Bayesian Hierarchical Models for Loss Development

Michael Cooney, Jake Morris and Cynon Sonkkila
“Modern Bayesian methods provide richer information, with greater flexibility and broader applicability than 20th century methods.

Bayesian methods are intellectually coherent and intuitive ...[and] readily computed...”

John K. Kruschke
Introduction to Bayesian methods

• Actuarial work is fundamentally assumptions-based:
  – Data + Models + Judgements = Predictions

• Key challenge: updating assumptions as new information arises
  – Are existing assumptions still relevant?
  – To what extent should we react? Are we consistent?

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Introduction to Bayesian methods

• Standard actuarial problems → credibility theory:

\[ \text{Estimate} = Z\bar{X}_i + (1 - Z)\mu \]

• OK, but we might also like…
  – Model flexibility, e.g. nonlinearities, time-series, …
  – Full distribution of estimates (reflecting uncertainty in \( \bar{X}_i \) & \( \mu \)):

  “\textbf{Given [our] estimate of future payments and ... current state of knowledge, what is the probability that final payments will be no larger than the given value?”\textbf{”}
Introduction to Bayesian methods

• Bayes’ theorem (probability):

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}
\]

• Bayes’ theorem (inference):

\[
p(\theta|X) \propto p(\theta)L(X; \theta)
\]

*Posterior \propto Prior \times Likelihood*

• For actuaries:

\[
p(ULR|Inc) \propto p(ULR)L(Inc; ULR)
\]
Introduction to Loss Development models
Development factors by Cohort Year
Grid Approximation

Marginal Probability of Observed Curve as Function of ULR

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2D Grid Approximation
2D Grid Approximation

Grid Approximation of Curve Fit

ULR

\( \lambda \)

Prob

0.009

0.006

0.003

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Observed vs Fitted Development Curves

Plot of Possible Curves vs Observed

Development Factor

Time

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5 Years Of Development Only
3 Years of Development
Sum of all Accident Years
Specifying the Model

Loss(t) = Premium × Ultimate Loss Ratio × GF(t)

- Model growth function as Weibull or Log-logistic
- Hierarchy by year
Posterior Sampling

• High-dimensional integrals

• Computationally infeasible

• Sample instead

• Stan (mc-stan.org)

• Hamiltonian Monte Carlo
Full Model Specification

\[
\text{Loss}(Y,t) \sim \text{Normal}(\mu(Y,t), \sigma_Y)
\]
\[
\mu(Y,t) = \text{Premium}(Y) \times LR(Y) \times GF(t)
\]
\[
\sigma_Y = \text{Premium}(Y) \times \sigma
\]
\[
LR_Y \sim \text{Lognormal}(\mu_{LR},\sigma_{LR})
\]
\[
\mu_{LR} \sim \text{Normal}(0,0.5)
\]
Outputs of MCMC

Sampled Loss Ratio for Accounting Year 1988
Outputs of MCMC

Sampled Loss Ratio for Accounting Year 1988

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Parameter Inference

Posterior Credibility Intervals for ULR

Accounting Year ULR

ULR[1]
ULR[2]
ULR[3]
ULR[4]
ULR[5]
ULR[6]
ULR[7]
ULR[8]
ULR[9]
ULR[10]
mu_ULR_exp

0.5  0.6  0.7  0.8
Value
Sanity Check for 1988

Plot of 1988 Year Loss Development Against Posterior Distribution
Predictions for 1993

![Plot of 1993 Year Loss Prediction](image-url)
Predictions for 1995

Plot of 1995 Year Loss Prediction

Loss

Time

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Recap

• We have seen how a time varying development pattern can be approximated using a CDF like curve.

• Relatively low number of parameters are needed.

• The example shown is for:
  – an exponential CDF fit
  – 2 parameters: Ultimate and Lambda.

\[ \text{Incurred} = \text{Ultimate} \times (1 - e^{-\lambda \times \text{time}}) \]
Recap

- Parameter Uncertainty can be represented using the likelihood function (of MLE fame).
- Presented here on a grid.
- MLE would be in the centre.
Recap

• Sampling is conducted on the Likelihood distribution

• Sample development curves give an envelope of reasonable development patterns that fit the data.
Recap

• Less data creates more uncertainty.
Recap

• More data creates less uncertainty.
Recap

• Sharing credibility from year to year is incorporated using a prior.

• One method is to think of the prior years as samples “what might happen” for a new year.

• So the “sum” of previous years is a suggestion for a new year i.e. a prior.

• 14 years of data shown here as an example.

• Perhaps there is a better way?
Recap

• The prior is better approximated as a smooth distribution.
• A lognormal prior for Lambda and Ultimate are shown here.
• The prior can be fitted as part of a hierarchical model.
Recap

• Applying a prior based on other years lends credibility to a year with limited samples.

• Reduced mean and spread can result when a combination of the prior and data are used to estimate the range of reasonable ultimates.
Extensions

• Instead of tracing the path of “incurred” development we can trace the path of some other parameters.

• Opposite is the path of a typical AY cohort of 100 claims as they develop.

• Mean and standard deviation tend to increase with time as heavier claims are reported later.
Time series of fitted parameters

- This behaviour can be represented as a trend in the fitted parameters.
- Here we fit a lognormal at each point in time and plot the parameters.
- Mu and Sigma trend much like a development curve.
Time series regression

• This behaviour can be modelled with a growth curve.

• Here mu and sigma are fitted using a lognormal CDF with a start and end parameter.

• For example:
  - \[ \text{Mu} = \text{start} + (\text{end}-\text{start}) \times \text{LNCDF}(\mu_{\text{development}}, \sigma_{\text{development}}) \]

• In this case the “ultimate” distribution of claims are given by the ultimate mu and sigma (the “end” parameter).
Time series regression

- Such a model does not require development factors.
- Bayesian techniques are best used to estimate mu and sigma.
- The ultimate expected mu and sigma are then given with parameter uncertainty.
- Recent years where there is limited data would utilise a credible prior based on previous years as before.
Further extensions to treat extreme events

- Some loss data may be described using Extreme Value Theory type distributions.
- A typical plot is a log-log survival plot or Hill plot shown opposite.
- Linear behaviour on the upper plot would better be modelled by a Pareto distribution.
- A Hill plot (lower right) would show stable fitted alpha above some level.
Fitting a Lognormal-Pareto distribution

• The method for fitting a lognormal distribution with a Pareto tail is outlined by Teodorescu, S. (2009).

• The model has three parameters:
  – Alpha, the Pareto distribution parameter
  – Theta, the level at which the Pareto distribution will be fitted
  – Sigma, one of the lognormal parameters

• Other parameters are fixed due to the requirement for the distributions to be continuous and smooth at theta.

• These three parameters are modelled through time just as mu and sigma previously.

• An example fit is shown opposite.
Estimating Ultimate CDF/LEV for a Typical Dataset

- After MCMC we calculate fitted CDFs and LEVs with error bounds.
- Of note is the level of error in the LEV (and therefore any ILF) for the upper layers.
Summary

- Long tailed claims can be modelled as a distribution that changes through time to some ultimate position.
- Each parameter of the distribution can be modelled through time using a growth curve.
- A Lognormal distribution with a Pareto distribution acting in the tail may be useful for including treatment for extreme events seamlessly in your severity model.
- Hierarchical models are useful for projecting undeveloped claims without development factors.
- MCMC methods can provide reasonable measures of uncertainty for parameters such as portfolio ILFs.
- Uncertainty in ILFs may then be useful for credibility based pricing for excess layers using frequency/severity models.
Getting started

• Ordinary Least Squares (OLS)
• Independent variables ($X$), parameters ($\beta$)

\[ y = \beta X + \epsilon \]
\[ \epsilon \sim \mathcal{N}(0, \sigma^2) \]

• Restate as a probability model:

\[ y \sim \mathcal{N}(\beta X, \sigma^2) \]

• Data are modelled as Normal with mean $\beta X$ and variance $\sigma^2$
  – Equivalent, yet more intuitive
Getting started

• Auto claims data:

<table>
<thead>
<tr>
<th>log_loss</th>
<th>lawyer</th>
<th>gender</th>
<th>seatbelt</th>
<th>age</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>yes</td>
<td>male</td>
<td>yes</td>
<td>50</td>
</tr>
<tr>
<td>2.4</td>
<td>no</td>
<td>female</td>
<td>yes</td>
<td>28</td>
</tr>
<tr>
<td>-1.1</td>
<td>no</td>
<td>male</td>
<td>yes</td>
<td>5</td>
</tr>
<tr>
<td>2.4</td>
<td>yes</td>
<td>male</td>
<td>no</td>
<td>32</td>
</tr>
<tr>
<td>2.0</td>
<td>no</td>
<td>male</td>
<td>yes</td>
<td>30</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

• Linear model for loss cost:

\[
\text{log_loss} = \beta_0 + \beta_1 \text{lawyer} + \beta_2 \text{gender} + \beta_3 \text{seatbelt} + \beta_4 \text{age}
\]

• R implementation:

```R
model_lm <- lm(log_loss ~ lawyer + seatbelt + gender + age, data = data)
```
### Colour palette for PowerPoint presentations

- **Dark blue**
  - R17  G52  B88

- **Gold**
  - R217  G171  B22

- **Mid blue**
  - R64  G150  B184

**Secondary colour palette**

- **Light grey**
  - R220  G221  B217

- **Pea green**
  - R121  G163  B42

- **Forest green**
  - R0  G132  B82

- **Bottle green**
  - R17  G179  B162

- **Cyan**
  - R0  G156  B200

- **Light blue**
  - R124  G179  B225

- **Violet**
  - R128  G118  B207

- **Purple**
  - R143  G70  B147

- **Fuscia**
  - R233  G69  B140

- **Red**
  - R200  G30  B69

- **Orange**
  - R238  G116  29

- **Dark grey**
  - R63  G69  B72

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**Getting started**

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Getting started

• Bayesian version?
  – Setting up from scratch in Stan → time/effort

• R package ‘rstanarm’ reduces coding requirements
  – Pre-built Stan models (e.g. linear models, GLMs, ANOVA…)
  – R syntax relatively simple:

```r
model_stanlm <- stan_lm(log_loss ~ lawyer + seatbelt + gender + age,
                         prior = R2(location = 0.8),
                         data = data)
```

• Offers various outputs…
Getting started
Getting started
Getting started
Conclusions

• Bayesian methods can offer a variety of benefits
  – Reflect uncertainty, model flexibility, external data/judgement, hierarchical models, …

• Numerous potential actuarial applications
  – Reserving, pricing, profitability studies, portfolio optimisation, …

• Learning curve → rstanarm a good place to start

“Scientific disciplines from astronomy to zoology are moving to Bayesian data analysis. We should be leaders of the move, not followers.”
- John K. Kruschke (2010)
Further Reading

Hierarchical Growth Curve Models for Loss Reserving - Guszcza (CAS Forum 2008)
Hierarchical Compartmental Models for Loss Reserving - Morris (CAS E-Forum, Summer 2016)
On the Truncated Composite Lognormal-Pareto Model - Teodorescu (2009)

Doing Bayesian Data Analysis - John Kruschke
Statistical Rethinking - Richard McElreath
Data Analysis Using Regression and Multi-level/Hierarchical Models - Gelman and Hill
An Introduction to Statistical Learning - Tibshirani and Hastie

Stan Documentation - (tutorials, case studies, etc)

Modelling Loss Curves in Insurance with RStan (Stan Case Study) - Cooney
Open Actuarial - Various
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