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Measuring the Value of Risk Cost Models

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Introduction I

Following innovations in machine learning and computational statistics a large variety of new modeling techniques are being applied to premium rating.

In order to carry out model comparison and selection in this regime it is particularly useful to develop metrics that allow us to evaluate predictive power without the knowledge of models' internal structure.

Common diagnostics used today include (Berry et al., 2009; Goldburd et al., 2016):

- calibration plots,
- quantile charts,
- double lift or loss ratio plots,
- Lorenz curves and the Gini index.

Introduction II

The relationships between these tools and the potential economic value of the models are not necessarily well understood (Meyers, 2008; Meyers and Cummings, 2009).

In this talk we establish a precise connection between the traditional diagnostics and the economic value then take advantage of the resulting intuition to motivate a new family of model-agnostic evaluation metrics.

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Single Period Optimal Pricing I

To illustrate the economic rationale for the Gini index and related diagnostics, we first consider a simple model of single period optimal pricing (e.g. Talluri and van Ryzin, 2004).

We seek to maximise the total profit objective for a cohort of n policies subject to a constraint on the minimum retention level D, where for the i-th policy with risk characteristics $\mathbf{x}_i$ the proposed premium is denoted p_i, the expected demand $d_i(p_i, \mathbf{x}_i)$ is a function of premium and $c(\mathbf{x}_i)$ corresponds to the expected cost of claims:

$$\begin{aligned} \underset{p_1,\dots,p_n}{\text{maximise}} \quad & \sum_{i=1}^{n} \big(p_i - c(\mathbf{x}_i)\big) d(p_i, \mathbf{x}_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} d_i(p_i, \mathbf{x}_i) \geq D. \end{aligned}$$

Here the decision variables are premiums $p_i \geq 0$.

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Single Period Optimal Pricing II

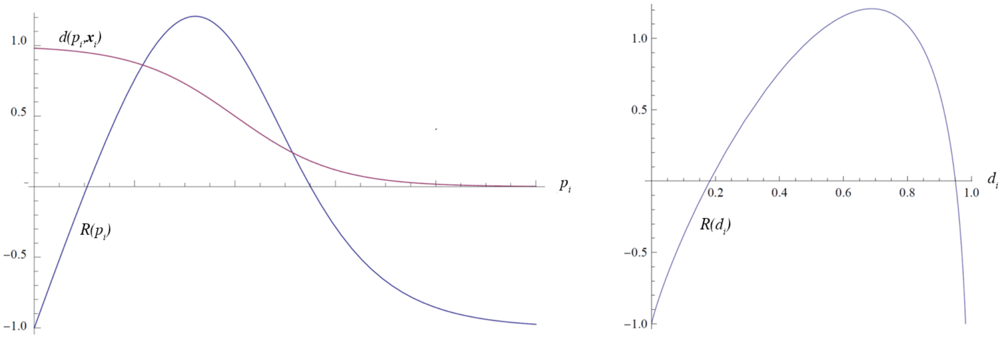

Figure 1: *Left:* Logistic demand $d(p_i, \mathbf{x}_i)$ and revenue $R(p_i) = \big(p_i - c(\mathbf{x}_i)\big)d(p_i, \mathbf{x}_i)$ as functions of price p. *Right:* Revenue $R(d_i) = \big(p(d_i, \mathbf{x}_i) - c(\mathbf{x}_i)\big)d_i$ as a function of demand d_i is concave for $d_i \in [0, 1)$.

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Single Period Optimal Pricing III

We can rewrite the same problem using policy demand as the decision variable, assuming one-to-one correspondence between premium and demand $p(d_i, \mathbf{x}_i) = d^{-1}(d_i, \mathbf{x}_i)$:

$$\begin{aligned} \underset{d_1,\ldots,d_n}{\text{maximise}} \quad & \sum_{i=1}^{n} \big(p(d_i, \mathbf{x}_i) - c(\mathbf{x}_i)\big) d_i = R(d_1, \ldots, d_n) \\ \text{subject to} \quad & \sum_{i=1}^{n} d_i \geq D, \end{aligned}$$

where $R(d_1, \ldots, d_n)$ denotes the total profit over the single period.

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Single Period Optimal Pricing IV

We can then formulate the Lagrangian:

$$L(d_1, \ldots, d_n, \lambda) = \sum_{i=1}^{n} \big(p(d_i, \mathbf{x}_i) - c(\mathbf{x}_i)\big) d_i + \lambda \Big(\sum_{i=1}^{n} d_i - D \Big)$$

and write the first order optimality conditions as:

$$\frac{\partial L}{\partial d_i} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \lambda \geq 0.$$

Observe that:

$$\frac{\partial L}{\partial d_i} = \frac{\partial R}{\partial d_i} + \lambda$$

and that therefore if the portfolio is priced optimally, marginal profit with respect to demand for each policy is constant:

$$\frac{\partial R}{\partial d_i} = -\lambda^*.$$

Single Period Optimal Pricing V

This condition is intuitive – should $\frac{\partial R}{\partial d_i} \neq \frac{\partial R}{\partial d_j}$ for some i and j, we can reallocate demand between contracts i and j in such a way as to increase total profit.

Finally, let us express $\frac{\partial R}{\partial d_i}$ as a function of premium p_i and $\epsilon_i = \frac{\partial d_i}{\partial p_i} \frac{p_i}{d_i}$, the price elasticity of demand:

$$\begin{aligned} \frac{\partial R}{\partial d_i} &= \frac{\partial p_i}{\partial d_i} d_i + p_i - c(\mathbf{x}_i) \\ &= \left(\frac{\partial d_i}{\partial p_i} \frac{p_i}{d_i} \right)^{-1} p_i + p_i - c(\mathbf{x}_i) \\ &= p_i \left(1 + \frac{1}{\epsilon_i} \right) - c(\mathbf{x}_i). \end{aligned}$$

We will use this representation several times in the rest of the paper.

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Model Diagnostics I

We now remind of the four common diagnostics that are used to evaluate risk cost models – the calibration plot, the quantile chart the Lorenz curve and the Gini index.

The calibration plot compares model predictions $c(X)$ with the claims outcome Y:

$$\begin{aligned} x_{\text{Calibration}}(t) &= t, \\ y_{\text{Calibration}}(t) &= \mathbb{E}\big[Y \,|\, c(X) = t\big]. \end{aligned}$$

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Model Diagnostics II

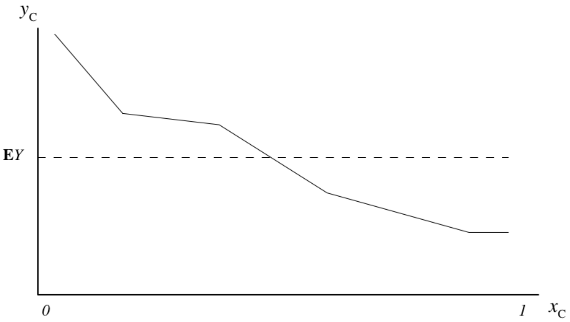

Figure 2: A quantile chart. The x axis denotes the proportion of total exposure, in descending order by model predictions $\mathbb{E}[\mathbb{I}(c(X) \geq t)]$, and the y axis represents the corresponding expected cost of claims $\mathbb{E}[Y \mid c(X) = t]$.

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Model Diagnostics III

The quantile chart is a rescaling of the calibration plot along the x axis to correspond to the proportion of policies with $c(X) \geq t$:

$$
\begin{aligned}
x_{\text{Quantile}}(t) &= \mathbb{E}\big[\mathbb{I}\big(c(X) \geq t\big)\big] \\
&= \Pr\big(c(X) \geq t\big) = S(t), \\
y_{\text{Quantile}}(t) &= \mathbb{E}\big[Y \mid c(X) = t\big].
\end{aligned}
$$

Finite sample approximations are used in practice, e.g. the so called “decile plot”.

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Model Diagnostics IV

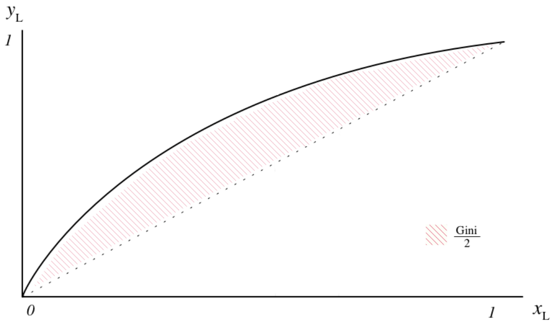

Figure 3: A Lorenz curve. The x axis denotes the proportion of total exposure, in descending order by model predictions $\mathbb{E}[\mathbb{I}(c(X) \geq t)]$, and the y axis represents the corresponding proportion of expected claims $\mathbb{E}[Y\mathbb{I}(c(X) \geq t)]/\mathbb{E}[Y]$. The Gini index $G = 2\int_0^1 y_{\text{Lorenz}} - x_{\text{Lorenz}}\, dx_{\text{Lorenz}}$ is twice the area between the Lorenz curve and line $y = x$.

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Model Diagnostics V

Define the Lorenz curve (also "lift curve") with respect to a threshold parameter t as follows:

$$
\begin{aligned}
x_{\text{Lorenz}}(t) &= S(t), \\
y_{\text{Lorenz}}(t) &= \frac{\mathbb{E}\left[Y \mathbb{I}\left(c(X) \geq t\right)\right]}{\mathbb{E}[Y]}.
\end{aligned}
$$

In the above, Y corresponds to the cost of claims, X to the risk characteristics and $c(X)$ to the predictions of the risk cost model being evaluated. Expectations are with respect to the joint distribution of (Y, X).

Model Diagnostics VI

In practice, the curve is approximated using data held out from model estimation $\{(y_i, e_i, \mathbf{x}_i)\}_{i=1}^n$ with e_i denoting per observation exposure:

$$\hat{x}_{\text{Lorenz}}(t) \;=\; \frac{\sum_{i=1}^n e_i \mathbb{I}\big(c(\mathbf{x}_i) \geq t\big)}{\sum_{i=1}^n e_i},$$

$$\hat{y}_{\text{Lorenz}}(t) \;=\; \frac{\sum_{i=1}^n y_i \mathbb{I}\big(c(\mathbf{x}_i) \geq t\big)}{\sum_{i=1}^n y_i}.$$

The Gini index is then commonly defined with reference to the Lorenz curve:

$$G \;= 2\int_0^1 y_{\text{Lorenz}}(t) - x_{\text{Lorenz}}(t)\, dS(t).$$

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Model Diagnostics VII

We show that the Lorenz curve is the *integral* of the corresponding quantile plot, scaled by the average cost of claims per policy $\mathbb{E}[Y]$:

$$\begin{aligned}
\int_0^{S(u)} y_{\text{Quantile}}\, dx_{\text{Quantile}} &= \int_0^{S(u)} \mathbb{E}\big[Y \mid c(X) = t\big]\, dS(t) \\
&= \int_\infty^{u} \mathbb{E}\big[Y \mid c(X) = t\big] \frac{dS}{dt}\, dt \\
&= \int_u^{\infty} \mathbb{E}\big[Y \mid c(X) = t\big] \Pr\big(c(X) = t\big)\, dt \\
&= \mathbb{E}\Big[Y \mathbb{I}\big(c(X) \geq u\big)\Big] \\
&= y_{\text{Lorenz}}(u)\, \mathbb{E}[Y].
\end{aligned}$$

Economic interpretations I

We now combine results from the two previous sections to gain an economic intuition for the diagnostic metrics.

First observe that if we take $\epsilon_i = -1$, then the equation $\frac{\partial R}{\partial d_i} = p_i\big(1 + \frac{1}{\epsilon_i}\big) - c(\mathbf{x}_i)$ reduces to:

$$\frac{\partial R}{\partial d_i} = -c(\mathbf{x}_i),$$

allowing the interpretation of the quantile plot in terms of marginal profit – namely we can take the values on the y axis to represent *actual* (measured using realised claims experience Y) negative marginal profit with respect to demand for that segment.

Unless the graph is perfectly flat, the optimality condition of constant marginal profit is not satisfied and we can improve portfolio performance by rebalancing demand through price adjustments.

Economic interpretations II

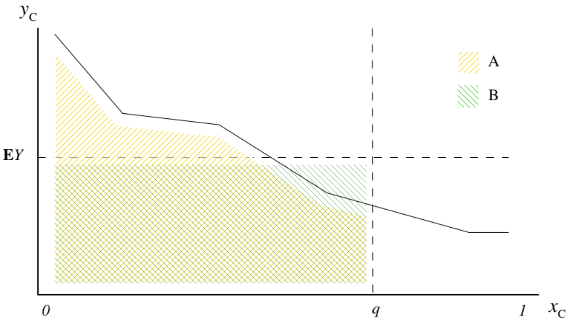

Figure 4: Quantile plot. Under assumption of $\epsilon = -1$, the area $A - B = \mathbb{E}\left[\frac{\partial R}{\partial d}\mathbb{I}\left(c(X) \geq S^{-1}(q)\right)\right] - q\mathbb{E}\left[\frac{\partial R}{\partial d}\right]$ represents the economic gain from a demand neutral price change where we raise prices for q of total policies with highest absolute marginal profit with respect to demand as to forgo q units of demand and then offset that loss of demand through a price reduction for the entire portfolio, gaining q units of demand. Here $\mathbb{E}(Y)$ corresponds to the average cost of claims per policy.

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Economic interpretations III

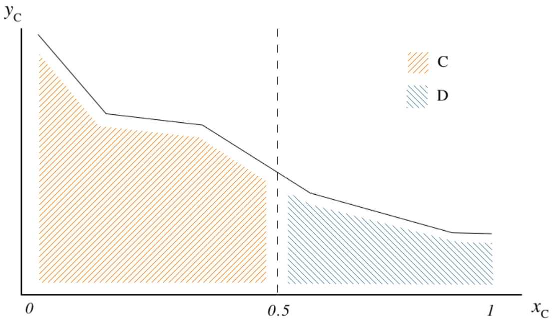

Figure 5: Quantile plot. Area $C - D = \mathbb{E}\big[\frac{\partial R}{\partial d}\mathbb{I}\big(c(X) \geq S^{-1}(0.5)\big)\big] - \mathbb{E}\big[\frac{\partial R}{\partial d}\mathbb{I}\big(c(X) \leq S^{-1}(0.5)\big)\big]$ corresponds to the economic value of a price adjustment where we increase premiums for policies in C and simultaneously reduce premiums for policies in D so as to effect offsetting demand changes of 0.5 and -0.5 units respectively.

Economic interpretations IV

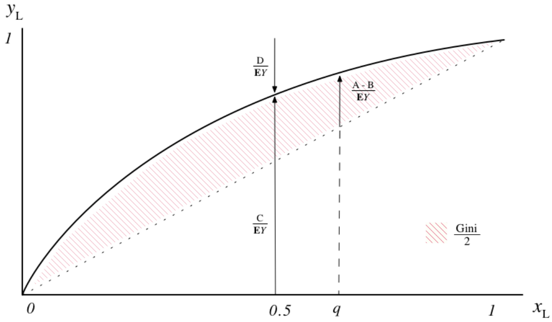

Figure 6: Lorenz Curve and the Gini index. We observe that the area $A - B$ from Figure 4 corresponds to the distance between the Lorenz curve and the line $y = x$ up to constant $\mathbb{E}[Y]$. The Gini coefficient is equal to twice the area between the Lorenz curve and the same line and can be said to represent the average economic value of price change decisions of type shown in Figure 4 as we vary the threshold q, scaled by the constant $\mathbb{E}[\mathbb{Y}]/2$.

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Marginal profit plots I

We can now define a new family of models diagnostics, parametrised through the choice of elasticity assumption ϵ.

The marginal profit plot (a generalisation of the quantile plot) is given by:

$$\begin{aligned} x^*_{\text{Quantile}}(t, \epsilon) &= \mathbb{E}\left[\mathbb{I}\left(c(X) - p(X)\left(1 + \frac{1}{\epsilon}\right) \geq t\right)\right] \\ &= \Pr\left(c(X) - p(X)\left(1 + \frac{1}{\epsilon}\right) \geq t\right) = S_\epsilon(t), \\ y^*_{\text{Quantile}}(t, \epsilon) &= \mathbb{E}\left[Y - p(X)\left(1 + \frac{1}{\epsilon}\right) \;\middle|\; c(X) - p(X)\left(1 + \frac{1}{\epsilon}\right) = t\right]. \end{aligned}$$

Note that we have introduced a new quantity $p(X)$, corresponding to the current premiums.

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Marginal profit plots II

We can define the associated marginal profit Lorenz curve and the marginal profit Gini index as follows:

$$
\begin{aligned}
x^*_{\text{Lorenz}}(t,\epsilon) &= S_\epsilon(t), \\
y^*_{\text{Lorenz}}(t,\epsilon) &= \int_0^{S_\epsilon(t)} y^*_{\text{Quantile}}\, dx^*_{\text{Quantile}} \\
&= \mathbb{E}\left[\mathbb{I}\left(c(X) - p(X)\left(1+\frac{1}{\epsilon}\right) \geq t\right)\left(Y - p(X)\left(1+\frac{1}{\epsilon}\right)\right)\right], \\
G^*(\epsilon) &= 2\int_0^1 y^*_{\text{Lorenz}}(t) - \mathbb{E}[Y]\, x^*_{\text{Lorenz}}(t)\ \ dS_\epsilon(t).
\end{aligned}
$$

We chose not to rescale y^*_{Lorenz} by $\mathbb{E}[Y]$. This is due to the difficulties with the definition of the Lorenz curve and associated quantities in situations where negative measurements are allowed

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Marginal profit plots III

(e.g. consider the case when $\mathbb{E}[Y] = 0$). This issue does not arise if we instead adopt unscaled "generalised" Lorenz curve (Shorrocks, 1983).

With the choice of $\epsilon = -1$ we recover the standard definitions up to the constant $\mathbb{E}[Y]$. As we let $\epsilon \to -\infty$, we have an analogue of the so called "loss ratio chart", a plot comparing expected vs. actual loss ratios, but defined for the dollar margin $p(X) - c(X)$, rather than the ratio $\frac{c(X)}{p(X)}$.

All of the economic interpretations also apply for the marginal profit and related plots. It can be particularly informative to compare $G^*(\epsilon)$ values for the candidate models across a realistic range of elasticities.

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Marginal profit plots IV

Finally we observe that it is possible to use per observation elasticity values $\hat{\epsilon}_i$, however some care needs to be taken in this situation as the resulting statistics can be quite sensitive to the predictive uncertainty in the estimates $\hat{\epsilon}_i$. We have addressed this setting in more depth in a separate publication (Semenovich and Petterson, 2019).

Conclusion I

In this presentation we have:

- reviewed first oder optimality conditions for the single period optimal pricing problem,
- demostrated the connection between quantile charts and the Lorentz curve,
- given a reinterpretation of the standard risk cost model diagnostics in terms of marginal profit,
- proposed a new family of metrics based on economic principles.

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Questions

Comments