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Calibrating and Communicating Dependence
Open Forum, Staple Inn

Paul Sweeting and Andrew Smith

Workshop Overview

- Design of a typical internal model
  - Four things called correlation matrices and why they are different
  - Elliptical copula
- Communication and Visualisation
  - The different ways of assessing extreme co-movement
  - How relationships between many data series can be displayed graphically
  - Transparency: understanding the link between inputs and outputs
- Quantitative calibration tools
  - How to build calibration tools using Excel
  - Empirical dependency patterns in real financial data
  - Measuring t-copula degrees of freedom from the data
  - Asymmetry and arachnitude

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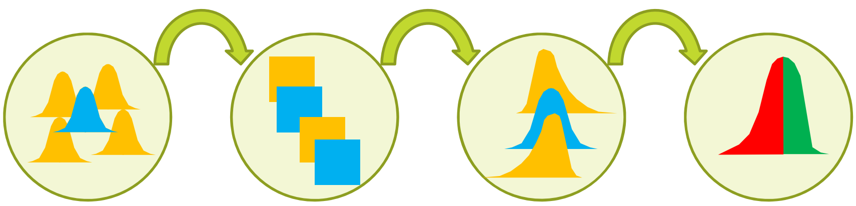
Design of a Typical Internal Model

Context: Internal Model Calculation Kernel



Copula component matrix (100 000 × 50)

Uniform component matrix: (100 000 × 50)

Risk driver matrix (100 000 × 50)

Net asset matrix (100 000 × 10)
P&L attribution cube (100 000 × 10 × 50)

Count	Low	Medium	High
Simulations	10 000	100 000	1 000 000
Risk drivers	10	50	500
Business units	1	10	200

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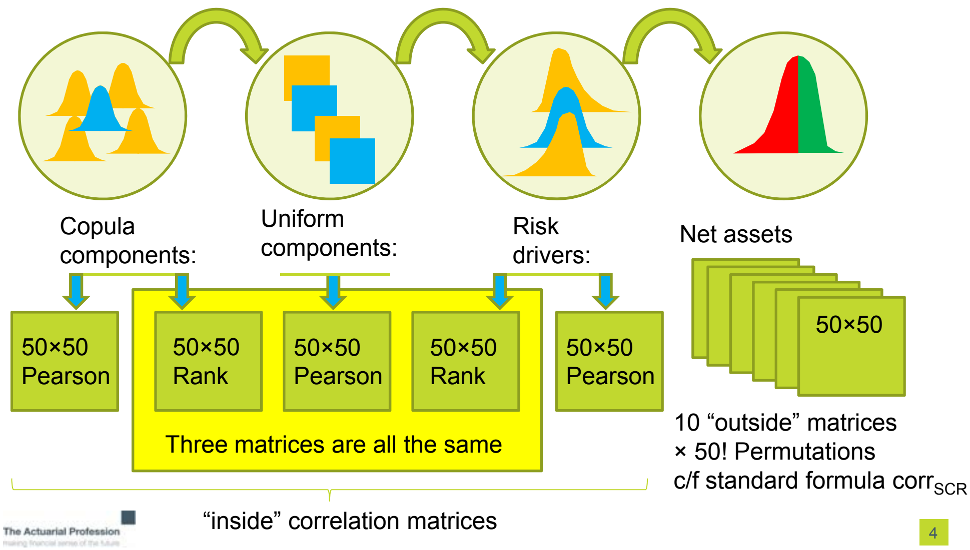
Counting Correlation Matrices



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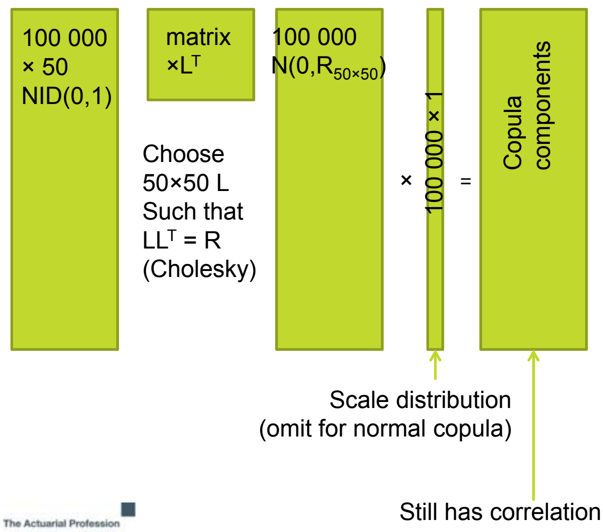
Elliptically Contoured Copula



Scale Dist squared	Copula Dist
1	Normal
Γ^{-1}	Student T
Inverse Gauss	NIG
Kolmogorov-Smirnov	Logistic
Γ	Variance Gamma

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Calibration Questions

- Given historic observation of risk drivers
  - We can estimate marginal distributions
  - And so the create a historic distribution of the uniform components
- Remaining questions:
  - How to estimate the copula cdf F_c
  - And the copula correlation matrix R_c
  - How to test for elliptically contoured copulas
  - And what to do when ellipticity is rejected?

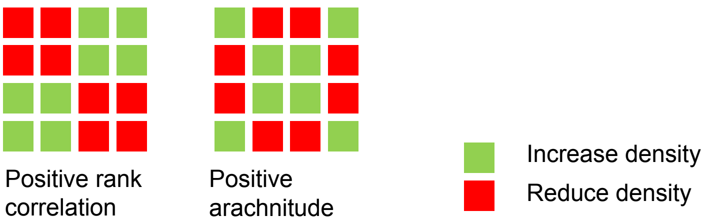
Communication and Visualisation

Tail correlation

- Broadly speaking, calculate correlation in a tail of the data
- All correlation types can be used
  - Linear only valid for elliptical data
  - Rank versions probably more robust
- Multi-dimensional versions do exist…
  - Collapsing the relationship between more than to variables into a single statistic
- …but often more than one version

Tail correlation – limitations

- Tail correlation looks – by definition – at the tail
  - What if the extreme values lie off the main diagonal?
  - What if there is "arachnitude"?



- Subjectivity also required
  - How is the tail defined?

Defining the Copula Function

- Consider the cumulative distribution function of uniform components
- Start by looking at a pair of uniform components
- Joint density c(u,v) = **Prob**{U≤u,V≤v}
- Equivalently, joint CDF C(u,v)

$$C(u,v) = \int_0^u \int_0^v c(x,y)\,dy\,dx$$

- This is called the *copula function*

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Tail dependence

- Broadly speaking, consider
  - The proportion of observations in a tail…
  - …relative to the maximum proportion possible
- Again several versions
- Most popular is the coefficient of tail dependence
- C(u,u)/u, as u tends to zero (for lower measure)
- Main issue: always zero for Gaussian copulas
- Transforms exist…
- …but with similar issues

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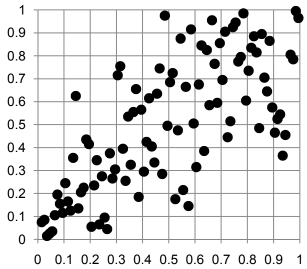
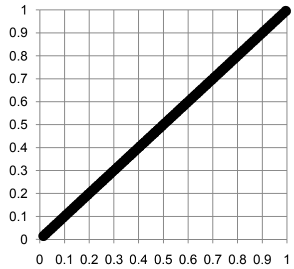
Tail dependence

- Broadly speaking, consider
  - The proportion of observations in a tail…
  - …relative to the maximum proportion possible…
  - …defined as C(u,u)/u





Tail dependence

- Again several versions
- Most popular is the coefficient of tail dependence
- C(u,u)/u, as u tends to zero (for lower measure)
- Main issue: always zero for Gaussian copulas
- Transforms exist…
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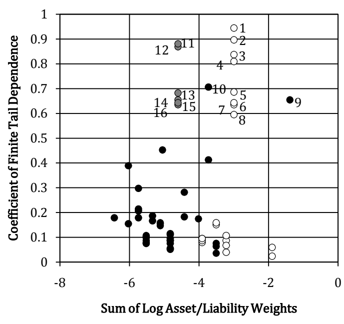
Tail dependence – alternative

- Why not use C(u,u)/u…
- …but use a finite value for u?
- This introduces subjectivity…
- …but can at least be used for all copulas
- Why not also consider for higher dimensions…
- …e.g. C(u,u,u)/u?

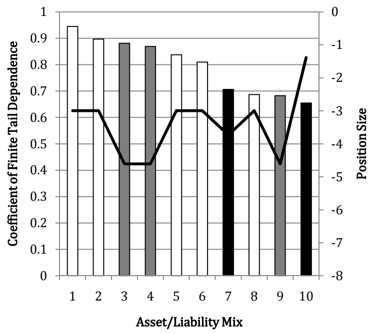
Scatter Plot



1. Real Liabilities/All IL Gilts
2. Nominal Liabilities/All Gilts
3. Real Liabilities/+15 Year IL Gilts
4. Nominal Liabilities/+15 Year Gilts
5. Real Liabilities/All Gilts
6. Real Liabilities/+15 Year Gilts
7. Nominal Liabilities/All IL Gilts
8. Nominal Liabilities/+15 Year IL Gilts
9. Real Liabilities/Nominal Liabilities
10. UK Equities/Europe ex UK Equities
11. All IL Gilts/ +15 Year IL Gilts
12. All Gilts/+15 Year Gilts
13. All IL Gilts/ All Gilts
14. +15 Year IL Gilts/All Gilts
15. All IL Gilts/+15 Year Gilts
16. +15 Year IL Gilts/+15 Year Gilts

- Black = two risk assets
- White = one asset, one liability
- Grey = two matching assets

Bar Chart



1. Real Liabilities/All IL Gilts
2. Nominal Liabilities/All Gilts
3. All IL Gilts/+15 Year IL Gilts
4. All Gilts/+15 Year Gilts
5. Real Liabilities/+15 Year IL Gilts
6. Nominal Liabilities/+15 Year Gilts
7. Europe ex UK Equities/UK Equities
8. Real Liabilities/All Gilts
9. All IL Gilts/All Gilts
10. Real Liabilities/Nominal Liabilities

- Black = two risk assets
- White = one asset, one liability
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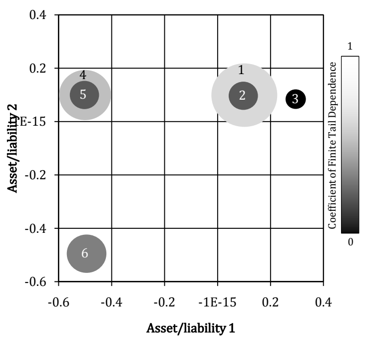
Balloon Plot



1. All IL Gilts/+15 Year IL Gilts
2. All IL Gilts/All
3. Europe ex UK Equities/UK Equities
4. Nominal Liabilities/+15 Year Gilts
5. Nominal Liabilities/+15 Year Gilts
6. Real Liabilities/Nominal Liabilities

- Black = low risk
- White = high risk

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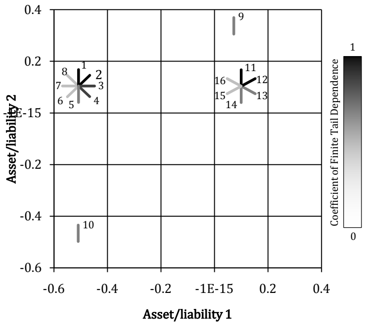
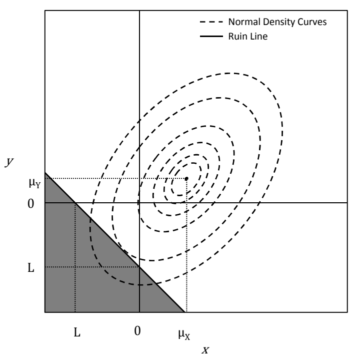
Sunflower Plot

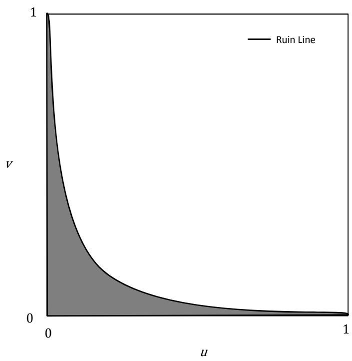


1. Real Liabilities/All IL Gilts
2. Nominal Liabilities/All Gilts
3. Real Liabilities/+15 Year IL Gilts
4. Nominal Liabilities/+15 Year Gilts
5. Real Liabilities/All Gilts
6. Real Liabilities/+15 Year Gilts
7. Nominal Liabilities/All IL Gilts
8. Nominal Liabilities/+15 Year IL Gilts
9. Real Liabilities/Nominal Liabilities
10. UK Equities/Europe ex UK Equities
11. All IL Gilts/ +15 Year IL Gilts
12. All Gilts/+15 Year Gilts
13. All IL Gilts/ All Gilts
14. +15 Year IL Gilts/All Gilts
15. All IL Gilts/+15 Year Gilts
16. +15 Year IL Gilts/+15 Year Gilts

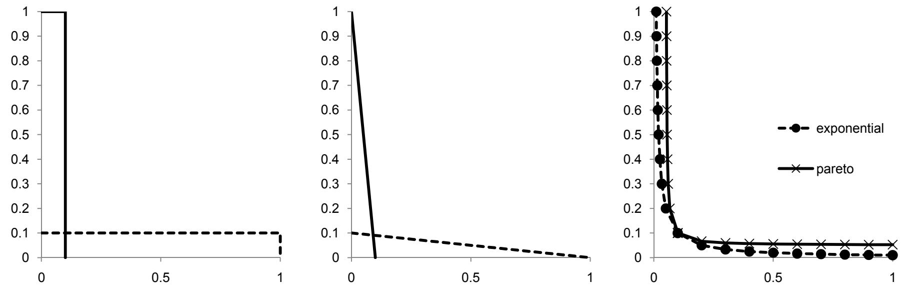
- White = low risk
- Black = high risk

Ruin lines – in absolute and copula space





Approximating loss functions with copulas



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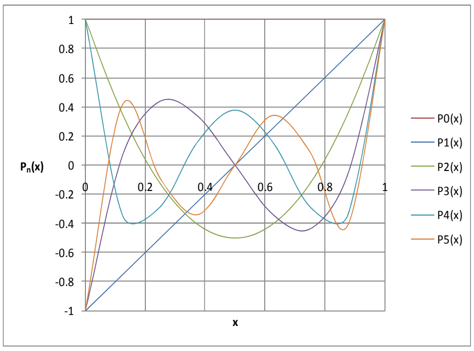
Quantitative Calibration Tools

What are (shifted) Legendre Polynomials?

n	n^{th} Legendre Polynomial $P_n(u)$
0	$P_0(u) = 1$
1	$P_1(u) = 2u-1$
2	$P_2(u) = 6u^2-6u+1$
3	$P_3(u) = 20u^3-30u^2+12u-1$
4	$P_4(u) =70u^4-140u^3+90u^2-20u+1$
5	$P_5(u) =252u^5-630u^4+560u^3 -210u^2+30u-1$



Recurrence: $(n+1)P_{n+1}(u) = (2n+1)(2u-1)P_n(u) - nP_{n-1}(u)$

Orthogonality: $\int_0^1 P_i(u)P_j(u)du = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases}$

Integration: $\int_0^u P_n(\xi)d\xi = \frac{P_{n+1}(u) - P_{n-1}(u)}{2(2n+1)}$

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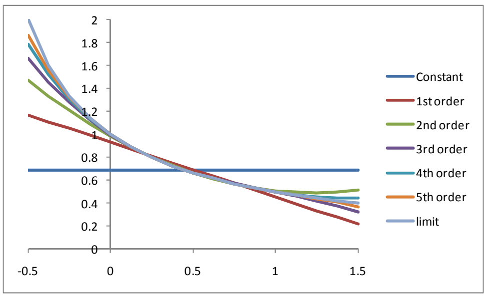
Approximating 1/(1+u) with Legendre Polynomials

- Guess an expansion of the form $\frac{1}{1+u} = \sum_{i=0}^{\infty} a_i P_i(u); 0 \le u \le 1$
- Pick out a_j using orthogonality $\int_0^1 \frac{P_j(u)}{1+u}du = \sum_{i=0}^{\infty}\left\{a_i \int_0^1 P_i(u)P_j(u)du\right\} = \frac{a_j}{2j+1}$
- Rapid convergence:



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Approximating Copula Density with Legendre Polynomials

- Guess an expansion and use orthogonality to estimate terms:

$$c(u,v)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}P_i(u)P_j(v)$$

$$\begin{aligned}a_{ij}&=(2i+1)(2j+1)\int_0^1\int_0^1 c(u,v)P_i(u)P_j(v)\,dv\,du\\&=(2i+1)(2j+1)\mathbf{E}\{P_i(U)P_j(V)\}\end{aligned}$$

- Uniform marginals imply $a_{01}=a_{02}=a_{03}\ldots=a_{10}=a_{20}=a_{30}\ldots=0$
- Integrate term by term to get copula function:

$$C(u,v)=\int_0^u\int_0^v c(x,y)\,dy\,dx=uv+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\frac{[P_{i+1}(u)-P_{i-1}(u)][P_{j+1}(v)-P_{j-1}(v)]}{4(2i+1)(2j+1)}$$

- Extensions to 3, 4 dimensions

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UK and Denmark Monthly Equity Returns 1970-2009

Source: Datastream

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Rank Correlations and Arachnitude

- Suppose we have 2 dimensions, N = 480 observations
- In each dimension, replace n^{th} smallest by $u = n/(N+1)$

Correlation matrix	$P_1(U_{dk})$	$P_1(U_{uk})$	$P_2(U_{dk})$	$P_2(U_{uk})$
$P_1(U_{dk})$	1	$3 \times a_{11}$ = Rank correlation	0	sqr(15)$\times a_{12}$ Cross correl
$P_1(U_{uk})$	$3 \times a_{11}$=Rank correlation	1	sqr(15)$\times a_{21}$ Cross correl	0
$P_2(U_{dk})$	0	sqr(15)$\times a_{21}$ Cross correl	1	$5 \times a_{22}$ = Arachnitude
$P_2(U_{uk})$	sqr(15)$\times a_{12}$ Cross correl	0	$5 \times a_{22}$ = Arachnitude	1

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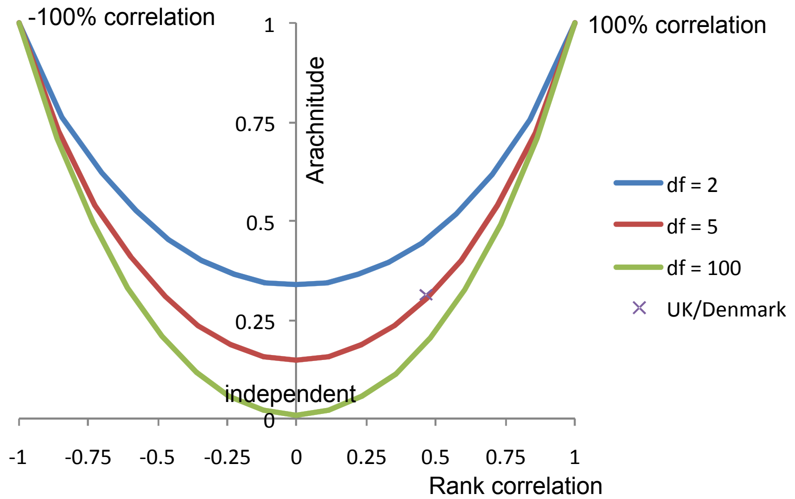
Rank Correlation and Arachnitude
T copula calibration for UK and Denmark



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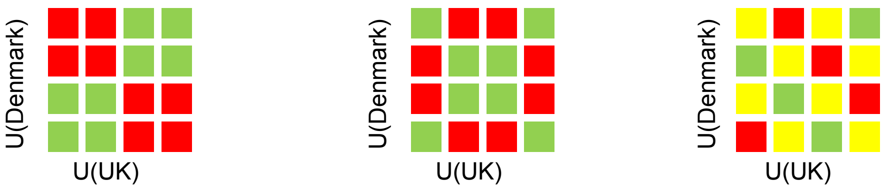
Cross Correlations: Denmark & UK

Correlation matrix	$P_1(U_{dk})$	$P_1(U_{uk})$	$P_2(U_{dk})$	$P_2(U_{uk})$
$P_1(U_{dk})$	1	46.2%	0	-12.0%
$P_1(U_{uk})$	46.2%	1	-12.5%	0
$P_2(U_{dk})$	0	-12.5%	1	31.4%
$P_2(U_{uk})$	-12.0%	0	31.4%	1

Empirical estimates for UK/Danish equity returns

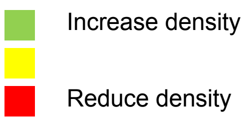
Interpreting Correlations and Arachnitude



Positive rank correlation

Positive arachnitude

Positive cross correlation



Notes:
Extreme case of squared rank correlation= 1 is attained for spider copula (mixture of increasing and decreasing copulas)
Individuated T copula (and so gauss and T copula) imply cross correlations are zero.

Purpose of Multiple Comparisons

- We have observed specific features of the UK / Danish data set
  - Are these real features of the underlying distribution or sampling artefact?
  - Difficult to analyse mathematically because of the need to start with a hypothesis about the “true” copula
- An alternative is to test consistency across multiple economies in the search for “stylised facts”.
  - We could also test robustness across different time periods
  - Recognise that the economies are not independent, so feature seen in all economies could still be statistical fluke

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Our Chosen Data Set

- MSCI equity indices
- 31/12/1969 – 31/12/2009
- Monthly total return indices, coverage for 480 months
- In US Dollars
- 18 series representing different countries

  Countries represented: Australia, Austria, Belgium, Canada, Denmark, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Singapore, Spain, Sweden, Switzerland, UK, US

- In this presentation we analyse only two-dimensional dependency. There are 153 pairs of countries for which this can be analysed. In the charts that follow, each country pair is represented by one point.

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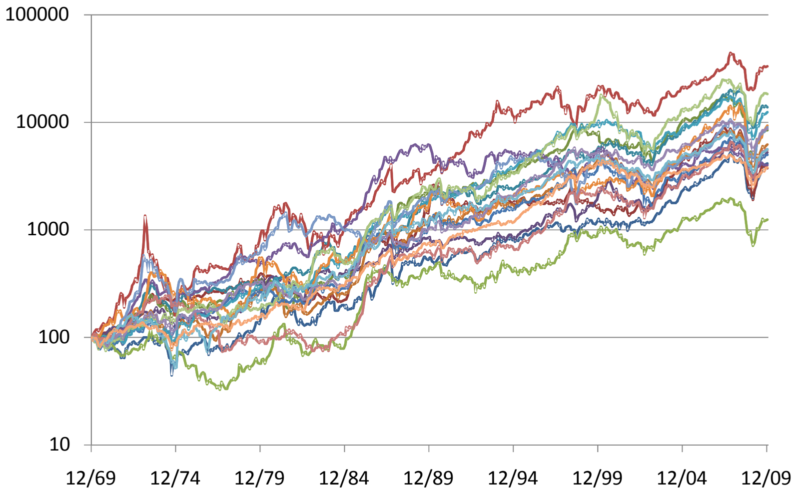
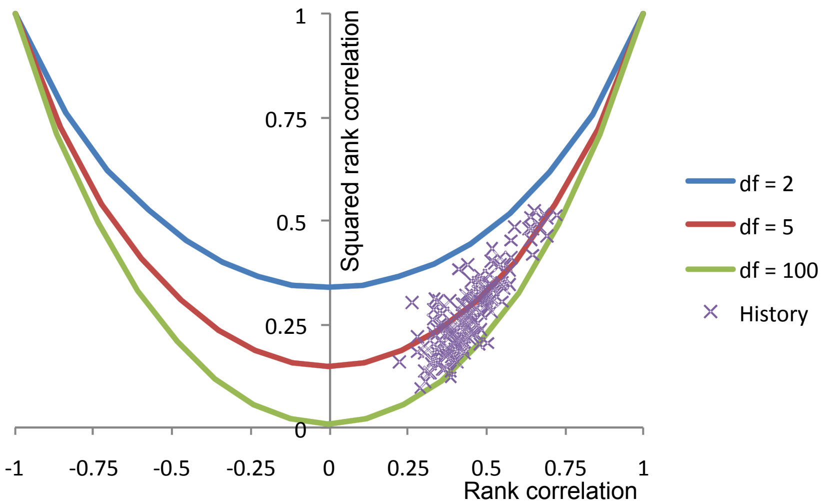
Equity Total Return Data 1970-2010

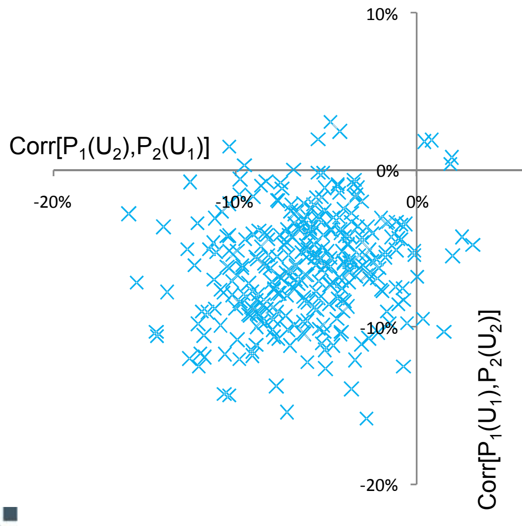


Source: Datastream

Fitting a T Copula: 5 df is typical Significant Rejection of Gauss Copula



Negative Cross Correlations: Systematic Feature Rejection of T copula (standard or individuated)



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