What type of pensions would most people prefer?

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1. Scottish Ministers' Widows' Fund

- Once in Edinburgh discussing life and pension insurance, it seems quite natural to recall that the first modern insurance fund, The Scottish Ministers' Widows' Fund, was established right here.

- Ministers Robert Wallace and Alexander Webster along with Colin Maclaurin, Professor of Mathematics at Edinburgh achieved to create the first viable insurance scheme for widows and children of deceased ministers of the Church of Scotland in the mid 1700.

2. Abstract

- We analyze optimal consumption and portfolio choice when the agent takes the market as given.

- Preferences are vital to our approach. First we recall how the results look like for the conventional additive and separable expected utility model, and we indicate that this model has problems in a temporal context.

- We then consider recursive utility, which in contrast has an axiomatic underpinning. With this in place, the anomalies met by the expected utility approach disappear.

- It follows from our model how aggregate consumption in society can be as smooth as implied by data, and at the same time be consistent with the relatively large, observed growth rate.

- We look at implications for pension insurance, as well as risk premia and the equilibrium short-term interest rate in the associated equilibrium model.

- Since the recursive model fits market data much more convincingly than the conventional model, this leaves more credibility to the former representation, and more weight to recommendations based on it.

3. Introduction

- The framework of this discussion is the Life Cycle Model.

- This model is useful when it comes to analyzing "optimal" pension plans (and life insurance contracts) where the benefits are state dependent.

- The viewpoint is that of the pension insurance customer.

- We compare basic results from two different specifications of preferences: Expected utility (EU), and recursive utility (RU).

4. The problem

- Consider an individual (U, e), with utility function $U(c)$ for a life-time consumption stream $c = \{c_t,\ 0 \le t \le \tau\}$, and with an endowment process $e = \{e_t,\ 0 \le t \le \tau\}$.

- Here $U : L_+ \to \mathbb{R}$, where

$$L = \Big\{c : c_t \text{ is } \mathcal{F}_t\text{-adapted, and } E\Big(\int_0^\tau c_s^2 ds\Big) < \infty\Big\}.$$

- For a price π_t of the consumption good, the problem is to solve

$$\sup_{c \in L} U(c), \tag{1}$$

subject to

$$E\Big\{\int_0^\tau \pi_t c_t \, dt\Big\} \le E\Big\{\int_0^\tau \pi_t e_t \, dt\Big\} := w. \tag{2}$$

- The quantity π_t is also known as the "state price deflator", or the Arrow-Debreu prices in units of probability.

- State prices reflect what the representative consumer is willing to pay for an extra unit of consumption; in particular is π_t high in "times of crises" and low in "good times".

- The pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. (This makes it possible to compound risk-free payments at a higher rate of interest than r_t.)

- The dynamic equation for the wealth W_t of the agent is the following

$$dW_t = \big(W_t(\varphi_t' \nu_t + r_t) - c_t\big)dt + W_t\varphi_t'\sigma_t \, dB_t, \qquad W_0 = w. \tag{3}$$

- Here
  ν_t = excess returns on the risky assets over the risk-free asset,
  φ_t = fractions of wealth invested in the various risky securities, and
  σ_t = volatilities of the risky assets in units of the prices of these assets.

5. Expected utility

- The preference structure of the representative agent is, in the conventional model (expected utility):

$$U(c) = E\Big\{ \int_0^{\mathcal{T}} e^{-\delta t} u(c_t) dt \Big\}. \tag{4}$$

- Here $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$. This is the CRRA utility function.

- γ = relative risk aversion, δ = impatience rate.

- The problem we want to solve is (1) subject to the budget constraint (2). The Lagrangian for this problem is

$$\mathcal{L}(c;\lambda) = E\Big\{\int_0^{\mathcal{T}} \big(u(c_t,t)dt - \lambda \pi_t (c_t - e_t)\big)\Big\} \tag{5}$$

- For expected utility, the solution technique we use is Kuhn-Tucker, with a Lagrange function, reducing the problem to an unconstrained maximization problem.

- Then we find the first order condition using directional derivatives in function space (Gateau-derivatives), and finally we determine the Lagrange multiplier that yields equality in the budget constraint.

- Finally we have the solution by the Saddle Point Theorem. Equivalently, we could have employed dynamic programming.

- The optimal consumption satisfies the dynamics

$$\frac{dc_t^*}{c_t^*} = \Big(\frac{r_t - \delta}{\gamma} + \frac{1}{2}\,\frac{1}{\gamma}\,(1 + \frac{1}{\gamma})\,\eta_t' \cdot \eta_t\Big)\,dt + \frac{1}{\gamma}\eta_t' \cdot dB_t. \qquad (6)$$

- The solution can be written

$$c_t^* = c_0\, e^{\frac{1}{\gamma}\{\int_0^t (r_s - \delta + \frac{1}{2}\eta_s' \cdot \eta_s)ds + \int_0^t \eta_s' \cdot dB_s\}}; \qquad t \geq 0.$$

- Here the state price deflator has the representation

$$\pi_t = e^{-\int_0^t r_s ds}\xi_t = e^{-\int_0^t r_s ds}\; e^{-\frac{1}{2}\int_0^t \eta_s' \eta_s ds - \int_0^t \eta_s dB_s}, \qquad (7)$$

where $\eta_t =$ the market-price of risk. The Sharpe ratio is $\eta = (\mu_M - r)/\sigma_M > 0$ (e.g., with only one risky asset).

- Alternatively the optimal consumption can be written as

$$c_t^* = c_0 \pi_t^{-\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma} t} = c_0 \xi_t^{-\frac{1}{\gamma}} e^{\int_0^t \frac{1}{\gamma}(r_s - \delta) ds}. \tag{8}$$

- Consider a "shock" to the economy, via the state price π_t.

- It is natural to think of this as stemming from a shock to the term $\int_0^t \eta_s dB_s$ via the process B.

- Assuming η positive, this lowers the state price and *increases* optimal consumption.

- In equilibrium this leads to the *mutuality* principle. It holds for all agents - they are all affected in the same "direction" by market movements.

- The optimal investment policy that goes along with this is the Mossin-Samuelson-Merton formula

$$\varphi = \frac{1}{\gamma}(\sigma_t \sigma_t')^{-1} \nu_t.$$

Here φ signifies the agent's fraction of wealth in the risky securities.

- Table 1 presents the summary statistics of the data used in the Mehra and Prescott (1985)-paper:

	Expectation	Standard dev.	covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{cM} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889 -1978. Continuous-time compounding. $\hat{\kappa}_{M,c} = .4033$.

- Consider an average household: Assuming a relative risk aversion of around two, the optimal fraction in equity, resulting from this standard formula, is 119%, using the summary statistics of Table 1, and assuming one single risky asset, the S&P-500 index itself.

- In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity.

- Conditional on participating in the stock market, this number increases to about 40% in financial assets (some recent estimate says 60%).

- One could object to this that the Eu-(equilibrium) model is consistent with a value for γ around 26 only. Using this value instead, the optimal fraction in equity is down to around 9%.

- In isolation this seems reasonable enough. However, such a high value for the relative risk aversion is considered implausible.

- In order to illustrate what a risk aversion of 26 really means, consider a random variable X with probability distribution given in Table 2:

- The equation

$$E\{u(100 + X)\} := u(100 + m)$$

defines its certainty equivalent m at initial fortune 100 for the utility function u.

- If u is of power type $u(x) = \frac{x^{(1-\gamma)}}{1-\gamma}$, the certainty equivalent m is illustrated in Table 2 for some values of γ.

X	0	100
Probability	0.5	0.5
$\gamma = 0$	$m = 50.00$	
$\gamma = 1$	$m = 41.42$	
$\gamma = 2$	$m = 33.33$	
$\gamma = 3$	$m = 26.49$	
$\gamma = 4$	$m = 21.89$	
$\gamma = 5$	$m = 17.75$	
$\gamma = 17$	$m = 4.42$	
$\gamma = 20$	$m = 3.71$	
$\gamma = 22$	$m = 3.55$	
$\gamma = 26$	$m = 2.81$	

Table 2: Certainty equivalents of X for CRRA-utility.

6. Pensions

- Let T_x be the remaining life time of a person who entered into a pension contract at age x. Let $[0, \tau]$ be the support of T_x.

- The single premium of an annuity paying one unit per unit of time is given by the formula

$$\bar{a}_x^{(r)} = \int_0^{\tau} e^{-rt} \frac{l_{x+t}}{l_x} dt, \tag{9}$$

where r is the short term interest rate.

- The single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^{n} e^{-rt} \frac{l_{x+t}}{l_x} dt. \tag{10}$$

- Consider the following income process e_t:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{11}$$

- Here y is a constant, interpreted as the consumer's salary when working, and n is the time of retirement for an x-year old.

- Equality in the budget constraint can then be written

$$E\Big(\int_0^{\mathcal{T}} (e_t - c_t^*) \pi_t P(T_x > t) dt \Big) = 0.$$

(The Principle of Equivalence).

- The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\tilde{r})}_x} \exp\left\{(\frac{1}{\gamma}(r - \delta) + \frac{1}{2\gamma}\eta^2)t + \frac{1}{\gamma}\eta B_t\right\}, \tag{12}$$

provided the agent is alive at time t (otherwise $c_t^* = 0$).

- The initial value c_0 is then

$$c_0 = y \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\tilde{r})}_x}$$

where

- $$\tilde{r} = r - \frac{1}{\gamma}(r - \delta) + \frac{1}{2}\frac{1}{\gamma}(1 - \frac{1}{\gamma})\,\eta'\eta. \tag{13}$$

- The premium intensity p_t at time t while working is given by $p_t = y - c_t^*$. This is an $\mathcal{F}_t$- adapted process.

- This shows that the same conclusions hold for the optimal pension as with optimal consumption with regard to the sensitivity of stock market uncertainty.

- The expected utility model may be taken as support for unit linked pension insurance, or, defined contribution (DC)-plans. Here all the financial risk resides with the customers.

- One may wonder: Are ordinary pension insurance customers best equipped to carry aggregate financial risk? See Aase, K. K., (2015). Life Insurance and Pension Contracts I: The Time Additive Life Cycle Model. *ASTIN Bulletin*, Volume 45(1), pp 1-47.

- Recall the theory of syndicates: In a Pareto optimum is the risk tolerance (rt_λ) of the syndicate (i.e., an insurance company) the sum of the risk tolerances (rt_i) of the individual members of the syndicate.

$$rt_\lambda(W) = \sum_{i=1}^{n} rt_i(c_i(W)).$$

- The interpretation of this result is precisely that the risk carrying capacity of a syndicate is larger than that of any of its members.

- Expected utility in a *temporal* context lacks an axiomatic underpinning (Mossin (1969)).

7. Recursive Utility

- Recursive utility was first formulated in the setting of discrete time. The basic notions of separating time and risk preferences are roughly summarized as follows:

- First consider a riskless economy, where preferences over consumption sequences $(c_0, c_1, \cdots, c_T)$ are characterized with Koopmans' (1960) time aggregation $g(\cdot, \cdot)$, where

$$U(c_t, c_{t+1}, \cdots, c_T) = g(u(c_t), U(c_{t+1}, c_{t+2} \cdots, c_T)).$$

- This framework is then generalized to evaluate uncertain consumption sequences by replacing the second argument in $g(\cdot,\cdot)$ by the period t certainty equivalent of the probability distribution over all possible consumption continuations.

- The resultant class of recursive preferences may be characterized as

$$U(c_t, c_{t+1}, \cdots, c_T) = g(u(c_t), m_{t+1}(U(c_{t+1}, c_{t+2} \cdots, c_T)),$$

- (This is not behavioral economics.)

- Here $m_{t+1}(\cdot)$ describes the certainty equivalent function based on the conditional probability distribution over consumption sequences beginning in period $t+1$. We use the notation $V_t = U(c_t, c_{t+1}, \cdots, c_T)$.

- $$V_t = g(u(c_t), m_{t+1}) = \left((1-\beta)c_t^{1-\rho} + \beta\big(E_t(V_{t+1}^{1-\gamma})\big)^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1}{1-\rho}}$$

- Here $0 < \beta < 1$, $1 \neq \gamma > 0$, $\rho > 0, \rho \neq 1$. γ is the relative risk aversion, ρ is time preference, the inverse of the EIS-parameter ψ.

- The parameter β is the impatience discount factor, with impatience rate $\delta = -\ln(\beta)$.

- When the parameter β is large, the agent puts more weight on the future and less weight on the present, in accordance with the impatience interpretation of this parameter.

- These preferences have an axiomatic underpinning (e.g., Chew and Epstein (1991), Kreps and Porteus (1978)).

- Such preferences are dynamically consistent (Johnsen and Donaldson (1985)).

- This framework has an extension to continuous time models (Duffie and Epstein (1992), two papers).

8. The continuous-time representation

- Recursive utility in continuous time: $U : L_+ \to \mathbb{R}$ is defined by two primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$.

- The function $f(c_t, V_t)$ represents a felicity index at time t, and A is associated with a measure of absolute risk aversion (of the Arrow-Pratt type) for the agent.

- In addition to current consumption c_t, the function f also depends on future utility V_t time t, a stochastic process with volatility Z_t at time t.

- The representation is: $V_\mathcal{T} = 0$, and

$$V_t = E_t\Big\{ \int_t^\mathcal{T} \Big(f(c_s, V_s) - \frac{1}{2} A(V_s)\, Z_s' Z_s \Big) ds \Big\}, \quad t \in [0, \tau] \qquad (14)$$

- Since $V_\mathcal{T} = 0$ and $\int Z_t dB_t$ is is assumed to be a martingale, (14) has the stochastic differential equation representation

$$dV_t = \left(-f(t, c_t, V_t) + \frac{1}{2} A(V_t)\, Z_t' Z_t \right) dt + Z_t' dB_t. \qquad (15)$$

- If, for each consumption process c_t, there is a well-defined *pair* (V, Z) of a utility process V *and* volatility Z_t satisfying the associated BSDE, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility.

- The pair (f, A) generating V is called an aggregator.

- One may think of the term $-\frac{1}{2}A(V_t)Z_t' Z_t$ as the Arrow-Pratt approximation to the certainty equivalent of V_t. For continuous processes in continuous time there is no element of approximation involved.

- The defining equation for utility is associated to the quadratic BSDE, and existence and uniqueness of solutions to such equations is in general far from granted.

- These topics have been dealt with in the original paper Duffie and Epstein (1992b).

- These questions are also part of contemporary research in applied mathematics, see e.g., Øksendal and Sulem (2014), or Peng (1990).

- For the particular BSDE that we end up with, existence and uniqueness follows from Duffie and Lions (1992). See also Schroder and Skiadas (1999) for the life cycle model.

- We work with the following specification, which corresponds to the aggregator (f, A) with the constant elasticity of substitution (CES) form

$$f(c,v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - v^{(1-\rho)}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{16}$$

- The parameters have the same interpretations as for the discrete-time model.

- This preference fall in the Kreps-Porteus class when the certainty equivalent is derived from expected utility.

9. Optimal consumption with RU

- We use the same technique as explained for the expected utility model, except that we employ the stochastic maximum principle instead of directional derivatives.

- The optimal consumption turns out to be

$$c_t^* = c_0\, e^{\int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c(s)'\sigma_c(s))ds + \int_0^t \sigma_c(s) dB_s} \tag{17}$$

- where $\mu_c(t)$ and $\sigma_c(t)$ are as determined as

$$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big) \tag{18}$$

- and

$$\mu_c(t) = \frac{1}{\rho}(r_t - \delta) + \frac{1}{2}\frac{1}{\rho}(1 + \frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma - \rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) + \frac{1}{2}\frac{(\gamma - \rho)\gamma(1 - \rho)}{\rho^2}\sigma_V'(t)\sigma_V(t) \quad (19)$$

- Here $V_t\sigma_V(t) := Z_t$ and V_t exist as a solution to the system of forward/backward stochastic differential equations.

- Thus $\sigma_V(t)$ is part of the recursive preferences, i.e., primitives of the model.
  (The quantity $\sigma_V(t)$ is the volatility of the growth rate of utility V_t.)

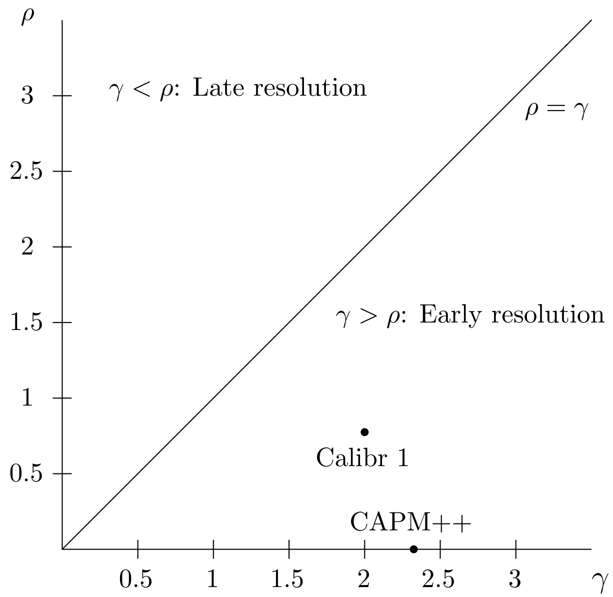

Figure 1: Calibration points in the (γ, ρ)-space

- Recall $\eta > 0$. Notice, if $\gamma > \rho$ the recursive utility agent has preference for *early resolution* of uncertainty.

- From (17) and (18) we see the following: First, a shock to the economy via B has the conventional effect via the market-price-of-risk-term η_t;

- Second, if $\sigma_V(t) > 0$, the shock has the opposite effect via the recursive utility term

  $$(\rho - \gamma)\sigma_V(t)$$

  provided $\gamma > \rho$.

- The volatility of wealth $\sigma_W(t)$ is internalized in equilibrium, as a linear combination of $\sigma_c(t)$ and $\sigma_V(t)$, i.e., of primitives of the model. (In the equilibrium context, aggregate consumption is taken as given.)

- The interpretation is that the agent uses wealth to dampen the effects of market movements on consumption. The Eu-model is simply too simplistic to account for this.

- This can be illustrated using the discrete time analogue:

$$\ln\left(\frac{c^*_{t+1}}{c^*_t}\right) = \frac{1}{\rho}\ln(\beta) - \frac{1}{\rho}\,\frac{1-\rho}{1-\gamma}\ln\left(\frac{\pi_{t+1}}{\pi_t}\right) - \frac{1}{\rho}\,\frac{\gamma-\rho}{1-\gamma}\ln(1+R^W_t). \qquad \text{(R-U)} \tag{20}$$

- It is instructive to compare this relationship to the corresponding one for expected utility, which is

$$\ln\left(\frac{c^*_{t+1}}{c^*_t}\right) = \frac{1}{\gamma}\ln(\beta) - \frac{1}{\gamma}\ln\left(\frac{\pi_{t+1}}{\pi_t}\right). \qquad \text{(E-U)} \tag{21}$$

- Recall, when times are good, the state price is low, and vice versa. With expected utility optimal consumption is then up, and this is the only source of uncertainty that affects consumption, meaning that optimal consumption is very sensitive to market variations.

- With recursive utility this is seen to be different. The agent's wealth can be shown to be negatively correlated with the state price. This means that when the state price is low, R^W tends to be up, and vice versa.

- From (20) we notice that when the sign of $\frac{1-\rho}{1-\gamma}$ is the same as the sign of $\frac{\gamma-\rho}{1-\gamma}$, then market variations will be dampened by opposite variations in the wealth, which then leads to a more stable consumption.

- This is seen to happen when (i) $\gamma > \rho$ and $\rho < 1$, or when (ii) $\gamma < \rho$, and $\rho > 1$.

- In the first case the agent has preference for early resolution of uncertainty, in the second for late. As we indicate below, the first case is typically better in accordance with market data than the latter case (but also the latter case may fit for certain choices of the parameters).

- That consumers use wealth to dampen variations in consumption seems both reasonable and also in line with what we observe. The expected utility model is simply too simple to capture this phenomenon.

- (Here it is tempting to recall Albert Einstein: "A problem should be studied through the simplest model possible, not simpler.")

- We then have the following:

Theorem 1 *Assume the preferences are such that σ_V is positive, and the market price of risk η is positive. The individual with recursive utility will then prefer to smooth market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

- The investment strategy that attains the optimal consumption of the agent is given below.

- In each period the agent both consumes and invests for future consumption. In order to average out consumption across time and state, less is consumed in good times, in which case more is invested, compared to the conventional consumer, who treats every period as if it were the last one.

- First we turn to pension insurance:

9.1. Pensions with RU

- The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$
\begin{aligned}
c_t^* = y \frac{\bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(\hat{r})}} \exp\Big\{ \big(\frac{1}{\rho}(r-\delta) + \frac{1}{2\rho}\eta^2 + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V^2\big)t \\
+ \frac{1}{\rho}(\eta + (\rho-\gamma)\sigma_V)B_t\Big\}, \quad (22)
\end{aligned}
$$

provided the agent is alive at time t (otherwise $c_t^* = 0$).

- Here

$$
\begin{aligned}
\hat{r} = r - \frac{1}{\rho}(r-\delta) + \frac{1}{2}\frac{1}{\rho}(1-\frac{1}{\rho})\,\eta'\eta + \frac{1}{\rho}(\frac{1}{\rho}-1)\,(\rho-\gamma)\,\eta\sigma_V \\
- \frac{1}{\rho}(\gamma-\rho)\Big(\frac{1}{\rho}(\gamma-\rho) + \frac{1}{2}(1-\gamma)\Big)\sigma_V^2. \quad (23)
\end{aligned}
$$

- The premium intensity is given by the F_t-adapted process $p_t := y - c_t^*$.

- As can be seen, the optimal pension with recursive utility is being "smoothened" in the same manner as the optimal consumption, summarized in Theorem 1.

- A positive shock to the economy via the term B_t increases the optimal pension benefits via the term ηB_t, which may be mitigated, or strenghtened, by the term $(\rho - \gamma)\sigma_V B_t$, depending on its sign.

- When $(\gamma > \rho)$, then $\sigma_V(t) > 0$ and shocks to the economy are smoothened in the optimal pension with RU.

- This indicates that the pensioner in this model can be considerably more sophisticated than the one modeled in the conventional way when $\rho = \gamma$. We summarize as follows:

- **Theorem 2** *Under the same assumptions as in Theorem 1, the individual with recursive utility will prefer a pension plan that smoothens market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

- This points in the direction of DB-pension plan rather than a DC-plan ($\gamma > \rho$).

- When $\rho > \gamma$ the agent has preference for late resolution of uncertainty, and the opposite conclusion may follow, depending on the sign of $\sigma_V(t)$.

10. Optimal investment with RU

- Consider an agent with recursive utility who takes the market as given. In this setting we now discuss optimal portfolio choice.

- At the beginning or each period, the agent allocates a certain proportion of wealth to immediate consumption, and then invests the remaining amount in the available securities for future consumption. Accordingly, the the optimal portfolio choice will depend on consumption.

- The growth rate of consumption, and its conditional variance, is known once consumption is determined. This is achieved by the agent using his/her preferences faced with the various market opportunities.

- **Theorem 3** *The optimal portfolio fractions in the risky assets are given by*

$$\varphi(t) = \frac{1-\rho}{\gamma-\rho}(\sigma_t \sigma_t')^{-1}\nu_t - \frac{\rho(1-\gamma)}{\gamma-\rho}(\sigma_t\sigma_t')^{-1}(\sigma_t \sigma_c^*(t)). \tag{24}$$

assuming $\gamma \neq \rho$.

- The volatility of the optimal consumption growth rate depends on preferences and market quantities only:

$$\sigma_{c^*}(t) = \frac{1}{\rho}\Big(\eta_t + (\rho-\gamma)\sigma_V(t)\Big). \tag{25}$$

- The agent first determines the optimal consumption growth rate and then the optimal portfolio choice in such a manner that the relationship (24) holds.

- The above formula answers the question of the insurance industry how to invest in order to satisfy the recursive utility pensioner.

- The optimal fractions with recursive utility depend on both risk aversion and time preference as well as the volatility σ_{c^*} of the optimal consumption growth rate dc^*_t/c^*_t of the agent.

- To illustrate, consider the standard situation with one risky and one risk-free asset, letting the S&P-500 index be the risky security.

- Consider the market data of Table 1 with the S&P-500 index playing the role of the risky asset, with the associated estimate of $\sigma_{c^*} = 0.0355$.

- The recursive model explains an average of 13% in risky securities for the following parameter values $\gamma = 2.6$ and $\rho = .90$.

- Given participation in the stock market, when $\varphi = .40$, this is consistent with $\gamma = 2.2$ and $\rho = .76$.

- If $\varphi = .60$, this can correspond to $\gamma = 2.0$ and $\rho = .66$, etc., a potential resolution of this puzzle.

- Most people invest in other risky assets than merely stocks. To capture this, consider instead the wealth portfolio. It has a lower variance that the market portfolio. Assume this to be 10 per cent (compared to 15 per cent for the market portfolio).

- Also suppose the growth rate of the wealth portfolio is lower than the growth rate of the market portfolio: .035 vs. .070, and the correlation coefficient between the wealth portfolio and the market portfolio is $\kappa_{W,M} = .30$.

- The recursive model explains an average of 13% in risky securities for the following parameter values $\gamma = 2.6$ and $\rho = .92$.

- Given participation in the stock market, when $\varphi = .40$, this is consistent with $\gamma = 2.2$ and $\rho = .79$.

- If $\varphi = .60$, this can correspond to $\gamma = 2.0$ and $\rho = .71$.

- In addition to the insurance industry, other interesting applications would be to management of funds that invests public wealth to the benefits of the citizens of a country, or the members of a society.

- If this country/society is large enough for an estimate of the volatility of the consumption growth rate of the group to be available, the application becomes particularly simple, as the above exercise shows.

- One such example is the Norwegian Government Pension Fund Global (formerly the Norwegian Petroleum Fund). Other net oil exporting countries have similar funds.

- See Aase, K. K. (2016). Life Insurance and Pension Contracts II: The life cycle model with recursive utility. *ASTIN Bulletin*, Volume 46(1), pp 71-102.

10.1. Mutual fund theory

- Separation results in continuous-time models exist, e.g., Kanna and Kulldorff (1999), which extend the Ross (1978) separation results in one period models.

- Our result in Theorem 3 indicates that the general results for additive expected utility do not hold for non-expected utility (perhaps, not surprising).

- From Theorem 3 we can perhaps view the two terms $(\sigma_t \sigma_t')^{-1} \nu_t$ and $(\sigma_t \sigma_t')^{-1} \sigma_t$ as two distinct 'mutual funds', and develop a similar theory as for the class of additive and separarble utility functions.

- Thus the standard separation results in continuous time are not as general as the authors sometimes want us to believe.

11. Equilibrium

- We may turn the life cycle model around and consider equilibrium as well.

- Without going into details, the expression for the risk premium of any risky security with return rate $\mu_R(t)$ is given by the following formula

-
$$\mu_R(t) - r_t = \rho \sigma_c'(t) \sigma_R(t) + (\gamma - \rho) \sigma_V'(t) \sigma_R(t). \quad (26)$$

- Provided a representative agent equilibrium exists, in equilibrium
$$\sigma_W(t) = (1 - \rho) \sigma_V(t) + \rho \sigma_c(t) \quad (27)$$

- This relationship determines the volatility of the wealth portfolio in terms of primitives of the model.

- These are the preferences (represented by $\sigma_V(t)$ and ρ) and the volatility of the growth rate of aggregate consumption ($\sigma_c(t)$), now taken as given.

- Recall: In the Lucas (1978) model prices are determined in equilibrium such the "representative agent" optimally consumes the aggregate endowment. In this perspective $\sigma_c(t)$ becomes exogenously given.

- Similarly, the equilibrium risk-free interest rate in terms of $\sigma_V(t)$ is

$$r_t = \delta + \rho\, \mu_c(t) - \frac{1}{2}\rho(1+\rho)\, \sigma_c' \sigma_c - \rho(\gamma - \rho)\sigma_c'(t)\sigma_V(t) - \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma_V'(t)\sigma_V(t). \quad (28)$$

- The final step is to turn equation (27) around and use the resulting expression for $\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_c(t))$ in these two formulas.

- The result for the risk premium of any risky security is

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\sigma_c'(t)\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\sigma_W'(t)\sigma_R(t). \tag{29}$$

- The first term on the right hand side corresponds to the consumption based CAPM of Breeden (1979), while the second term corresponds to the market based CAPM of Mossin (1966).

- The latter is only valid in a "timeless" setting, i.e., a one period model with consumption only on the terminal time, in its original derivation.

- The result for the equilibrium spot rate is

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\frac{\rho-\gamma}{1-\rho}\,\sigma_W'(t)\sigma_W(t). \quad (30)$$

- The two first terms on the right-hand side are the familiar terms in the Ramsay (1928)-model (a deterministic model).

- The third term corresponds to the "precautionary savings term" in the conventional model:

$$\frac{1}{2}\rho(1+\rho)\,\sigma_c'(t)\sigma_c(t).$$

  If $\rho = \gamma$ in (30), this expression results.

- The last term in (30) is new, and comes from the recursive specification of utility.

- When $\gamma > \rho$ and $\rho < 1$ this term is negative. When the wealth uncertainty increases, the "prudent" RU maximizer saves, and the interest rate falls.

- This helps explaining the "low" real interest rate observed during the 90-year period of the data.

- Consider a calibration to the data summarized in Table 1.

- By fixing the impatience rate δ to some reasonable value, $\delta = 0.03$ say, one solution to the two equations (29) and (30) with $R = M$ (M is the market portfolio) and $W = M$, this yields $\gamma = 1.74$ and $\rho = 0.48$.

- Using $R = M$ and another proxy for the wealth portfolio instead, for example with $\sigma_W(t) = .10$ and with an instantaneous correlation with the market portfolio M, $\kappa_{W,M} = .80$, we obtain for $\delta = 0.02$ that $\gamma = 2.11$ and $\rho = 0.74$.

- The resulting preference parameters seem plausible, and many other reasonable combinations fit the equations as well.

- In contrast, a similar calibration of the expected utility model leads to the (unique) values $\gamma = 26$ and $\delta = -.015$. This is the equity premium puzzle.

- See Aase, K. K (2016): Recursive utility using the stochastic maximum principle. *Quantitative Economics*. To appear. It is listed on http://qeconomics.org/ under Papers to appear.

- We present the results of the Norwegian economy. This is a relatively small, open economy in which the central statistical agent, Statistisk sentralbyrå, has provided us with the data needed, also related to the wealth portfolio (from 1985 to 2013).

- Table 6 contains the data corresponding to Table 2, which was organized by Hjetland (2015).

	Expectat.	Standard dev.	Covariances
Consumption growth	1.794%	1.390%	$\text{cov}(M, c) = .00078684$
Return OBX	10.70%	32.025%	$\text{cov}(M, b) = .00180603$
Government bills	2.141%	3.618%	$\text{cov}(c, b) = 1.0873E\text{-}05$
Equity premium	8.559%	31.703%	

Table 3: Key Norwegian-data for the time period 1971-2014.

- The estimates provided by Statistisk sentralbyrå (2014) are restricted to include capital that is measurable in units of account: (i) human capital; (ii) real capital; (iii) financial capital (including the Sovereign Pension Fund of Norway); (iv) natural resources. For the whole period 72-75 per cent of the national wealth can be attributed to human capital.

- The estimates related to the growth rate of the wealth portfolio (log terms) are as follows: $\sigma_W = .01849$, $\mu_W = .0219$, $\sigma_{W,M} = .00142$, $\sigma_{W,c} = .000127$. Below only the first and the third estimate are needed.

- The data on the wealth portfolio naturally represents a challenge to collect, and is associated with a fair amount of uncertainty; the presented estimates still gives a good indication of the national wealth. This gives us the calibrations of Table 7.

Parameters	γ	ρ	EIS	δ
$\gamma = 0.50$	0.50	1.004	.997	.013
$\gamma = 1.50$	1.50	.996	1.004	.013
$\gamma = 2.00$	2.00	.992	1.008	.013
$\gamma = 2.50$	2.50	.989	1.011	.013
$\gamma = 3.00$	3.00	.985	1.015	.014
$\gamma = 3.50$	3.50	.981	1.019	.014
$\gamma = 4.00$	4.00	.977	1.023	.014
$\gamma = 5.00$	5.00	.969	1.032	.014

Table 4: Calibrations of the recursive model to the Norwegian economy.

- The parameter estimates are reasonable over most of the range shown. When $\gamma = 0$, $\delta = 0.0128$ and $\rho = 1.0075$. When $\gamma = 25$, $\delta = .019$ and $\rho = .77$. This indicates a time preference $\rho \in (0.8, 1.0)$ and an impatience rate $\delta \in (0.012, 0.020)$.

- Thus relatively large variations in γ are associated with relatively small variations in the other two parameters.

- In conclusion, this indicates that the average Norwegian is reasonably patient, has an EIS just above 1 and has a relative risk aversion within a reasonable range.

- This is in accordance with Dagsvik et.al. (2006), who estimate EIS to be between 1 and 1.5 for the Norwegian population.

- The expected utility model calibrates to $\delta = -0.776$ and $\gamma = \rho = 108.78$ for this data set.

- The assumption that the economy is closed is restrictive; imposes that consumption equals domestic output. If exports and imports balance, this could still be reasonable.