The APCI model — a stochastic implementation.

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1 Contributors

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2 Background
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- CMI released new projection spreadsheet.
- Calibration is done by new APCI model.
- See Continuous Mortality Investigation [2017].
2 Background

- CMI intended APCI model for calibrating deterministic targeting spreadsheet.
- Richards et al. [2017] show how to implement it as a fully stochastic model.
- Presented at sessional meeting of IFoA on 16th October 2017.
- Paper and materials at www.longevitas.co.uk/apci
3 APCI model
3 APCI model

\[ \log m_{x,y} = \alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x} \]  (1)
3 Related models for log $m_{x,y}$

Age-Period : $\alpha_x + \kappa_y$  \hspace{1cm} (2)

APC : $\alpha_x + \kappa_y + \gamma_{y-x}$  \hspace{1cm} (3)

Lee-Carter : $\alpha_x + \beta_x \kappa_y$  \hspace{1cm} (4)

APCI : $\alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x}$  \hspace{1cm} (5)
3 Model relationships

Age-Period:

\[ \alpha_x + \kappa_y \]
3 Model relationships

Age-Period:
\[ \alpha_x + \kappa_y \]

Add \( \gamma_{y-x} \)

APC:
\[ \alpha_x + \kappa_y + \gamma_{y-x} \]
3 Model relationships

Age-Period:
\[ \alpha_x + \kappa_y \]

Add \( \gamma_{y-x} \)

Add \( \beta_x \)

APC:
\[ \alpha_x + \kappa_y + \gamma_{y-x} \]

Lee-Carter:
\[ \alpha_x + \beta_x \kappa_y \]
3 Model relationships

Age-Period:
\[ \alpha_x + \kappa_y \]

Add \( \gamma_{y-x} \)

APC:
\[ \alpha_x + \kappa_y + \gamma_{y-x} \]

Add \( \beta_x \)

Lee-Carter:
\[ \alpha_x + \beta_x \kappa_y \]

Add \( \beta_x \)
Change nature of \( \kappa_y \)

APCI:
\[ \alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x} \]
3 Model relationships

**Age-Period:**
\[ \alpha_x + \kappa_y \]

Add \( \gamma_{y-x} \)

**APC:**
\[ \alpha_x + \kappa_y + \gamma_{y-x} \]

Add \( \beta_x \)

**Lee-Carter:**
\[ \alpha_x + \beta_x \kappa_y \]

Add \( \gamma_{y-x} \)
Change nature of \( \beta_x \)
Change nature of \( \kappa_y \)

**APCI:**
\[ \alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x} \]
APCI model can be viewed superficially as either:

- An APC model with added Lee-Carter-like $\beta_x$ term, or
- A Lee-Carter-like model with added $\gamma_{y-x}$ cohort term.
3 APCI model

…but there are important differences:

- In the Lee-Carter model the change in mortality is age-dependent: $\beta_x \kappa_y$.
- In the APCI model only the expected change is age-dependent: $\beta_x (y - \bar{y})$.
- $\kappa_y$ in the APCI model is very different to $\kappa_y$ in the other models.
Although related to the APC or Lee-Carter models, the APCI model is not a generalization of either.
4 Fitting and constraints
All of these models have an infinite number of possible parameterisations.

Pick the Age-Period model as a simple example...
If we set:

\[ \alpha'_x = \alpha_x + v, \forall x \]
\[ \kappa'_y = \kappa_y - v, \forall y \]

then the model will have the same fitted values for any real-valued \( v \).
4 Identifiability — solution

- Use an identifiability constraint to impose desired behaviour without changing fit.
- Choice of identifiability constraints helps interpretation and can make parameters like $\kappa_y$ forecastable.
Age-Period model:

- Imposing $\sum_y \kappa_y = 0$ does not change the fit...
- ...but it means that $\alpha_x$ is (broadly) the average of $\log \mu_{x,y}$ over the period.
4 Constraints used

\[ \text{AP} : \sum \kappa_y = 0 \quad (6) \]

\[ \text{LC} : \sum \kappa_y = 0, \sum \beta_x = 1 \quad (7) \]

\[ \text{APC} : \sum \kappa_y = 0, \sum \gamma_c = 0, \sum (c - c_{\text{min}} + 1) \gamma_c = 0 \quad (8) \]

where \( c = y - x \).
APCI model uses five identifiability constraints:

\[ \sum_y \kappa_y = 0 \quad (9) \]

\[ \sum_y (y - y_1) \kappa_y = 0 \quad (10) \]

\[ \sum_{x,y} \gamma_c = 0 \quad (11) \]

\[ \sum_{x,y} (c - c_{\text{min}} + 1) \gamma_c = 0 \quad (12) \]

\[ \sum_{x,y} (c - c_{\text{min}} + 1)^2 \gamma_c = 0 \quad (13) \]
Continuous Mortality Investigation [2017] uses (for example) $\sum_c \gamma_c = 0$.

⇒ Cohort with one observation gets same weight as cohort with thirty observations?
Cairns et al. [2009] weight according to number of observations, i.e. $\sum_{x,y} \gamma_c = \sum_c w_c \gamma_c = 0$.

Cairns et al. [2009] approach preferable.

See also Richards et al. [2017, Appendix C].
The Age-Period, APC and APCI models:
- are linear,
- use identifiability constraints, and
- have parameters that can be smoothed.
4 Fitting

- Assume $D_{x,y} \sim \text{Poisson}(E_{x,y} \mu_{x,y})$.
- AP, APC and APCI models are penalized, smoothed GLMs.
- Lee-Carter model can fitted as pairwise conditional penalized, smoothed GLMs.
Currie [2013] sets out generalized GLM-fitting algorithm to:

- maximise likelihood,
- apply linear identifiability constraints, and
- smooth parameters.

Note that the Currie algorithm achieves these simultaneously, not in separate stages as in Continuous Mortality Investigation [2017].
Identifiability constraints do not always have to be linear; see Girosi and King [2008], Cairns et al. [2009] and Richards and Currie [2009].

However, proving that a constraint is an identifiability constraint is harder if it is non-linear.

The Currie [2013] algorithm works with linear constraints only.
5 Parameter estimates
Parameter estimates $\hat{\alpha}_x$ for four unsmoothed models.

**Age-Period $\hat{\alpha}_x$**

**APC $\hat{\alpha}_x$**

**Lee-Carter $\hat{\alpha}_x$**

**APCI $\hat{\alpha}_x$**
$\Rightarrow \alpha_x$ plays the same role across all four models, i.e. average log mortality by age.

...as long as $\sum_y \kappa_y = 0$.

$\Rightarrow \alpha_x$ could be smoothed to reduce effective dimension of model.
Parameter estimates $\hat{\beta}_x$ for Lee-Carter and APCI models (both unsmoothed).
Parameter estimates $\hat{\beta}_x$ for Lee-Carter and $-\hat{\beta}_x$ for APCI models (both unsmoothed).
\( \beta_x \) plays an analogous role in the Lee-Carter and APCI models, namely an age-related modulation of the time index.

\( \beta_x \) in APCI model operates on a quite different scale due to \((y - \bar{y})\) term.

\( \beta_x \) in APCI model would be better multiplied by \((\bar{y} - y)\) term...

...and have a constraint on \( \beta_x \) analogous to the Lee-Carter one.
Like $\alpha_x$, $\beta_x$ could be smoothed to reduce effective dimension of model.

Smoothing $\beta_x$ also improves forecasting properties; see Delwarde et al. [2007].
Note that the APCI model has two time-varying components:

1. An age-dependent central linear trend, \((y - \bar{y})\), and
2. An unmodulated, non-linear term, \(\kappa_y\).
5 Conclusions for $\alpha_x$ and $\beta_x$

- $\alpha_x$ and $\beta_x$ play similar roles across all models.
- What about $\kappa_y$ and $\gamma_{y-x}$?
Parameter estimates $\hat{\kappa}_y$ for four unsmoothed models.

Age-Period $\hat{\kappa}_y$

APC $\hat{\kappa}_y$

Lee-Carter $\hat{\kappa}_y$

APCI $\hat{\kappa}_y$
\( \kappa_y \) plays a similar role in the Age-Period, APC and Lee-Carter models.

\( \kappa_y \) plays a very different role in the APCI model.

APCI \( \hat{\kappa}_y \) values have less of a clear trend pattern for forecasting.

APCI \( \hat{\kappa}_y \) values are strongly influenced by structural decisions made elsewhere in the model.
Parameter estimates $\hat{\gamma}_{y-x}$ for APC and APCI models (both unsmoothed).
The $\gamma_{y-x}$ values appear to play analogous roles in the APC and APCI models. . .

. . .yet the values taken and the shapes displayed are very different.

If values and shapes are so different, what do $\gamma_{y-x}$ values represent?

$\gamma_{y-x}$ don’t have an interpretation independent of the other parameters in the same model. . .

. . . $\gamma_{y-x}$ don’t describe cohort effects in any meaningful way.
6 To smooth or not to smooth?

- Continuous Mortality Investigation [2017] smooths all parameters.
- However, only $\alpha_x$ and $\beta_x$ exhibit regular behaviour.
- Does it make sense to smooth $\kappa_y$ and $\gamma_{y-x}$?
6 To smooth or not to smooth?

- CMI’s smoothing parameter for \( \kappa_y \) is \( S_\kappa \).
- Smoothing penalty for \( \kappa_y \) is
  \[
  10^{S_\kappa} \sum_{y=3}^{n_y} \left( \kappa_y - 2\kappa_{y-1} + \kappa_{y-2} \right)^2.
  \]
- Value for \( S_\kappa \) is set subjectively.
- What is the impact of smoothing \( \kappa_y \)?
life expectancies are [...] very sensitive to the choice made for $S_\kappa$, with the impact varying across the age range. At ages above 45, changing $S_\kappa$ by 1 has a greater impact than changing the long-term rate by 0.5%.”

Continuous Mortality Investigation [2016, page 42]

See also [https://www.longevitas.co.uk/site/informationmatrix/signalornoise.html](https://www.longevitas.co.uk/site/informationmatrix/signalornoise.html)
6 Impact of smoothing APCI $\kappa_y$

- $S_\kappa$ has a large impact because $\kappa_y$ collects features left over from other parts of the model structure.
- Indeed, $\kappa_y$ collects every remaining period effect and applies it without any age modulation.
- If $\kappa_y$ is a “left-over”, should one smooth it at all?
7 Value-at-Risk (VaR)
“Whereas a catastrophe can occur in an instant, longevity risk takes decades to unfold”

The Economist [2012]
7 Trend risk v. one-year view

Solution from Richards et al. [2014]:
- Simulate next year’s experience.
- Refit the model.
- Value liabilities.
- Repeat...
7 Sensitivity of forecast

Observed male mortality at age 70 in E&W

Central projections based on simulated 2011 experience

Source: Lee-Carter example from Richards et al. [2014].
www.longevitas.co.uk
Approach from Kleinow and Richards [2016] for parameter uncertainty:

- $\gamma_{y-x}$: use ARIMA model without mean.
- $\kappa_y$ under AP, APC and LC models: use ARIMA model with mean.
- $\kappa_y$ under APCI model: use ARIMA model without mean.
Value-at-risk capital requirements for annuities payable to male 70-year-olds. Source: Richards et al. [2017, Table 4].

See also https://www.longevitas.co.uk/site/informationmatrix/twinpeaks.html
Variety of density shapes.
⇒ not all unimodal.
Considerable variability between models.
⇒ need to use multiple models.
VaR99.5% capital-requirement percentages by age for four models. Source: Richards et al. [2017].
Q. Why do capital requirements reduce with age for Lee-Carter, but not with APCI?
A. $\kappa_y$ is unmodulated by age in APCI model.
8 Conclusions

- APCI model is implementable as a fully stochastic model.
- APCI model shares features and drawbacks with Age-Period, APC and Lee-Carter models.
- Smoothing APCI $\hat{\alpha}_x$ and $\hat{\beta}_x$ seems sensible.
- Smoothing APCI $\hat{\kappa}_y$ and $\hat{\gamma}_{y-x}$ is not sensible.
- Currie [2013] algorithm makes fitting penalized, smoothed GLMs straightforward.


More on longevity risk at [www.longevitas.co.uk](http://www.longevitas.co.uk)
10 Constraints (again)
10 Corner cohorts

Number of observations for each cohort in the data region.
10 Constraints (again)

- Both Continuous Mortality Investigation [2017] and Richards et al. [2017] avoid estimating “corner cohorts”.
- This means not all constraints are required for identifiability.
- Continuous Mortality Investigation [2017] and Richards et al. [2017] both fit over-constrained APCI models.
- What impact does this have?
Over-constrained models reduce the goodness-of-fit... 
...but can be used to impose desirable behaviour on parameters.
Parameter estimates $\hat{\kappa}_y$ APC(S) model

$\hat{\kappa}_y$ (over-constrained)

$\hat{\kappa}_y$ (minimal constraints)
Parameter estimates $\hat{\gamma}_{y-x}$ APC(S) model

$\hat{\gamma}_{y-x}$ (over-constrained)

$\hat{\gamma}_{y-x}$ (minimal constraints)
\( \hat{\kappa}_y \) robust to over-constrained model.

Values for \( \hat{\gamma}_{y-x} \) differ, but shape similar.
Parameter estimates $\hat{\kappa}_y$ APCI(S) model

$\hat{\kappa}_y$ (over-constrained)

$\hat{\kappa}_y$ (minimal constraints)
10 APCI model — $\gamma_{y-x}$

Parameter estimates $\hat{\gamma}_{y-x}$ APCI(S) model

$\hat{\gamma}_{y-x}$ (over-constrained)  

$\hat{\gamma}_{y-x}$ (minimal constraints)
Neither $\hat{\kappa}_y$ nor $\hat{\gamma}_{y-x}$ robust to over-constrained model.

$\kappa_y$ in APCI model is a term which picks up left-over aspects of fit.

$\hat{\gamma}_{y-x}$ changes radically depending on constraint choices.

⇒ What are the implications for the CMI spreadsheet of using $\hat{\gamma}_{y-x}$ from APCI model?