Agenda

1. What is LPI?
2. The inflation options market
3. The inflation volatility smile
4. Modelling the inflation volatility smile
5. SABR
6. A simple LPI model
7. Conclusion and discussion
Types of LPI

LPI, Limited Price Indexation, is the RPI link in pensions

Following Wilkie(1988), we define various types of LPI indexation:

- Type 1: \( LPI_t = RPI_t \)
- Type 2: \( LPI_t = \max[RPI_0 \times (1+\text{floor})^t, RPI_t] \)
- Type 3: \( LPI_t = \max[RPI_0, RPI_1, \ldots, RPI_t] \)
- Type 4: \( LPI_t = LPI_{t-1} \times \min[\max[1+\text{floor}, RPI_t/RPI_{t-1}], 1+\text{cap}] \)
- Type 5: \( LPI_t = LPI_{t-1} \times [1+\text{participation} \times (RPI_t/RPI_{t-1}-1)] \)

For 0% floors, Type 1 < Type 2 < Type 3 < Type 4

Illustration of types of LPI

<table>
<thead>
<tr>
<th>Year</th>
<th>RPI(_t)</th>
<th>RPI(_{t-1})</th>
<th>0% floor</th>
<th>Type 2 floor at 0%</th>
<th>Type 3 floor at 0%</th>
<th>Type 4 floor at 0%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Type 1)</td>
<td>y/y% index</td>
<td></td>
<td>pension increase</td>
<td>pension increase</td>
<td>pension increase</td>
</tr>
<tr>
<td>0</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>1</td>
<td>101.00</td>
<td>1.0%</td>
<td>100.00</td>
<td>101.00 1.0%</td>
<td>101.00 1.0%</td>
<td>101.00 1.0%</td>
</tr>
<tr>
<td>2</td>
<td>103.53</td>
<td>2.5%</td>
<td>100.00</td>
<td>103.53 2.5%</td>
<td>103.53 2.5%</td>
<td>103.53 2.5%</td>
</tr>
<tr>
<td>3</td>
<td>99.90</td>
<td>-3.5%</td>
<td>100.00</td>
<td>100.00 -3.4%</td>
<td>103.53 0.0%</td>
<td>103.53 0.0%</td>
</tr>
<tr>
<td>4</td>
<td>98.90</td>
<td>-1.0%</td>
<td>100.00</td>
<td>100.00 0.0%</td>
<td>103.53 0.0%</td>
<td>103.53 0.0%</td>
</tr>
<tr>
<td>5</td>
<td>104.84</td>
<td>6.0%</td>
<td>100.00</td>
<td>104.84 4.8%</td>
<td>104.84 1.3%</td>
<td>109.74 6.0%</td>
</tr>
</tbody>
</table>
The UK inflation options market

- The building blocks of the RPI derivatives market are zero coupon inflation swaps. These are a hedge for type-1 LPI.
- Three forms of “vanilla” RPI inflation options trade:
  1. **Year-on-year (y/y) RPI caps and floors.** The cap has $T$ caplets with payoffs $\max[0, (\text{RPI}_t/\text{RPI}_{t-1} - 1) - K\%]$ at times $t=1,2,\ldots,T$.
  2. **RPI index caps and floors** with a single payoff at maturity. The cap payoff is $\max[0, \text{RPI}_T/\text{RPI}_0 - (1+K\%)^T]$. Hedge for LPI type-2 liabilities.
  3. **LPI swaps** hedge LPI type-4 liabilities. The most common collar strikes traded are $[0\%,5\%]$, $[0\%,3\%]$ and $[0\%,\infty]$ (i.e. no cap).

The UK inflation options market

- Participants
- Maturities
- Liquidity
- Equilibrium
- Execution costs
- Price transparency
The implied inflation volatility smile

- Inflation options trade on price, not volatility
- Prices can be inverted to implied volatilities using the conventional options pricing formulae for each market
- For RPI index options, it is natural to use the Black Scholes (lognormal) model to price since the index is always positive and is expected to grow exponentially:

\[
C = SN(d_1) - Ke^{-(r-t)}N(d_2)
\]

\[
P = Ke^{-(r-t)} - S + C(S, t)
\]

\[
d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]
Implied RPI index volatility smile

Type-2 LPI liabilities can be priced directly from the implied volatility given the strike and maturity

![Graph showing implied index volatility smiles for different maturities, with strike rates from 0% to 7.5% and volatility rates from 0% to 17.5%.

The inflation implied volatility smile: y/y options

- For RPI y/y options, the y/y rate may be negative so the Black Scholes lognormal model is not appropriate.
- The market convention is to assume the underlying y/y rate has a normal distribution and use the Bachelier(1900) model. The resulting vol $\sigma$ is called the normal vol or basis point vol:

\[
C = e^{-r(T-t)}[(F - K)N(d_1) + \frac{\sigma \sqrt{T-t}}{\sqrt{2\pi}} e^{-d_1^2/2}]
\]

\[
P = e^{-r(T-t)}[K - F]N(-d_1) + \frac{\sigma \sqrt{T-t}}{\sqrt{2\pi}} e^{-d_1^2/2}
\]

where

\[
d_1 = \frac{F - K}{\sigma \sqrt{T-t}}
\]

See VBA code in the spreadsheet on the conference website.
The RPI y/y inflation options market

RPI y/y atm inflation vol > GBP LIBOR atm caplet floorlet vol out to 5 years

Source: RBS, 6 June 2010

The RPI y/y inflation options market

RPI y/y inflation vol smile compared with LIBOR cap floor vol smile

Source: Bloomberg composite 6 June 2010
Modelling the implied volatility smile

- Historically, the implied volatility smile first appeared in equity option markets
- Techniques developed for equities have subsequently been used in interest rate and inflation markets
  - Alternate stochastic process
  - Local volatility models
  - Stochastic volatility models
  - Lévy processes

Volatility smiles - alternate stochastic process

- The presence of a smile may interpreted as a deviation from lognormality in the dynamics of the underlying
- Popular processes include the CEV (constant elasticity of variance) process
  \[ dS = \mu dt + \sigma S^\beta dW \quad \text{where} \quad 0 \leq \beta \leq 1 \]
- And the shifted-lognormal (displaced-diffusion) process
  \[ d(S + \alpha) = \mu dt + (S + \alpha)\sigma dW \]
- Both processes allow a monotonically downward sloping skew, however alternate techniques are needed to introduce curvature.
Volatility smiles – local volatility models

- Volatility is a function of the underlying itself, hence
  \[ dS = \mu S dt + \sigma(S, t) dW \]
- First introduced by Dupire (1994) and Derman and Kani (1994).

Stochastic volatility models

- The underlying and its volatility are driven by correlated Brownian motions
- A well known stochastic volatility model is due to Heston (1993)
  \[ \begin{align*}
  dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW^1_t \\
  dv_t &= \kappa [\theta - v_t] dt + \sigma \sqrt{v_t} dW^2_t \\
  \end{align*} \]
  where \( dW^1_t dW^2_t = \rho dt \)
- Semi-analytical option prices may be found via Fourier transform techniques as
  \[ C(S, v, t) = SP_1 - KP(t, T)P_2 \]
  where \( P(t, T) \) is the \( T \)-maturity discount bond and \( P_1 \) and \( P_2 \) are the associated probabilities evaluated via numerical integration.
Lévy processes

- General class of processes with independent and stationary increments
- This includes familiar processes such as Brownian motion and Poisson process which introduces random jumps into the dynamics
- All other Lévy processes are generalisations of a Brownian motion and a possibly infinite number of Poisson processes.

Lévy processes that have become popular in finance include:
- **Jump-diffusion** – dynamics are driven by a diffusion and a finite number of Poisson processes,
- **Variance-gamma** – may be interpreted as a Brownian motion evaluated at a time given by a gamma process, see Madan et. al. (1990,1991,1998),
- **Normal-Inverse-Gaussian** – may be interpreted as a Brownian motion evaluated at a time given by an Inverse-Gaussian process, see Barndorff-Nielsen (1997,1998).
SABR (Stochastic Alpha Beta Rho) model

- This stochastic volatility model is the market standard for fitting an implied volatility smile,

\[ dF_t = \alpha F_t^\beta dW_t^1, \quad F_0 = f \]
\[ d\alpha_t = \nu \alpha_t dW_t^2, \quad \alpha_0 = \alpha \]

where \( dW_t^1 dW_t^2 = \rho dt \) and \( \nu, \beta \) are constants such that \( 0 \leq \beta \leq 1 \) and \( \nu \geq 0 \).

- It is not a true stochastic volatility model since a separate set of parameters \( \nu, \beta \) and \( \rho \) are associated with each maturity.

- The great advantage is that the equivalent Black-Scholes (lognormal) implied volatility may be approximated analytically by \( \sigma_{\text{B}}(K,f) \) defined as follows:

\[
\sigma_{\text{B}}(K,f) = \frac{\alpha}{(fK)^{\gamma/2}} \left\{ 1 + \frac{\nu^2}{24} \log^2 \frac{f}{K} + \frac{6 \nu^4}{1920} \log^4 \frac{f}{K} + \cdots \right\} \left( \frac{z}{\chi(z)} \right)
\]

\[
\times \left\{ 1 + \left[ \frac{\omega^2}{24} \left( \frac{fK}{\nu} \right)^{2} + \frac{1}{4} \frac{\rho \nu \alpha}{(fK)^{\gamma/2}} + \frac{2 - 3 \rho^2}{24} \nu^2 \right] T + \cdots \right\}
\]

where \( \omega = 1 - \beta \) and

\[
z = \frac{\nu}{\alpha} (fK)^{\gamma/2} \log \frac{f}{K} \quad \chi(z) = \log \left\{ \frac{\sqrt{1 - 2 \rho z + z^2} + z - \rho}{1 - \rho} \right\}
\]
SABR Normal model

- For options on the RPI y/y rate, we use a normal model of the underlying and the normal option pricing formula of Bachelier. Hence:

\[ dF_t = \alpha_t dW_t^1, \quad F_0 = f \]
\[ d\alpha_t = \nu \alpha_t dW_t^2, \quad \alpha_0 = \alpha \]

where \( dW_t^1, dW_t^2 = \rho dt \) and \( \nu \) is constant such that \( \nu \geq 0 \).

- The equivalent normal implied volatility may be approximated analytically as

\[ \sigma_N(K, f) = \alpha \left( \frac{z}{\chi(z)} \right) \cdot \left\{ 1 + \frac{2 - 3 \rho^2}{24} \nu^2 T + \cdots \right\} \]

where

\[ z = \frac{\nu}{\alpha}(f - K), \quad \chi(z) = \log \left\{ \frac{1 - 2 \rho z + z^2 + z - \rho}{1 - \rho} \right\} \]

Weaknesses of the SABR model

- These analytic approximations are derived via singular perturbation techniques relying on a ‘small volatility’ expansion, hence assuming both volatility \( \alpha \) and volatility-of-volatility \( \nu \) are small.

- For extreme parameter values and strikes far away-from-the-money, these approximations break down.

- This is well known by the market and each market participant has their own set of proprietary ‘fixes’.

- The breakdown of these approximations is most clearly visible if one examines the probability density function derived from call spreads using implied volatilities, which becomes negative at extreme parameter values and away-from-the-money.
SABR model: negative probabilities

SABR model implied density for $F=3.63\%$, $\alpha=1.25\%$, $\beta=50\%$, $\rho=15\%$, $\nu=22\%$

Simple LPI model: RPI y/y history and forward rates

historical rate $\leftarrow$ market forward rate

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SABR normal volatility smiles: effect of parameters

- **Base**: atmvol=0.60%, ρ=0%, ν=40%
- **Higher atmvol**: atmvol=0.80%, ρ=0%, ν=40%
- **Negative skew**: atmvol=0.60%, ρ=-40%, ν=40%
- **Less smile**: atmvol=0.80%, ρ=0%, ν=20%

**Simple LPI model**

- **Type 4**: LPI\(_t\) = LPI\(_{t-1}\) \* min[max[1+floor\(_t\), RPI\(_t\)/RPI\(_{t-1}\)], 1+cap\(_t\)]
- **A 30y LPI swap has 60 RPI y/y options embedded in the swap**:
  
  LPI\(_{30}\) = LPI\(_0\) \* \( \frac{RPI\(_1\)}{RPI\(_0\)} - 1 + \text{floor}\(_1\) - \text{cap}\(_1\) \)
  \* \( \frac{RPI\(_2\)}{RPI\(_1\)} - 1 + \text{floor}\(_2\) - \text{cap}\(_2\) \)
  :          :          :
  \* \( \frac{RPI\(_{30}\)}{RPI\(_{29}\)} - 1 + \text{floor}\(_{30}\) - \text{cap}\(_{30}\) \)

- Most LPI swap trades use 3 strikes: [0,5], [0,3] and [0,∞]. SABR RPI y/y normal model has 3 parameters: \( \alpha, \rho \) and \( \nu \) (in our example spreadsheet we reparameterise as: atm vol, \( \rho \) and \( \nu \))
- In practice a unique fit can be usually identified
Simple LPI model: \( LPI[0,5] \) fit good
Simple LPI model: $LPI_{[0,\infty]}$ fit good

![Graph showing the fit of the LPI model to the market data for different maturities.]

Simple LPI model: $LPI_{[0,3]}$ fit good

![Graph showing the fit of the LPI model to the market data for different maturities.]

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Simple LPI model: \( \text{LPI}[0,2.5] \) fit good

Simple LPI model: \( \text{LPI}[3,5] \) fit poor in 50y
Simple LPI model: pros and cons

+ recovers RPI and main LPI swap rates and allows alternative strikes, maturities and RPI reference dates to be priced quickly
+ risk to the model parameters (the greeks) is quick and simple

Simple LPI model: the greeks

<table>
<thead>
<tr>
<th>DELTA and SABR VEGA LADDERS</th>
<th>per £1m notional LPI swap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DELTA notional(£m)</td>
</tr>
<tr>
<td>RPI zc</td>
<td></td>
</tr>
<tr>
<td>30y</td>
<td>498</td>
</tr>
<tr>
<td>10y</td>
<td>944</td>
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</tr>
<tr>
<td>25y</td>
<td>2186</td>
</tr>
<tr>
<td>30y</td>
<td>2620</td>
</tr>
<tr>
<td>40y</td>
<td>3436</td>
</tr>
<tr>
<td>50y</td>
<td>4204</td>
</tr>
<tr>
<td>TOTAL</td>
<td>2003</td>
</tr>
</tbody>
</table>

RPI zc swap notionals (in £'m) to delta hedge £1million LPI[0,5] liability
**Simple LPI model: pros and cons**

- recovers RPI and main LPI swap rates and allows alternative strikes, maturities and RPI reference dates to be priced quickly
- risk to the model parameters (*the greeks*) is quick and simple
- effects on LPI swap rates and greeks of RPI swap scenarios or curve moves are readily explored

- is not a true model, recovers RPI zc swap rates but does not recover market prices of y/y and index options
- does not price other types of LPI swaps or other inflation derivatives

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**RPI vs LPI[0,5] swap market rates**

![Graph showing RPI vs LPI[0,5] swap market rates]
Conclusion and discussion

- The implied RPI volatility smile is an important feature of the inflation options market. The skew towards expensive floors/cheaper caps is extreme as a result of lack of natural supply of floors.
- LPI models proposed in literature have had far more general applicability, but have not emphasised the effect of the smile.
- The simple type-4 LPI model presented may assist with “interpolating” values for LPI liabilities, calculation delta and vega risks and understanding dealer execution charges for these greeks.

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and its staff are encouraged.
The views expressed in this presentation are those of the presenter.
References

LPI

OPTION PRICING MODELS

VOLATILITY MODELS

VOLATILITY MODELS cont.