The Design of Pension Contracts, on the perspective of customers

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• Is the largest expected value optimal to the customers?
What contract is attractive?

- Is the largest expected value optimal to the customers?
- Expectation = \( \sum x_ip_i \)

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What contract is attractive?

**St. Petersburg Paradox**

- Tossing a fair coin until the first head appears. What you will receive is \( £2^n \) if the first head appears at \( n \)th coin tossing.
- How much would you pay to play this game?
What contract is attractive?

St. Petersburg Paradox

• Tossing a fair coin until the first head appears. What you will receive is £2^n if the first head appears at n^th coin tossing.

• How much would you pay to play this game?

$$E = \frac{1}{2} \times 2 + \frac{1}{2^2} \times 4 + \frac{1}{2^3} \times 8 + \ldots + \frac{1}{2^n} \times 2^n$$

$$= 1 + 1 + 1 + \ldots + 1 = \infty$$
Expected Utility Theory

St. Petersburg Paradox

• Daniel Bernoulli (1738) proposed a solution using utility function.

Characteristics:
• prefer more to less
  \[ U'(x) > 0. \]
• diminishing marginal utility
  \[ U''(x) < 0. \]

Expected Utility Theory

• Von Neumann and Morgenstern (1947) show that rational people will always prefer actions that maximize expected utility.

• Mathematically, the expected utility is calculated as
  \[ E = \sum U(x_i)p_i \]
Expected Utility Theory

• $U(x_i) = \log_2(x)$

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>...</th>
<th>$2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>...</td>
<td>$\frac{1}{2^n}$</td>
</tr>
<tr>
<td>$U(x_i)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>$n$</td>
</tr>
</tbody>
</table>

$E = \frac{1}{2} \times \log_2 2 + \frac{1}{2^2} \times \log_2 4 + \frac{1}{2^3} \times \log_2 8 + ... + \frac{1}{2^n} \times \log_2 2^n$

$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + ... + \frac{n}{2^n}$

$= 2$

Product design

• Determine the optimal investment strategy:
  • how much to invest in equities to maximize the expected utility

• Test which product design gives the highest expected utility.
Optimal investment strategies

• Merton’s portfolio
  • Assume simple financial market model throughout that there is only an equity (risky asset) and a bond (risk-free asset) available for investment.
  • Results based on Black-Scholes model.
• Aim: find an investment strategy which maximizes the expected utility of wealth at a fixed time,
  \[ E(U(X_T^T)). \]

Optimal investment strategies under EUT

• Merton’s (1969) solution based on \( U(x) = \frac{x^{1-\gamma}}{1-\gamma} \).
• Calculated as \( \frac{\mu - r}{\gamma \sigma^2} \),
• Optimal to put a fixed proportion in equities at all times, which is not dependent on time or wealth.
Optimal investment strategies under EUT

- $r_f = 4\%$
- $\mu = 6.5\%$
- $\sigma = 15\%$
- $\gamma = 5$

Dynamic optimal portfolio for two assets under EUT

Constant Proportion

Risk-free Asset
Risky asset
Limitations of EUT

- Framing effect
- Non-linear preference (Allais's paradox)
- Source dependence (Ellsberg's paradox)
- Risk seeking
- Loss aversion

Imagine that we are preparing for the outbreak of an unusual disease, which is expected to kill 600 people.

<table>
<thead>
<tr>
<th>Treatment A</th>
<th>Treatment B</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Saves 200 lives&quot;</td>
<td>&quot;A 33% chance of saving all 600 people, 66% possibility of saving no one.&quot;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Treatment C</th>
<th>Treatment D</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;400 people will die&quot;</td>
<td>&quot;A 33% chance that no people will die, 66% probability that all 600 will die.&quot;</td>
</tr>
</tbody>
</table>
Limitations of EUT

- Allais's paradox

<table>
<thead>
<tr>
<th>Investment A</th>
<th>Investment B</th>
<th>Investment C</th>
<th>Investment D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Win</td>
<td>Chance</td>
<td>Win</td>
<td>Chance</td>
</tr>
<tr>
<td>£1 million</td>
<td>100%</td>
<td>£1 million</td>
<td>89%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nothing</td>
<td>90%</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>£1 million</td>
<td>11%</td>
</tr>
<tr>
<td>£5 million</td>
<td>10%</td>
<td></td>
<td>£5 million</td>
</tr>
</tbody>
</table>

- Ellsberg's paradox

Consider an urn containing 30 red balls and 60 other balls that are either black or yellow. It is unknown how many black or yellow balls there are, but the total number of black balls and yellow balls equals 60.

<table>
<thead>
<tr>
<th>Gamble A</th>
<th>Gamble B</th>
</tr>
</thead>
<tbody>
<tr>
<td>You receive £100 if you</td>
<td>You receive £100 if you</td>
</tr>
<tr>
<td>draw a red ball</td>
<td>draw a black ball</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Gamble C</th>
<th>Gamble D</th>
</tr>
</thead>
<tbody>
<tr>
<td>You receive £100 if you</td>
<td>You receive £100 if you</td>
</tr>
<tr>
<td>draw a red or yellow ball</td>
<td>draw a black or yellow ball</td>
</tr>
</tbody>
</table>
Cumulative Prospect Theory

- Tversky and Kahneman (1992) proposed the Cumulative Prospect Theory (CPT). It is viewed as a better model in explaining people's behaviour in decision making under uncertainty.

- CPT utility is calculated by value function and weighting function.

\[ E = \sum V(x_i)W(p_i) \]
Cumulative Prospect Theory

- Weighting function

![Weighting function graph](image1)

Optimal investment strategies under CPT

- $r_f = 4\%$
- $\mu = 6.5\%$
- $\sigma = 15\%$

![Optimal portfolio graph](image2)
**Optimal investment strategies under CPT**

- $r_f = 4\%$
- $\mu = 6.5\%$
- $\sigma = 15\%$

**Danish Time Pension**

- Danish “Time Pension”.
- Annual customer value calculated as

$$D'_n = \begin{cases} (1 + g')D'_{n-1} + P, & n = 0 \\ \alpha [A_n - (1 + g')D'_{n-1}], & n \in \{1, 2, ..., n\} \end{cases}$$

Underlying value

Guarantee rate

Participation ratio
The new contract

• Danish “Time Pension”. Annual customer value calculated as

\[ D'_n = \begin{cases} P, & n = 0 \\ (1 + g')D'_{n-1} + \alpha' A_n - (1 + g')D'_{n-1}, & n \in \{1, 2, \ldots, n\} \end{cases} \]

• The new contract. Annual customer value calculated as

\[ D_n = \begin{cases} P, & n = 0 \\ (1 + g)D_{n-1} + \alpha \max\{A_n - (1 + g)D_{n-1}, 0\}, & n \in \{1, 2, \ldots, n\} \end{cases} \]

Sample path of new contract

• Bull Market
Sample path of new contract

• Bear Market

Parameter sensitivity

• $r_f = 4\%$
• $\mu = 6.5\%$
• $\sigma = 15\%$
• $T = 20$
### Results

**Expected TV (terminal value)**

<table>
<thead>
<tr>
<th></th>
<th>Our new contract</th>
<th>Danish Time Pension</th>
<th>100% Bond</th>
<th>100% Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>29 November 2018</td>
<td>2.8793</td>
<td>3.0612</td>
<td>2.1911</td>
<td>3.5243</td>
</tr>
</tbody>
</table>

**Standard deviation**

- 1.4889
- 1.6829
- 0
- 2.6340

**Mean CPT utility**

- 1.3330
- 1.2816
- 1.0914
- 1.2785

**Proportion of Optimal portfolio**

<table>
<thead>
<tr>
<th></th>
<th>Our new contract</th>
<th>Danish Time Pension</th>
<th>100% Bond</th>
<th>100% Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>29 November 2018</td>
<td>61%</td>
<td>0</td>
<td>0</td>
<td>39%</td>
</tr>
</tbody>
</table>

### Results

- $P = 1$
- $\alpha = 13\%$
- $g' = 4\%$
- $g = 2\%$
- $r_f = 4\%$
- $\mu = 6.5\%$
- $\sigma = 15\%$
- $T = 20$

### Graphs

**Static optimal portfolio under CPT**

- $r_f = 4\%$
- $\mu = 6.5\%$
- $\sigma = 15\%$
Conclusion

• In this paper, we introduce a new pension contract with the features of guarantees and bonuses. It has transparent structure and clear distribution rule.

• Under cumulative prospect theory, the contract generates higher utility than the Time Pension. The result provides the evidence why the guarantees should be included in the pension contract.