Institute and Faculty of Actuaries

CHANGES TO THE SYLLABUS AND CORE READING FOR SUBJECT CT8 FOR THE 2018 EXAMINATIONS

Changes to the Syllabus and their impact on Core Reading

Syllabus objective (ix) 1.

This has been clarified to read:

State what is meant by arbitrage.

Changes to Core Reading

UNIT 3

Minor amendments have been made to this Unit and a revised Unit is attached.

UNIT 9

Amendments have been made to this Unit and a revised Unit is attached.

The other changes that have been made to the Core Reading are to correct typographical errors and improve the style.

Attachments: Unit 3 and 9

UNIT 3 — PORTFOLIO THEORY

Syllabus objectives

(iii) Describe and discuss the assumptions of mean-variance portfolio theory and its principal results.

1. Describe and discuss the assumptions of mean-variance portfolio theory.

2. Discuss the conditions under which application of mean-variance portfolio theory leads to the selection of an optimum portfolio.

3. Calculate the expected return and risk of a portfolio of many risky assets, given the expected return, variance and covariance of returns of the individual assets, using mean-variance portfolio theory.

4. Explain the benefits of diversification using mean-variance portfolio theory.

Some sections of this Unit have been adapted from lecture notes originally written by David Wilkie.

1 Portfolio theory

1.1 Introduction

Mean-variance portfolio theory, sometimes called modern portfolio theory (MPT), specifies a method for an investor to construct a portfolio that gives the maximum return for a specified risk, or the minimum risk for a specified return. However, the theory relies on some strong and limiting assumptions about the properties of portfolios that are important to investors. In the form described here the theory ignores the investor's liabilities, although it is possible to extend the analysis to include them.

The application of the mean-variance framework to portfolio selection falls conceptually into two parts. First, the definition of the properties of the portfolios available to the investor — the opportunity set. Second, the determination of how the investor chooses one out of all the feasible portfolios in the opportunity set.

1.2 Specification of the opportunity set

In specifying the opportunity set it is necessary to make some assumptions about how investors make decisions. Then the properties of portfolios can be specified in terms of relevant characteristics. It is assumed that investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon. Thus all the relevant properties of a portfolio can be specified with just two numbers — the mean return and the variance of the return. The variance (or standard deviation) is known as the risk of the portfolio.

To calculate the mean and variance of return for a portfolio it is necessary to know the expected return on each individual security and also the variance/covariance matrix for the available universe of securities.

1.3 Efficient portfolios

Two further assumptions about investor behaviour allow the definition of **efficient portfolios**.

The assumptions are:

(i) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(ii) Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance.

Once the set of efficient portfolios has been identified, all others can be ignored.

A portfolio is inefficient if the investor can find another portfolio with the same expected return and lower variance, or the same variance and higher expected return. A portfolio is efficient if the investor cannot find a better one in the sense that it has both a higher expected return and a lower variance. However, an investor may be able to rank efficient portfolios by using a utility function, as shown in Section 1.4 below.

Suppose an investor can invest in any of the N securities, $i = 1, ..., N$. A proportion x_i is invested in security S_i. The return on the portfolio R_P is

$$R_P = \Sigma_i \, x_i.R_i ,$$

where R_i is the return on security i.

The expected return on the portfolio is

$$E = E[R_P] = \Sigma_i \, x_i .E_i ,$$

where E_i is the expected return on security i.

The variance is

$$V = \text{Var}[R_P] = \Sigma_i \, \Sigma_j \, x_i .x_j .C_{ij} ,$$

where C_{ij} is the covariance of the returns on securities i and j and we write $C_{ii} = V_i$.

If there are just two securities, S_A and S_B , the above expressions reduce to:

$$E = x_A .E_A + x_B .E_B ,$$

and

$$V = x_A^2 V_A + x_B^2 V_B + 2\,x_A\,x_B\,C_{AB}\ .$$

As the proportion invested in S_A is varied, a curve is traced in E-V space.

The minimum variance can easily be shown to occur when:

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}\ .$$

As an example, consider the case where:

$E_A = 4\%$	$V_A = 4\%\%$	$(\sigma_A = 2\%)$
$E_B = 8\%$	$V_B = 36\%\%$	$(\sigma_B = 6\%)$

We now let the covariance between the two securities vary by considering C_{AB} equal to –0.75, 0, and +0.75 in turn. The results are plotted in Figure 1, where the vertical axis represents expected values of return and the horizontal axis represents standard deviation of return. In this space $(E - \sigma)$ the curves representing possible portfolios of the two securities are hyperbolae. It is possible to plot the same results in $E - V$ space, where the lines would be parabolae.

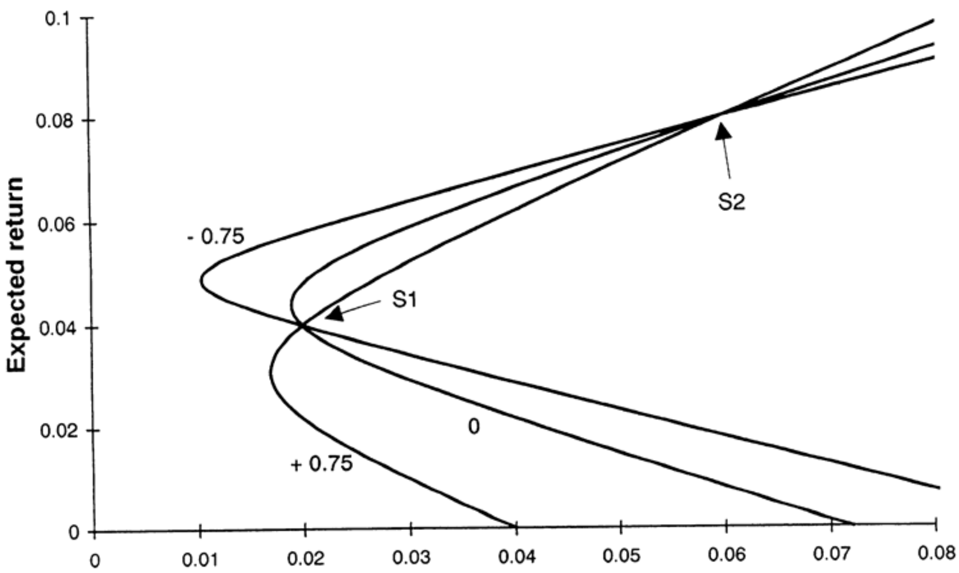

Figure 1

Figure 2 shows combinations of securities with correlation coefficients of +1, 0 and –1. For coefficients of +1 and –1, it is possible to obtain risk-free portfolios with zero standard deviation of return.

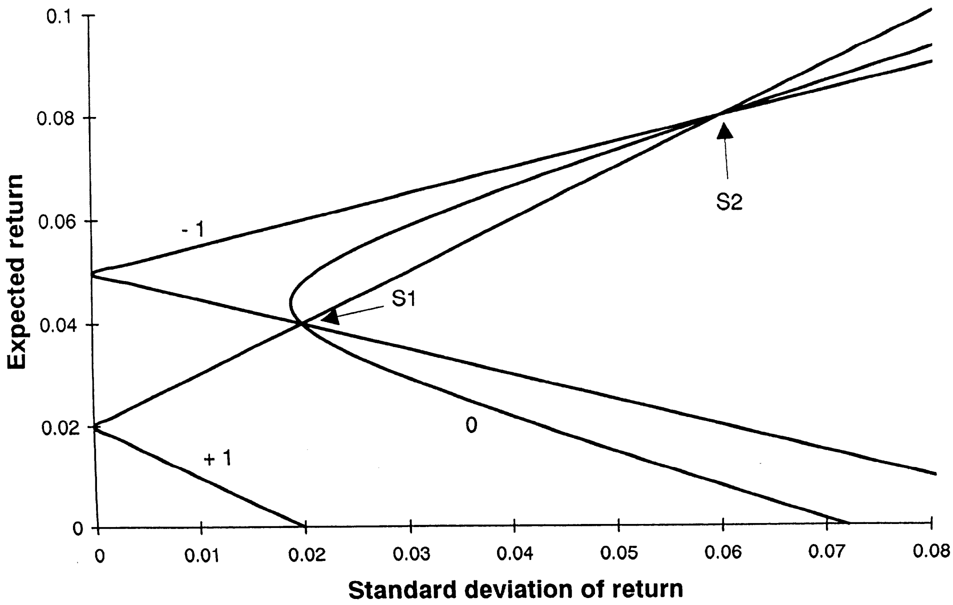

Figure 2

When there are N securities, the aim is to choose x_i to minimise V subject to the constraints

$$\Sigma_i x_i = 1$$

and

$$E = E_P, \text{ say,}$$

in order to plot the minimum variance curve.

One way of solving such a minimisation problem is the method of Lagrangian multipliers.

The Lagrangian function is

$$W = V - \lambda(E - E_P) - \mu(\Sigma_i x_i - 1).$$

To find the minimum, we set the partial derivatives of W with respect to all the x_i and λ and μ equal to zero. The result is a set of linear equations that can be solved.

The partial derivative of W with respect to x_i is:

$$\partial W/\partial x_i = 2\Sigma_j C_{ij} x_j - \lambda E_i - \mu$$

With respect to λ it is:

$$\partial W/\partial\lambda = -(\Sigma_i E_i x_i - E_P)$$

And with respect to μ it is:

$$\partial W/\partial\mu = -(\Sigma_i x_i - 1)$$

Setting each of these to zero gives:

$2\Sigma_j C_{ij} x_j - \lambda E_i - \mu = 0$ (one equation for each of n securities)

$$\Sigma_i E_i x_i = E_P$$

$$\Sigma_i x_i = 1$$

We now generalise to any E and V. The solution to the problem shows that the minimum variance V is a quadratic in E and each x_i is linear in E.

The usual way of representing the results of the above calculations is by plotting the minimum standard deviation for each value of E_P as a curve in expected return - standard deviation (E - σ) space. In this space, with expected return on the vertical axis, the efficient frontier is the part of the curve lying above the point of the global minimum of standard deviation.

All other possible portfolios are inefficient. In fact, it can be shown that normality of returns is not a necessary condition for the selection of optimal portfolios. There is a more general class of distributions called the elliptically symmetrical family which also result in optimality. All the distributions in this class have the property that the higher order moments can be expressed in terms of just their mean and variance.

1.4 Choosing an efficient portfolio

A series of indifference curves (curves which join all outcomes of equal utility) can be plotted in expected return – standard deviation space.

Portfolios lying along a single curve all give the same value of expected utility and so the investor would be indifferent between them.

Utility is maximised by choosing the portfolio on the efficient frontier at the point where the frontier is at a tangent to an indifference curve.

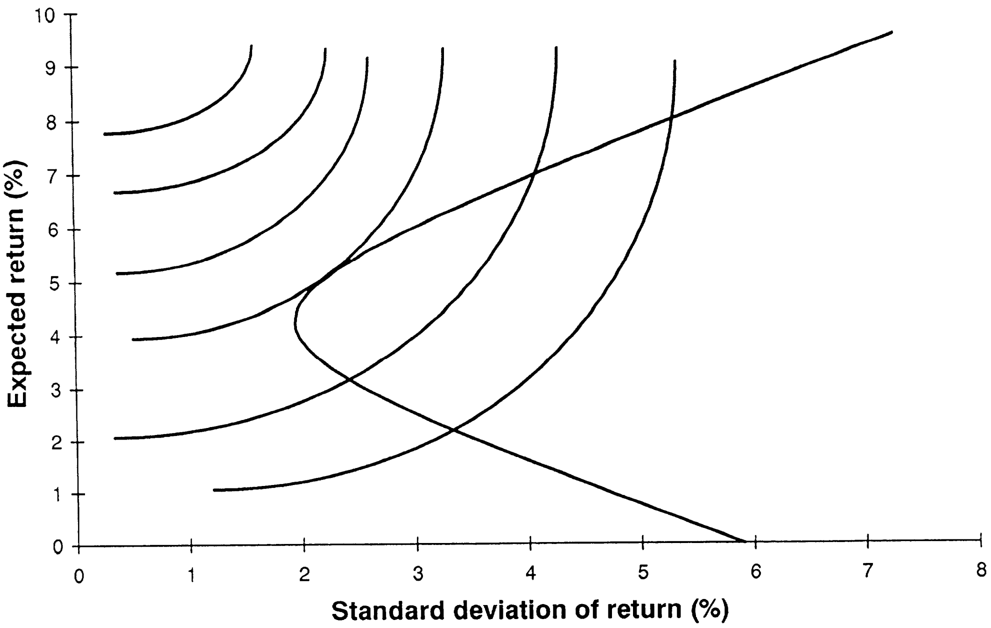

Figure 3

For quadratic utility functions the process described above produces optimal portfolios whatever the distribution of returns, because expected utility is uniquely determined if we know the mean and variance of the distribution.

If it is felt that the assumptions leading to a two-dimensional mean-variance type portfolio selection model are inappropriate, it is possible to construct models with higher dimensions. For example, skewness could be used in addition to expected return and a dispersion measure. It would then be necessary to consider an efficient *surface* in three dimensions rather than an efficient frontier in two. Clearly, the technique can be extended to more than three dimensions.

Although such models have been constructed, they do not appear to be widely used. It is doubtful whether the additional mathematical complexity, input data requirements and difficulty of interpretation in a pragmatic way, are compensated by real improvements in value added.

2 Benefits of diversification

The expression for the variance of the portfolio can be rewritten as:

$$V = \Sigma_i \; x_i^2 \; V_i + \Sigma_i \, \Sigma_{j,j \neq i} \, x_i . x_j . C_{ij}$$

Where all assets are independent, the covariance between them is zero and the formula for variance becomes:

$$V = \Sigma_i \, x_i^2 \; V_i$$

If we assume that equal amounts are invested in each asset, then with N assets the proportion invested in each is $1/N$. Thus:

$$V = \Sigma_i \, (1/N)^2 \; V_i \; = 1/N \, [\Sigma_i \; V_i /N] \; = 1/N \; \overline{V}$$

where $\overline{V}$ represents the average variance of the stocks in the portfolio. As N gets larger and larger, the variance of the portfolio approaches zero. This is a general result — if we have enough *independent* assets, the variance of a portfolio of these assets approaches zero.

In general, we are not so fortunate. In most markets, the correlation coefficient and the covariance between assets is positive. In these markets, the risk on the portfolio cannot be made to go to zero, but can be much less than the variance of an individual asset. With equal investment, the proportion invested in any one asset x_i is $1/N$ and the formula for the variance of the portfolio becomes

$$V = \Sigma_i \; (1/N)^2 \; V_i + \Sigma_i \, \Sigma_{j,j \neq i} \, (1/N)(1/N).C_{ij}$$

Factoring out $1/N$ from the first summation and $(N - 1)/N$ from the second gives:

$$V = 1/N \, \Sigma_i \; V_i /N + (N - 1)/N \, \Sigma_i \, \Sigma_{j,j \neq i} \; C_{ij} /N(N - 1)$$

Replacing the summation by averages, we have

$$V = 1/N.\overline{V} \; + (N - 1)/N.\overline{C}$$

The contribution to the portfolio variance of the variances of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches the average covariance as N gets large. The individual risk of securities can be diversified away, but the contribution to the total risk caused by the covariance terms cannot be diversified away.

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UNIT 9 — INTRODUCTION TO THE VALUATION OF DERIVATIVE SECURITIES

Syllabus objectives

(ix) Demonstrate a knowledge and understanding of the properties of option prices, valuation methods and hedging techniques.

1. State what is meant by arbitrage.

2. Outline the factors that affect option prices.

3. Derive specific results for options which are not model dependent:

   - Show how to value a forward contract.

   - Develop upper and lower bounds for European and American call and put options.

   - Explain what is meant by put-call parity.

0 Recap

A *derivative* is a security or contract which promises to make a payment at a specified time in the future, the amount of which depends upon the behaviour of some *underlying* security up to and including the time of the payment.

A *European* option is an option that can only be exercised at expiry. An *American* option is one that can be exercised on any date before its expiry.

1 Arbitrage

One of the central concepts in this area of financial economics is that of *arbitrage*.

Put in simple terms, an *arbitrage opportunity* is a situation where we can make a certain profit with no risk. This is sometimes described as a *free lunch*. Put more precisely, an arbitrage opportunity means that:

(a) We can start at time 0 with a portfolio which has a net value of zero (implying that we are long in some assets and short in others).

(b) At some future time T:

- the probability of a loss is 0
- the probability that we make a strictly positive profit is greater than 0

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired. The problem with this is that all of the active participants in the market would do the same and the market prices of the assets in the portfolio would quickly change to remove the arbitrage opportunity.

The *principle of no arbitrage* states simply that arbitrage opportunities do not exist.

If we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price. This is sometimes called the "law of one price".

2 Notation

The following notation will be used:

- t is the current time
- S_t is the underlying share price at time t
- K is the strike or exercise price
- T is the option expiry date
- c_t is the price at time t of a European call option
- p_t is the price at time t of a European put option
- C_t is the price at time t of an American call option
- P_t is the price at time t of an American put option
- r is the risk-free continuously compounding rate of interest (assumed constant)

Another useful definition is the *intrinsic value* of a derivative, which is the value assuming expiry of the contract immediately rather than at some time in the future. For a call option, for example, the intrinsic value at time t is simply $\max\{S_t - K, 0\}$.

3 Factors affecting option prices

A number of mathematical models are used to value options. One of the more widely used is the *Black-Scholes* model. This uses five parameters to value an option on a non-dividend-paying share. The five parameters are:

- **Underlying share price:** The effect of the price of the underlying share on a typical call option is shown in Figure 1. Note that the price for a call option is always greater than the intrinsic value. This follows on from the lower bound derived in Section 5 below: namely that

  $$c_t \geq S_t - Ke^{-r(T-t)} > S_t - K.$$

  In the case of a call option, a higher share price means a higher intrinsic value (or, where the intrinsic value is currently zero, a greater chance that the option is *in the money* at maturity). A higher intrinsic value means a higher premium. For a put option, a higher share price will mean a lower intrinsic value and a lower premium. In each case, the change in the value of the option will not match precisely the change in the intrinsic value because of the later timing of the option payoff.

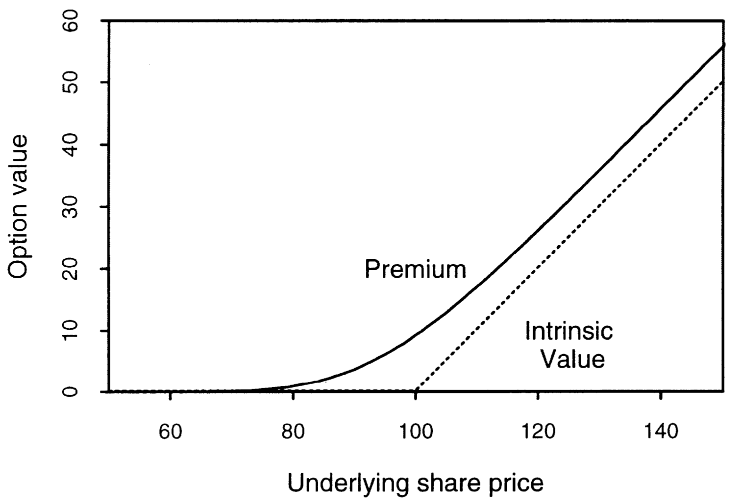

Figure 1: Call option premium and intrinsic value as a function of the current share price, S_t (with strike price 100)

- **Strike price:** In the case of a call option, a higher strike price means a lower intrinsic value. A lower intrinsic value means a lower premium. For a put option, a higher strike price will mean a higher intrinsic value and a higher premium. In each case the change in the value of the option will not match precisely the change in the intrinsic value because of the later timing of the option payoff.

- **Time to expiry:** The longer the time to expiry, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with term to maturity. This increase is moderated slightly by the change in the time value of money.

- **Volatility of the underlying share:** The higher the volatility of the underlying share, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with the volatility of the underlying share.

- **Interest rates:** An increase in the risk-free rate of interest will result in a higher value for a call option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option. For a put option, higher interest means a lower value.

The basic Black-Scholes model can be adapted to allow for a sixth factor determining the value of an option:

- **Income received on the underlying security:** In many cases the underlying security might provide a flow of, say, dividend income. Normally such income is not passed onto the holder of an option. Then the higher the level of income received, the lower is the value of a call option, because by buying the option instead of the underlying share the investor foregoes this income. The reverse is true for a put.

4 Forward pricing

A forward contract is the most simple form of derivative contract. It is also the most simple to price in the sense that the forward price can be established without reference to a model for the underlying share price.

Suppose that, besides the underlying share, we can invest in a cash account which earns interest at the continuously compounding rate of r per annum. Recall that the forward price K should be set at a level such that the value of the contract at time 0 is zero (that is, no money changes hands at time 0).

Proposition

The fair or economic forward price is $K = S_0 e^{rT}$.

Proof (a)

Suppose, first, that we have set the forward price at $K = S_0 e^{rT}$.

At the same time we can borrow an amount S_0 in cash (subject to interest at rate r) and buy one share. The net cost at time 0 is then zero.

At time T we will have:

- one share worth S_T on the open market
- a cash debt of S_0e^{rT}
- a contract to sell the share at the forward price K

Therefore we hand over the one share to the holder of the forward contract and receive K. At the same time we repay the loan: an amount S_0e^{rT}. Since $K = S_0e^{rT}$ we have made a profit of exactly 0. There is no chance of losing money on this transaction, nor is there any chance of making a positive profit. It is a risk-free trading strategy.

Now suppose instead that $K > S_0e^{rT}$.

We can issue one forward contract and, at the same time, borrow an amount S_0 in cash (subject to interest at rate r) and buy one share. The net cost at time 0 is zero. At time T we will have:

- one share worth S_T on the open market
- a cash debt of S_0e^{rT}
- a contract to sell the share at the forward price K

Therefore we hand over the one share to the holder of the forward contract and receive K. At the same time we repay the loan: an amount S_0e^{rT}. Since $K > S_0e^{rT}$ we have made a guaranteed profit, having made no outlay at time 0.

This is an example of *arbitrage*: that is, for a net outlay of zero at time 0 we have a probability of 0 of losing money and a strictly positive probability (in this case equal to 1) of making a profit greater than zero.

Instead of issuing one contract at this price, why not issue lots of them and we will make a fortune? In practice a flood of sellers would come in immediately, pushing down the forward price to something less than or equal to S_0e^{rT}. In other words the arbitrage possibility could exist briefly, but it would disappear very quickly before any substantial arbitrage profits could be made.

Now suppose that $K < S_0e^{rT}$.

We follow the same principles: at time 0

- buy one forward contract
- sell one share at a price S_0
- invest an amount S_0 in cash

The net value at time 0 is zero.

At time T we have cash of S_0e^{rT}; we pay K ($K < S_0e^{rT}$) for one share, after which our net holding of shares is zero. So the shareholding has zero value and we have $S_0e^{rT} - K > 0$ cash. Again this is an example of arbitrage, meaning that we should not, in practice, find that $K < S_0e^{rT}$.

Proof (b)

Let K be the forward price. Now compare the setting up of the following portfolios at time 0:

A: one long forward contract.
B: borrow Ke^{-rT} cash and buy one share at S_0.

If we hold both of these portfolios up to time T then both have a value of $S_T - K$ at T. By the principle of no arbitrage, these portfolios must have the same value at all times before T. In particular, at time 0, portfolio B has value $S_0 - Ke^{-rT}$ which must equal the value of the forward contract. This can only be zero (the value of the forward contract at $t = 0$) if $K = S_0e^{rT}$.

5 Bounds for option prices

5.1 Lower bounds on option prices

Consider a portfolio, A, consisting of a European call on a non-dividend-paying share and a sum of money equal to $Ke^{-r(T-t)}$. At time T, portfolio A has a value which is equal to the value of the underlying share, provided the share price S_T is greater than K. If S_T is less than K then the payoff from portfolio A is greater than that from the share. Since the option plus cash produces a payoff that is at least as great as that from the share, it must have a value greater than or equal to S_t. This gives us a lower bound for c_t:

$$c_t \geq S_t - Ke^{-r(T-t)}$$

A similar argument can be used for put options: portfolio B contains a European put option and a share. Compare this with the alternative of cash, currently worth $Ke^{-r(T-t)}$. At time T, portfolio B will be worth at least as much as the cash alternative. Thus:

$$p_t + S_t \geq Ke^{-r(T-t)}$$

$$\Rightarrow p_t \geq Ke^{-r(T-t)} - S_t$$

The lower bound for an American put option can be increased above that derived above for a European put option. Since early exercise is always possible, we have:

$$P_t \geq K - S_t.$$

5.2 Upper bounds on option prices

A call option gives the holder the right to buy the underlying share for a certain price. The payoff $\max\{S_T - K, 0\}$ is always less than the value of the share at time T, S_T. Therefore the value of the call option must be less than or equal to the value of the share:

$$c_t \leq S_t$$

For a European put option the maximum value obtainable at expiry is the strike price K. Therefore the current value must satisfy:

$$p_t \leq Ke^{-r(T-t)}$$

For certain types of stochastic model for S_t we find that we are able to write down explicit formulae for the prices of European call and put options.

From this section we can see that this applies also to an American call option on a non-dividend-paying stock. On the other hand, the possibility of early exercise of an American put option presents us with much more complexity. There are no simple rules for deciding upon the time to exercise. Partly as a result of this, there is no explicit formula for the price of an American put option.

6 Put-call parity

Consider the argument we used to derive the lower bounds for European call and put options on a non-dividend-paying stock. This used two portfolios:

- A: one call plus cash of $Ke^{-r(T-t)}$.
- B: one put plus one share.

Both portfolios have a payoff at the time of expiry of the options of $\max\{K, S_T\}$. Since they have the same value at expiry and since the options cannot be exercised before then they should have the same value at any time $t < T$. That is:

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

This relationship is known as *put-call parity*.

If the result was not true then this would give rise to the possibility of *arbitrage*. That is, for a net outlay of zero at time 0 we have a probability of 0 of losing money and a strictly positive probability (in this case) of making a profit greater than zero. In this case, the failure of put-call parity would allow an investor to sell calls and cash and buy puts and shares with a net cost of zero at time 0 and certain profit at time T.

In contrast to forward pricing, put-call parity does not tell us *what* c_t and p_t are individually: only the relationship between the two. To calculate values for c_t and p_t we require a model.

In all of these sections, the pricing of derivatives is based upon the principle of *no arbitrage*.

Note that we have made very few assumptions in arriving at these results. No model has been assumed for stock prices. All we have assumed is that we will make use of buy-and-hold investment strategies. Any model that we propose for pricing derivatives must, therefore, satisfy both put-call parity and the forward-pricing formula. If a model fails one of these simple tests then it is not arbitrage-free.

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