The variance of the trended mean gives the estimation variance
- We can still keep the Poisson assumption for the process distribution, just with a trended mean
  - Simulate from a Poisson distribution, conditional on the simulated mean
  - The variance of the forecasts gives the prediction variance
- Note: We have used standard linear regression in this example for simplicity – ideally we would fit a Poisson GLM
Stochastic Reserving

Over-dispersed Poisson Model

Doing it the HARD way
Over-Dispersed Poisson Model

\[ C_{ij} = \text{Incremental claims in origin year } i \text{ and development year } j \]

\[ C_{ij} \sim ODP(\mu_{ij}, \phi_j) \]

\[ E[C_{ij}] = \mu_{ij} \]

\[ Var[C_{ij}] = \phi_j \mu_{ij} \]

Variance proportional to expected value
Example Predictor Structures

\[ \log(\mu_{ij}) = \eta_{ij} \]

Log “link” function

\[ \eta_{ij} = c + a_i + b_j \]

Chain Ladder

\[ \eta_i(t) = c + a_i + b.t + d \log(t) \]

Hoerl Curve

\[ \eta_i(t) = c + a_i + s_1(t) + s_2(\log(t)) \]

Smoother
Parameter estimation

- Write down joint density of the data given the parameters – the “Likelihood”
- Treat as a function of the parameters
- Maximise the (log) Likelihood with respect to the parameters
Variability in Claims Reserves

- Variability of a forecast
- Includes estimation variance and process variance

\[
\text{prediction error} = \left(\text{process variance} + \text{estimation variance}\right)^{\frac{1}{2}}
\]

- Problem reduces to estimating the two components
- This is difficult analytically, but possible (see, for example, E & V 2002)
- Note: “prediction error” is also known as “root mean square error of prediction” (RMSEP)
Stochastic Reserving

Over-dispersed Poisson Model

Doing it the EASY way
Stochastic Reserving: Bootstrapping

- Bootstrapping assumes the data are independent and identically distributed.
- With regression type problems, the data are often assumed to be independent but are not identically distributed (the means are different for each observation).
- However, the residuals are usually i.i.d, or can be made so.
- Therefore, with regression problems, it is common to bootstrap the residuals instead.
Reserving and Bootstrapping

Any model that can be clearly defined can be bootstrapped

- Define and fit statistical model
- Obtain residuals and pseudo data
- Re-fit statistical model to pseudo data
- Obtain forecast, including process error
Bootstrapping the Chain Ladder
Over-dispersed Poisson model

1. Fit chain ladder model
2. Obtain Pearson residuals $r_{ij} = \frac{C_{ij} - \mu_{ij}}{\sqrt{\phi_j \mu_{ij}}}$
3. Resample residuals
4. Obtain pseudo data, given $r_{ij}^*, \mu_{ij}$
   
   $C_{ij}^* = r_{ij}^* \sqrt{\phi_j \mu_{ij}} + \mu_{ij}$

5. Use chain ladder to re-fit model, and estimate future incremental payments
6. Simulate observation from process distribution assuming mean is incremental value obtained at Step 5
7. Repeat many times, storing the reserve estimates (this gives the predictive distribution)
8. Prediction error is then standard deviation of results

Note: Where curve fitting has been used for smoothing and extrapolation (for tail estimation), replace the chain ladder model in steps 1 and 5 by the actual model used
Excel Example
### Taylor & Ashe Data

**Observed incremental values**

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<tr>
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<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
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<td>766,940</td>
<td>610,542</td>
<td>482,940</td>
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| Dev Factors | 3.49061 | 1.74733 | 1.45741 | 1.17385 | 1.10382 | 1.08627 | 1.05387 | 1.07656 | 1.01772 | 1.00000 |
## Taylor & Ashe Data
### Fitted incremental values (chain ladder model)

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<td>94,634</td>
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<td>375,833</td>
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<td>469,511</td>
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<td>566,731</td>
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| **Total** | 18,680,856 |
## Taylor & Ashe Data

**Scaled residuals : ODP with constant scale parameter**

<table>
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<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</table>

| Scale^0.5 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 | 229.3 |
Taylor & Ashe Data
Scaled residuals: ODP with constant scale parameter

Note that the volatility is lower at the earlier and later development periods.
## Taylor & Ashe Data

**Scaled residuals : ODP with non-constant scale parameter**

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<th>10</th>
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<tr>
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<td>-0.513</td>
<td>-0.576</td>
<td>0.762</td>
<td></td>
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</tr>
<tr>
<td>7</td>
<td>0.559</td>
<td>-0.913</td>
<td>0.706</td>
<td>-0.090</td>
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<tr>
<td>8</td>
<td>-1.149</td>
<td>-0.702</td>
<td>1.288</td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>9</td>
<td>-0.159</td>
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<tr>
<td>10</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Scale^0.5 | 139.9 | 142.3 | 153.0 | 318.1 | 282.6 | 386.6 | 296.7 | 83.9  | 99.6  | 83.9  |
Taylor & Ashe Data
Scaled residuals: ODP with non-constant scale parameter

Note that the residuals are standardised better when using non-constant scale parameters
## Taylor & Ashe Data
### ODP: Constant vs Non-Constant Scale Parameters

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Constant Scale</th>
<th>Non-Constant Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Prediction Error</td>
<td>Prediction Error %</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.0%</td>
</tr>
<tr>
<td>2</td>
<td>112,552</td>
<td>119.0%</td>
</tr>
<tr>
<td>3</td>
<td>217,547</td>
<td>46.2%</td>
</tr>
<tr>
<td>4</td>
<td>262,934</td>
<td>36.9%</td>
</tr>
<tr>
<td>5</td>
<td>306,595</td>
<td>31.0%</td>
</tr>
<tr>
<td>6</td>
<td>375,745</td>
<td>26.4%</td>
</tr>
<tr>
<td>7</td>
<td>500,332</td>
<td>22.9%</td>
</tr>
<tr>
<td>8</td>
<td>791,481</td>
<td>20.1%</td>
</tr>
<tr>
<td>9</td>
<td>1,060,473</td>
<td>24.7%</td>
</tr>
<tr>
<td>10</td>
<td>2,025,898</td>
<td>43.3%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>2,992,296</strong></td>
<td><strong>15.9%</strong></td>
</tr>
</tbody>
</table>
Over-dispersed Poisson model

- Note the possibility of obtaining negative pseudo incremental values when using non-parametric bootstrapping (resampling residuals), which could in turn lead to negative pseudo cumulative values.
- This is a known issue with non-parametric bootstrapping. For example:

“Although the [non-parametric] bootstrap/simulation procedure provides prediction errors that are consistent with their analytic counterparts, the predictive distribution produced in this way might have some undesirable properties. For example, for some origin year reserves, the minimum values of the predictive distribution could be negative.”

“If [non-parametric bootstrapping] is not without its difficulties, for example: a small number of sets of pseudo data may be incompatible with the underlying model…”

- If \( r_{ij} = \frac{C_{ij} - \mu_{ij}}{\sqrt{\phi_j \mu_{ij}}} \) then \( C_{ij}^* = r_{ij}^* \sqrt{\phi_j \mu_{ij}} + \mu_{ij} \)

\( C_{ij}^* < 0 \) if \( r_{ij}^* < -\sqrt{\frac{\mu_{ij}}{\phi_j}} \)

- \( C = \) incremental amounts
- \( \mu = \) expected incremental amounts
- \( \phi = \) scale parameter
- \( r = \) scaled Pearson residual

This issue disappears with parametric bootstrapping

ODP Bootstrapping – Practical Issues

- Most suitable for paid amounts
- Can result in negative incrementals/reserves in some simulations when the pseudo data are generated by re-sampling residuals and inverting
  - If this is a problem, use parametric bootstrapping instead
- E&V (1999) only considered using a constant dispersion parameter: in general it is better to use non-constant scale parameters
- Choice of process distribution?
  - Ideally we want an “over-dispersed Poisson” distribution
  - In practice just use a proxy (eg Gamma) with the same mean and variance properties
ODP Model – Characteristics

- It is a model of *incremental* amounts
- It is not suitable when development factors are less than 1
- When forecasting, by using a distribution that only allows positive values (e.g., Gamma or Lognormal), forecast incremental values will be positive
  - That is, simulated cumulative amounts will be strictly increasing
  - Simulated reserves can never be negative
  - The ultimate claims will be at least as big as the observed cumulative paid for each origin period
- (Although note comments on parametric vs non-parametric bootstrapping above)
Stochastic Reserving

Mack’s Model

Doing it the HARD way
Mack’s Model


\[ D_{ij} = \text{Cumulative claims in origin year } i \text{ and development year } j \]

Specified mean and variance only:

\[ E(D_{ij}) = \lambda_j D_{i, j-1} \]

\[ V(D_{ij}) = \sigma_j^2 D_{i, j-1} \]

- Expected value proportional to previous cumulative
- Variance proportional to previous cumulative
Mack’s Model

\[ \hat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} w_{ij} f_{ij}}{\sum_{i=1}^{n-j+1} w_{ij}} \]

Estimator for lambda

\[ \hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} w_{ij} \left( f_{ij} - \hat{\lambda}_j \right)^2 \]

Estimator for sigma squared

\[ w_{ij} = D_{i,j-1} \text{ and } f_{ij} = \frac{D_{ij}}{D_{i,j-1}} \]
Variability in Claims Reserves

- Variability of a forecast
- Includes estimation variance and process variance

\[
\text{prediction error} = (\text{process variance} + \text{estimation variance})^{\frac{1}{2}}
\]

- Problem reduces to estimating the two components. For example, for the reserves in origin year \(i\):

\[
RMSEP\left[\hat{R}_i\right] \approx \sqrt{\hat{D}_{in}^2 \sum_{k=n-i+1}^{n-1} \frac{\sigma_{k+1}^2}{\hat{\lambda}_{k+1}^2}} \left(\frac{1}{\hat{D}_{ik}} + \frac{1}{\sum_{q=1}^{n-k} D_{qk}}\right)
\]
Mack’s Model as a GLM

Step 1: Reformulate Mack’s model as a model of the ratios

Step 2: Recognise that Mack’s “Scale” parameters are derived from the squared residuals of a weighted normal regression model

\[
E\left(\frac{D_{ij}}{D_{i,j-1}}\right) = E\left(f_{ij}\right) = \lambda_j
\]

\[
V\left(\frac{D_{ij}}{D_{i,j-1}}\right) = V\left(f_{ij}\right) = \frac{\sigma_j^2}{w_{ij}}
\]

\[
w_{ij} = D_{i,j-1} \text{ and } f_{ij} = \frac{D_{ij}}{D_{i,j-1}}
\]

\[D_{ij} = \text{Cumulative claims in origin year } i \text{ and development year } j\]
Mack’s Model as a GLM

\[ f_{ij} \sim N \left( \lambda_j, \frac{\sigma_j^2}{w_{ij}} \right) \]

Weighted normal errors GLM with weights \( w \)

\[ r_{ij} = \sqrt{w_{ij}} \left( f_{ij} - \hat{\lambda}_j \right) \]

Pearson residual (unscaled)

\[ \hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} w_{ij} \left( f_{ij} - \hat{\lambda}_j \right)^2 \]

Note: Mack’s model was not derived as a weighted normal GLM, but a weighted normal GLM gives the same estimator of sigma
Stochastic Reserving

Mack’s Model

Doing it the EASY way
Reserving and Bootstrapping

- Define and fit statistical model
- Obtain residuals and pseudo data
- Re-fit statistical model to pseudo data
- Obtain forecast, including process error

Any model that can be clearly defined can be bootstrapped
1. Fit chain ladder model to the observed link ratios

2. Obtain (scaled) Pearson residuals

$$r_{ij} = \frac{w_{ij} (f_{ij} - \lambda_j)}{\sigma_j}$$

3. Resample residuals

4. Obtain pseudo data, given $r_{ij}^*$, $\lambda_j$

$$f_{ij}^* = \frac{r_{ij}^* \sigma_j}{\sqrt{w_{ij}}} + \lambda_j$$

5. Use chain ladder model to re-estimate the development factors (as a weighted average of the pseudo-link ratios, using the original weights $w$)
6. Given the observed cumulative payments to date, move 1 period ahead by multiplying the previous cumulative claims by the appropriate simulated development factor obtained at Step 5
   • Include the process error by sampling a single observation from the underlying process distribution

7. Move to the next period, where the forecast cumulative amounts are now conditional on the simulated 1 period ahead forecast obtained at Step 6 (including the process error)

8. Repeat many times, storing the reserve estimates (this gives the predictive distribution)

9. Prediction error is then standard deviation of results

Note: Where curve fitting has been used for smoothing and extrapolation (for tail estimation), replace the chain ladder model in steps 1 and 5 by the actual model used
Excel Example
## Taylor & Ashe Data

**Scaled residuals: Mack’s model**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.519</td>
<td>-1.117</td>
<td>-1.152</td>
<td>0.772</td>
<td>1.490</td>
<td>-0.850</td>
<td>-1.189</td>
<td>-0.759</td>
<td>0.000</td>
</tr>
<tr>
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<td>0.030</td>
<td>0.047</td>
<td>0.632</td>
<td>-0.608</td>
<td>-0.322</td>
<td>0.939</td>
<td>0.346</td>
<td>0.651</td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td>0.006</td>
<td>0.850</td>
<td>-1.141</td>
<td>0.758</td>
<td>0.682</td>
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<td>1.840</td>
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<td>-0.282</td>
<td>-0.907</td>
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</tr>
<tr>
<td>5</td>
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<td>0.689</td>
<td>-0.683</td>
<td>0.005</td>
<td>0.543</td>
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<td></td>
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<tr>
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<td>-0.636</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>9</td>
<td>0.197</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that the $\alpha$ parameters decrease rapidly.

| Mack's alpha | 400.4 | 194.3 | 204.9 | 123.2 | 117.2 | 90.5 | 21.1 | 33.9 | 21.1 |

This parameter is highly influential on the overall variability.
Note that the $\alpha$ parameters decrease rapidly.
## Taylor & Ashe Data

### Bootstrapping Mack’s Model

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Prediction Error</th>
<th>Prediction Error %</th>
<th>Simulated Prediction Error</th>
<th>Simulated Prediction Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.0%</td>
<td>0</td>
<td>0.0%</td>
</tr>
<tr>
<td>2</td>
<td>75,535</td>
<td>79.8%</td>
<td>75,001</td>
<td>78.4%</td>
</tr>
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<td>121,699</td>
<td>25.9%</td>
<td>121,578</td>
<td>26.0%</td>
</tr>
<tr>
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<td>133,549</td>
<td>18.8%</td>
<td>132,939</td>
<td>18.7%</td>
</tr>
<tr>
<td>5</td>
<td>261,406</td>
<td>26.5%</td>
<td>261,911</td>
<td>26.5%</td>
</tr>
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<td>414,910</td>
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<td>7</td>
<td>558,317</td>
<td>25.6%</td>
<td>558,639</td>
<td>25.7%</td>
</tr>
<tr>
<td>8</td>
<td>875,328</td>
<td>22.3%</td>
<td>880,184</td>
<td>22.4%</td>
</tr>
<tr>
<td>9</td>
<td>971,258</td>
<td>22.7%</td>
<td>979,052</td>
<td>22.8%</td>
</tr>
<tr>
<td>10</td>
<td>1,363,155</td>
<td>29.5%</td>
<td>1,368,720</td>
<td>29.4%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>2,447,095</strong></td>
<td><strong>13.1%</strong></td>
<td><strong>2,454,616</strong></td>
<td><strong>13.1%</strong></td>
</tr>
</tbody>
</table>
Practical Issues

- Choice of process distribution?
  - Normal: theoretically correct, but allows negative cumulative amounts
  - Gamma: pragmatic alternative

- When used with incurred data:
  - Provides distribution of ultimate claims
  - Provides distribution of IBNR+IBNER (not outstanding claims)
  - Also provides a distribution of Ultimates
    - By subtracting the observed paid amounts gives a distribution of outstanding amounts
  - Requires (simulated) paid to incurred ratios if paid cash-flows are required
Bootstrapping Mack’s Model
Characteristics

- It is a model of the *cumulative* amounts
- It will work with negative incremental observed claims where development factors are less than 1
- Although it is possible to force simulated cumulative amounts to be positive, there is nothing to stop a simulated cumulative amount being less than the previous amount. That is, negative incremental amounts are always possible.
- This may be beneficial with incurred data, but possibly a disadvantage with paid data
- Note: bootstrapping provides predictive distributions for Mack’s model (including cash-flows)
Stochastic Reserving

Lognormal Models
Lognormal Models

- It is also possible to fit linear regression models to the log of the incremental claims (log-normal models)
- Again, the prediction error can be calculated the hard way (analytically) or the easy way (using bootstrap or MCMC methods)
- To bootstrap the lognormal models, simply follow the steps outlined above for reserving and bootstrapping (see E&V 2006)
Stochastic Reserving

The one-yr view of risk
A Projected Balance Sheet View

When projecting Balance Sheets for solvency, we have an opening balance sheet with *expected* outstanding liabilities.

We then project one year forwards, simulating the payments that emerge in the year.

We then require a closing balance sheet, with (simulated) *expected* outstanding liabilities conditional on the payments in the year.

In a multi-year model, the closing balance sheet after one year becomes the opening balance sheet in the second year, and so on.
For Solvency II, a 1 year perspective is taken, requiring a distribution of the expected value of the liabilities after 1 year, for the 1 year ahead balance sheet in internal capital models.

If the standard formula is used, a 1 year-ahead “reserve risk” standard deviation % is required. This could be:

- The standard parameter for the line-of-business
- An undertaking specific parameter

The 1 year-ahead “reserve risk” standard deviation is the SD of the distribution of profit/loss on reserves after 1 year.
The one-year run-off result (undiscounted)
(the view of profit or loss on reserves after one year)

For a particular origin year, let:

The opening reserve estimate be \( R_0 \)

The reserve estimate after one year be \( R_1 \)

The payments in the year be \( C_1 \)

The run-off result (claims development result) be \( CDR_1 \)

Then

\[
CDR_1 = R_0 - C_1 - R_1 = U_0 - U_1
\]

Where the opening estimate of ultimate claims and the estimate of the ultimate after one year are \( U_0, U_1 \)
The one-year run-off result
(the view of profit or loss on reserves after one year)

Merz & Wuthrich (2008) derived analytic formulae for the standard deviation of the claims development result after one year assuming:

- The opening reserves were set using the pure chain ladder model (no tail)
- Claims develop in the year according to the assumptions underlying Mack’s model
- Reserves are set after one year using the pure chain ladder model (no tail)
- The mathematics is quite challenging. This is the HARD way

The M&W method is gaining popularity, but has limitations. What if:

- We need a tail factor to extrapolate into the future?
- Mack’s model is not used – other assumptions are used instead?
- We want another risk measure (say, VaR @ 99.5%)?
- We want a distribution of the CDR (not just a standard deviation)?
<table>
<thead>
<tr>
<th>Accident Year</th>
<th>12m</th>
<th>24m</th>
<th>36m</th>
<th>48m</th>
<th>60m</th>
<th>72m</th>
<th>84m</th>
<th>96m</th>
<th>108m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,202,584</td>
<td>3,210,449</td>
<td>3,468,122</td>
<td>3,545,070</td>
<td>3,621,627</td>
<td>3,644,636</td>
<td>3,669,012</td>
<td>3,674,511</td>
<td>3,678,633</td>
</tr>
<tr>
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<td>2,171,487</td>
<td>3,165,274</td>
<td>3,395,841</td>
<td>3,466,453</td>
<td>3,515,703</td>
<td>3,548,422</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2,140,328</td>
<td>3,157,079</td>
<td>3,399,262</td>
<td>3,500,520</td>
<td>3,585,812</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2,290,664</td>
<td>3,338,197</td>
<td>3,550,332</td>
<td>3,641,036</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2,148,216</td>
<td>3,219,775</td>
<td>3,428,335</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2,143,728</td>
<td>3,158,581</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2,144,738</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Expressed as a percentage of the opening reserves, this forms a basis of the reserve risk parameter under Solvency II (QIS 5 Technical Specification)
QIS 5: Undertaking Specific Parameters

For the reserving risk parameter:

Method 2: \[ \sigma_{U,lob} = \frac{\sqrt{MSEP_{lob}}}{PCO_{lob}} \]

Method 3: \[ \sigma_{U,lob} = \frac{\sqrt{MSEP_{lob}}}{CLPCO_{lob}} \]

There is also a credibility weighting between this and the standard parameter:

\[ \sigma_{res,lob} = c \cdot \sigma_{U,res,lob} + (1 - c) \cdot \sigma_{M,res,lob} \]

*MSEP* is from the Merz-Wuthrich formulae. Clearly there are some inconsistencies here:

- *PCO* is discounted, but *MSEP* is calculated using undiscounted amounts
- *MSEP* is only valid under the chain-ladder model and Mack’s assumptions
- What if other assumptions are used?
The one-year run-off result in a simulation model
The EASY way

For a particular origin year, let:

The opening reserve estimate be \( R_0 \)

The expected reserve estimate after one year be \( R_1^{(i)} \)

The payments in the year be \( C_1^{(i)} \)

The run-off result (claims development result) be \( CDR_1^{(i)} \)

Then

\[
CDR_1^{(i)} = R_0 - C_1^{(i)} - R_1^{(i)} = U_0 - U_1^{(i)}
\]

Where the opening estimate of ultimate claims and the expected ultimate after one year are \( U_0, U_1^{(i)} \)

for each simulation \( i \)
The one-year run-off result in a simulation model
The EASY way

1. Given the opening reserve triangle, simulate all future claim payments to ultimate using bootstrap (or Bayesian MCMC) techniques.

2. Now forget that we have already simulated what the future holds.

3. Move one year ahead. Augment the opening reserve triangle by one diagonal, that is, by the simulated payments from step 1 in the next calendar year only. An actuary only sees what emerges in the year.

4. For each simulation, estimate the outstanding liabilities, conditional only on what has emerged to date. (The future is still “unknown”).

5. A reserving methodology is required for each simulation – an “actuary-in-the-box” is required*. We call this re-reserving.

6. For a one-year model, this will underestimate the true volatility at the end of that year (even if the mean across all simulations is correct).

* The term “actuary-in-the-box” was coined by Esbjörn Ohlsson
The standard actuarial perspective: forecasting outcomes over the lifetime of the liabilities, to their ultimate position.

A single accident year, 4 years developed.

“Actual” simulated future amounts.
"Actual" simulated future amounts

Expected payments conditional on year 1 position
EMB ResQ Example
ResQ Example
Bootstrap Results Summary – “Ultimo” perspective
### ResQ Example

#### 1 Year ahead – Simulation 1

![ResQ Example Simulation 1](image)

<table>
<thead>
<tr>
<th>Year</th>
<th>12m</th>
<th>24m</th>
<th>36m</th>
<th>48m</th>
<th>60m</th>
<th>72m</th>
<th>84m</th>
<th>96m</th>
<th>108m</th>
<th>120m</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3,157,076</td>
<td>3,399,252</td>
<td>3,500,520</td>
<td>3,585,812</td>
<td>3,599,248</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2002</td>
<td>2,148,216</td>
<td>3,219,775</td>
<td>3,428,335</td>
<td>3,496,277</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2003</td>
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<td>3,156,531</td>
<td>3,394,672</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>2,144,738</td>
<td>3,221,989</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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</table>
ResQ Example
1 Year ahead – Simulation 2

<table>
<thead>
<tr>
<th>Year</th>
<th>12m</th>
<th>24m</th>
<th>36m</th>
<th>48m</th>
<th>60m</th>
<th>72m</th>
<th>84m</th>
<th>96m</th>
<th>108m</th>
<th>120m</th>
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<tbody>
<tr>
<td>1999</td>
<td>2,171,487</td>
<td>3,165,274</td>
<td>3,385,841</td>
<td>3,466,455</td>
<td>3,515,703</td>
<td>3,546,422</td>
<td>3,571,793</td>
<td>3,571,793</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>2,140,328</td>
<td>3,157,079</td>
<td>3,399,252</td>
<td>3,500,520</td>
<td>3,585,812</td>
<td>3,610,563</td>
<td>3,571,793</td>
<td>3,571,793</td>
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<tr>
<td>2002</td>
<td>2,148,216</td>
<td>3,219,775</td>
<td>3,428,335</td>
<td>3,484,910</td>
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<td>3,156,381</td>
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<tr>
<td>2004</td>
<td>2,144,738</td>
<td>3,232,164</td>
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</table>
ResQ Example
Bootstrap Run-off Results Summary – 1 year perspective

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Avg Latest Cumulative Amount</th>
<th>Avg Closing Expected Reserve</th>
<th>StDev Closing Expected Reserve</th>
<th>StDev %</th>
<th>Avg Closing Expected Ultimate</th>
<th>Avg Opening Expected Reserve</th>
<th>Expected Run-Off Result</th>
<th>StDev Run-Off Result</th>
<th>Expected Run-Off Result</th>
<th>StDev Run-Off Result Ratio</th>
<th>Expected Payment</th>
<th>Avg Opening Expected Ultimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996</td>
<td>3,785,633</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3,785,633</td>
</tr>
<tr>
<td>1997</td>
<td>3,906,604</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3,906,604</td>
</tr>
<tr>
<td>1998</td>
<td>3,903,790</td>
<td>4,380</td>
<td>233</td>
<td>6.6%</td>
<td>3,903,170</td>
<td>9,245</td>
<td>0</td>
<td>598</td>
<td>12.0%</td>
<td>4,370</td>
<td>3,903,170</td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>3,560,257</td>
<td>6,594</td>
<td>527</td>
<td>6.1%</td>
<td>3,575,811</td>
<td>20,389</td>
<td>0</td>
<td>3,516</td>
<td>13.0%</td>
<td>19,835</td>
<td>3,575,811</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>3,668,419</td>
<td>28,864</td>
<td>1,081</td>
<td>3.7%</td>
<td>3,637,284</td>
<td>51,472</td>
<td>0</td>
<td>9,243</td>
<td>18.3%</td>
<td>22,607</td>
<td>3,637,284</td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td>3,699,870</td>
<td>53,127</td>
<td>2,272</td>
<td>4.2%</td>
<td>3,752,997</td>
<td>111,951</td>
<td>0</td>
<td>28,428</td>
<td>25.9%</td>
<td>58,834</td>
<td>3,752,997</td>
<td></td>
</tr>
<tr>
<td>2002</td>
<td>3,507,728</td>
<td>107,777</td>
<td>5,081</td>
<td>4.7%</td>
<td>3,615,505</td>
<td>187,170</td>
<td>0</td>
<td>20,286</td>
<td>11.2%</td>
<td>79,393</td>
<td>3,615,505</td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>3,385,653</td>
<td>184,615</td>
<td>5,884</td>
<td>3.9%</td>
<td>3,570,268</td>
<td>411,687</td>
<td>0</td>
<td>28,110</td>
<td>6.6%</td>
<td>227,072</td>
<td>3,570,268</td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>3,165,496</td>
<td>412,685</td>
<td>9,109</td>
<td>2.3%</td>
<td>3,576,181</td>
<td>1,453,443</td>
<td>0</td>
<td>53,406</td>
<td>3.7%</td>
<td>1,020,750</td>
<td>3,576,181</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>32,424,651</strong></td>
<td><strong>800,002</strong></td>
<td><strong>19,608</strong></td>
<td><strong>2.45%</strong></td>
<td><strong>33,224,653</strong></td>
<td><strong>2,237,846</strong></td>
<td><strong>0</strong></td>
<td><strong>81,226</strong></td>
<td><strong>3.63%</strong></td>
<td><strong>1,437,841</strong></td>
<td><strong>33,224,653</strong></td>
<td></td>
</tr>
</tbody>
</table>
ResQ Example
99.5th percentile of the Bootstrap Run-off Result

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
<td>0.50%</td>
<td>0.28%</td>
<td>0.43%</td>
<td>0.31%</td>
<td>0.58%</td>
<td>0.58%</td>
<td>0.69%</td>
<td>0.61%</td>
<td>0.49%</td>
<td>0.49%</td>
</tr>
</tbody>
</table>

- Minimum: 0
- 0.1%: 0
- 0.2%: 0
- 0.3%: 0
- 0.4%: 0
- 0.5%: 0
- 0.6%: 0
- 0.7%: 0
- 0.8%: 0
- 0.9%: 0
- 1.0%: 0
- 1.1%: 0

Var @ 99.5% = - (0.5th percentile) = 208,912
<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Analytic Prediction Errors</th>
<th>Simulated Prediction Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 Year Ahead CDR</td>
<td>Mack Ultimate</td>
</tr>
<tr>
<td>0</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>1</td>
<td>567 567 568 568</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1,488 1,566 1,486 1,564</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3,923 4,157 3,916 4,147</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9,723 10,536 9,745 10,569</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>28,443 30,319 28,428 30,296</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>20,954 35,967 20,986 35,951</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>28,119 45,090 28,110 44,996</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>53,320 69,552 53,406 69,713</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>81,080 108,401</td>
<td>81,226 108,992</td>
</tr>
</tbody>
</table>
The input to a Bootstrap Run-off Result can be another Bootstrap Run-off Result.
The input to a Bootstrap Run-off Result can be another Bootstrap Run-off Result. This can be used to give the CDR between the 1st and 2nd years ahead, and so on.
### ResQ Example

#### Cascading Bootstrap Run-off Results

![ResQ Example](image)

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Avg Latest Cumulative Amount</th>
<th>Avg Closing Expected Reserve</th>
<th>StDev Closing Expected Reserve</th>
<th>StDev %</th>
<th>Avg Closing Expected Ultimate</th>
<th>Avg Opening Expected Reserve</th>
<th>Expected Run-Off Result</th>
<th>StDev Run-Off Result</th>
<th>StDev Run-Off Result Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996</td>
<td>5,678,633</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
<td>3,678,633</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
</tr>
<tr>
<td>1997</td>
<td>3,906,804</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
<td>3,906,804</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
</tr>
<tr>
<td>1998</td>
<td>3,903,170</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
<td>3,903,170</td>
<td>4,380</td>
<td>0</td>
<td>467</td>
<td>11.12%</td>
</tr>
<tr>
<td>1999</td>
<td>3,572,805</td>
<td>4,006</td>
<td>308</td>
<td>7.70%</td>
<td>3,576,811</td>
<td>6,554</td>
<td>0</td>
<td>1,366</td>
<td>15.27%</td>
</tr>
<tr>
<td>2000</td>
<td>3,628,590</td>
<td>8,694</td>
<td>640</td>
<td>7.36%</td>
<td>3,637,284</td>
<td>28,864</td>
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<td>29,807</td>
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<td>4.75%</td>
<td>3,752,997</td>
<td>53,127</td>
<td>0</td>
<td>9,579</td>
<td>18.22%</td>
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<tr>
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<td>3,564,374</td>
<td>51,131</td>
<td>2,822</td>
<td>5.52%</td>
<td>3,615,505</td>
<td>107,777</td>
<td>0</td>
<td>27,438</td>
<td>25.46%</td>
</tr>
<tr>
<td>2003</td>
<td>3,464,036</td>
<td>106,232</td>
<td>6,566</td>
<td>6.18%</td>
<td>3,570,268</td>
<td>184,615</td>
<td>0</td>
<td>20,404</td>
<td>11.05%</td>
</tr>
<tr>
<td>2004</td>
<td>3,393,018</td>
<td>185,163</td>
<td>8,104</td>
<td>4.38%</td>
<td>3,578,181</td>
<td>412,688</td>
<td>0</td>
<td>27,798</td>
<td>6.74%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>32,839,619</strong></td>
<td><strong>385,034</strong></td>
<td><strong>16,351</strong></td>
<td><strong>4.25%</strong></td>
<td><strong>33,224,653</strong></td>
<td><strong>800,002</strong></td>
<td><strong>0</strong></td>
<td><strong>52,344</strong></td>
<td><strong>6.54%</strong></td>
</tr>
</tbody>
</table>
Creating cascading CDRs over all years gives the following results:

The sum of the variances of the repeated 1 yr ahead CDRs (over all years) equals the variance over the lifetime of the liabilities

- Under Mack’s assumptions/chain ladder, this can be proved
- (The simulation based results are approximate due to numerical error)

This means that we expect the risk under the 1 year view to be lower than the standard “ultimo” perspective

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Number of years ahead</th>
<th>1 Yr</th>
<th>2 Yrs</th>
<th>3 Yrs</th>
<th>4 Yrs</th>
<th>5 Yrs</th>
<th>6 Yrs</th>
<th>7 Yrs</th>
<th>8 Yrs</th>
<th>Sqrt(Sum of Squares)</th>
<th>Mack Ultimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>568</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>568</td>
<td>568</td>
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<td>3</td>
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<td>487</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1,564</td>
<td>1,564</td>
</tr>
<tr>
<td>4</td>
<td>3,916</td>
<td>1,306</td>
<td>431</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4,151</td>
<td>4,147</td>
</tr>
<tr>
<td>5</td>
<td>9,745</td>
<td>3,837</td>
<td>1,277</td>
<td>425</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10,560</td>
<td>10,569</td>
</tr>
<tr>
<td>6</td>
<td>28,428</td>
<td>9,679</td>
<td>3,824</td>
<td>1,272</td>
<td>425</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30,303</td>
<td>30,296</td>
</tr>
<tr>
<td>7</td>
<td>20,986</td>
<td>27,438</td>
<td>9,343</td>
<td>3,693</td>
<td>1,226</td>
<td>409</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>35,998</td>
<td>35,951</td>
</tr>
<tr>
<td>8</td>
<td>28,110</td>
<td>20,404</td>
<td>26,922</td>
<td>9,162</td>
<td>3,613</td>
<td>1,208</td>
<td>402</td>
<td>0</td>
<td>0</td>
<td>45,055</td>
<td>44,996</td>
</tr>
<tr>
<td>9</td>
<td>53,406</td>
<td>27,798</td>
<td>20,236</td>
<td>26,687</td>
<td>9,111</td>
<td>3,600</td>
<td>1,203</td>
<td>402</td>
<td>0</td>
<td>69,600</td>
<td>69,713</td>
</tr>
<tr>
<td>Total</td>
<td>81,226</td>
<td>52,344</td>
<td>38,513</td>
<td>29,010</td>
<td>10,120</td>
<td>3,879</td>
<td>1,285</td>
<td>402</td>
<td>0</td>
<td>108,543</td>
<td>108,992</td>
</tr>
</tbody>
</table>

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Re-reserving in Simulation-based Capital Models

The advantage of investigating the claims development result (using re-reserving) in a simulation environment is that the procedure can be generalised:

- Not just the chain ladder model
- Not just Mack’s assumptions
- Can include curve fitting and extrapolation for tail estimation
- Can incorporate a Bornhuetter-Ferguson step
- Can be extended beyond the 1 year horizon to look at multi-year forecasts
- Provides a *distribution* of the CDR, not just a standard deviation
- Can be used to help calibrate Solvency II internal models
Conclusions

- Stochastic reserving has a reputation for being difficult
  - Attempted analytically, it can be, and the limitations of the formulae should be recognised
- Simulation techniques can simplify the modelling enormously, giving results that are analogous to the analytic results (when applied correctly), as well as providing additional information and allowing the models to be generalised
- An understanding of both the analytic and simulation based approaches can be obtained by following the key principles in each case
- The characteristics of the models and the effect of their key parameters should be understood. This will help with interpretation of outputs, especially when things go wrong.
- A reconciliation between the 1 year view and the “ultimo” view can be obtained by understanding the differences between the perspectives.
“Modern computer simulation techniques open up a wide field of practical applications for risk theory concepts, without the restrictive assumptions, and sophisticated mathematics, of many traditional aspects of risk theory”.

- Daykin, Pentikainen and Pesonen, 1996

“I believe that stochastic modelling is fundamental to our profession. How else can we seriously advise our clients and our wider public on the consequences of managing uncertainty in the different areas in which we work?”

- Chris Daykin, 1995
Recent developments in stochastic reserving

- Wuthrich & Merz (book & papers)
- Esbjorn Ohlsson (one yr view)
- Dorothea Diers (one yr view)
- Tom Wright (LMAG presentation one yr view)
  - They [M&W] use a completely different approach to the one described here, but the two methods give exactly the same variance for the one-year CDR
- Schnieper/Liu (splitting IBNR into new reported claims and IBNER)
- Quarg & Mack/Liu (combining information in paid and incurred)
- Stochastic BF – Mack, Verrall, Alai, Wuthrich
- Greg Taylor et al (individual claims)
- Magda Schiegl (3D)
- Verrall & Brydon (calendar year trends)
- Jens Perch-Nielsen (calendar year trends)
- Susanna Bjorkwall (parametric bootstrap, separation technique)
- Pavel Shevchenko (Bayesian)
- Fabrizio Restione (Bayesian)
- Anders Jessen (Bayesian - incorporating claim numbers)
- Piet de Jong (estimating correlations for multiple triangles)
- Murphy and McLennan (projecting individual open large claims and netting down)
- + many others
References


CP71 and CP75 (2009). http://www.ceiops.eu/content/view/14/18/
Peter graduated from City University with a BSc and PhD in Actuarial Science. After completing his PhD, entitled “Statistical Modelling of Excess Mortality of Medically Impaired Insured Lives”, Peter worked as a medical statistician at the London School of Hygiene and Tropical Medicine, conducting research into risk factors associated with Sudden Infant Death Syndrome (SIDS/Cot Death) and lecturing to post-graduate students in Medical Statistics.

Peter then returned to actuarial work, within the Group Non-Life Technical Department at Commercial Union (now Aviva), supporting the Executive Directors in worldwide reserve monitoring, business plan monitoring, and outwards catastrophe reinsurance modelling, amongst other activities.

Peter then moved to Lloyd’s as “Manager, Capital Modelling” in the Market Risk Unit, where he was jointly responsible for the Risk Based Capital system used for setting member capital requirements at Lloyd’s.

After working at Lloyd’s, Peter joined EMB in November 1999, specialising in research and statistical modelling, particularly financial risk modelling using simulation techniques. His main areas of work are:

• Risk based capital modelling
• Reserve variability methodologies
• Liability model parameterisation, including parameter uncertainty
• Catastrophe risk aggregation and reinsurance modelling
• Asbestos liability modelling
• Pricing using simulation techniques
• Generalised linear and non-linear modelling techniques

Peter is also involved in the development of EMB software, staff and client training, and is a regular speaker at seminars. He is a Chartered Statistician, a Senior Visiting Fellow at the Cass Business School, London, and is the author (or co-author) of numerous papers, including the prize-winning Institute of Actuaries paper "Stochastic Claims Reserving in General Insurance".