Minimum Reversion in Multivariate Time Series - Application to Human Mortality Data

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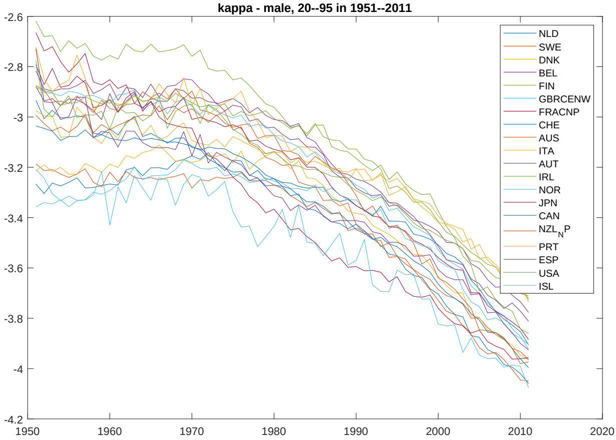
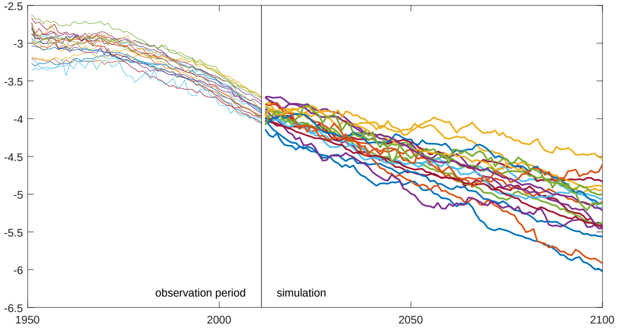
Fitted log mortality rates at age 70



Aim: projection of mortality for all populations simultaneously

Scenario for projected log mortality rates at age 70



Projections based on multivariate random walk with common drift

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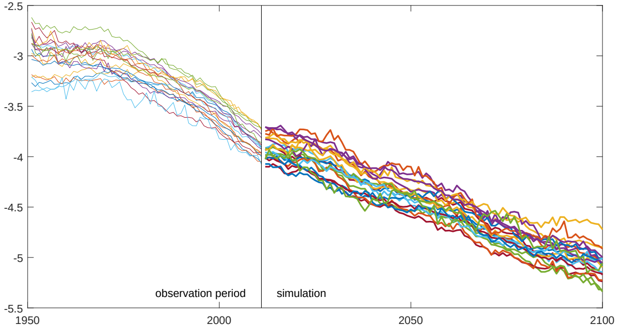
Scenario for projected log mortality rates at age 70



Projections based on our model

Introduction

- There are a number of models for the mortality experience in multiple populations available
- Such models have typically population specific period effects in addition to some common period or age effects

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Introduction

- There are a number of models for the mortality experience in multiple populations available
- Such models have typically population specific period effects in addition to some common period or age effects
- Focus of this talk is on a model for projecting mortality rates and generating mortality scenarios simultaneously for many countries
- multivariate time series model for period effects

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Motivation

- improvements in survival probabilities are driven by similar changes that do not stop at the border
- medical innovations: reduced death rates from cardio-vascular diseases, ...
- life style factors: smoking ban, sugar tax, minimum price per unit of alcohol

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- if true, survival probabilities in any country will show a tendency to move towards those of the country with highest survival rates
- we include a term in our model to incorporate that tendency ...

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- we include a term in our model to incorporate that tendency ...
- ... and investigate whether such a "'learning"' effect is signifcant

Model

- The number of deaths, D_{xtc}, in population $c \in \mathcal{C}$ at age $x \in \mathcal{X}$ in calendar year $t \in \mathcal{T}$ has a Poisson distribution:

$$D_{xtc} \sim \text{Pois}\left(\mu_{xtc} E_{xtc}\right)$$

- μ_{xtc} is the force of mortality
- E_{xtc} refers to the central exposed to risk.

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Common Age Effects

- Our model for the force of mortality is a modification, Kleinow (2015), of the Lee-Carter model:

$$\log \mu_{xtc} = \alpha_x + \beta_x \kappa_{t,c} \tag{1}$$

- Common age effects, α_x and β_x, ensure that period effects are comparable across populations since they are all rescaled with the same (age-dependent) constant.

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- Common age effects, α_x and β_x, ensure that period effects are comparable across populations since they are all rescaled with the same (age-dependent) constant.
- The parameters in (1) are not identifiable
- impose constraints on α and β:

$$\alpha_{x_r} = 0 \text{ and } \beta_{x_r} = 1 \tag{2}$$

  for a fixed reference age $x_r \in \mathcal{X}$.
  That means, fitted log mortality $\log \mu_{xtc} = \kappa_{t,c}$ for $x = x_r$ in every population $c \in \mathcal{C}$.
- In our empirical study we set $x_r = 70$.

Data

- mortality data for male populations in 20 countries:
  The Netherlands, Sweden, Denmark, Belgium, Finland, England & Wales, France, Switzerland, Australia, Italy, Austria, Ireland, Norway, Japan, Canada, New Zealand, Portugal, Spain, USA, Iceland
- ages: 20 - 95 ($\mathcal{X} = \{20, 21, \ldots, 95\}$),
- years 1951 - 2011 ($\mathcal{T} = \{1951, \ldots, 2011\}$)
- source: Human Mortality Database

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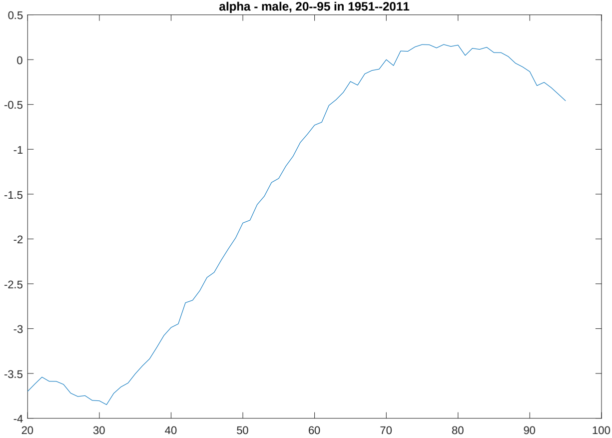
Common Age Effects - Empirical Results - alpha

$$\log \mu_{xtc} = \alpha_x + \beta_x \kappa_{t,c}$$



Common Age Effects - Empirical Results - beta

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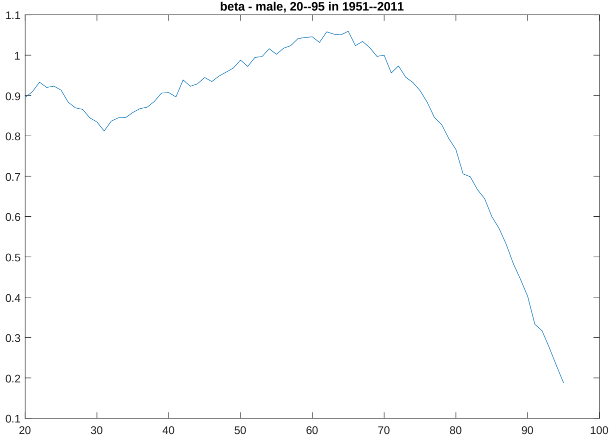

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Common Age Effects - Empirical Results - kappa

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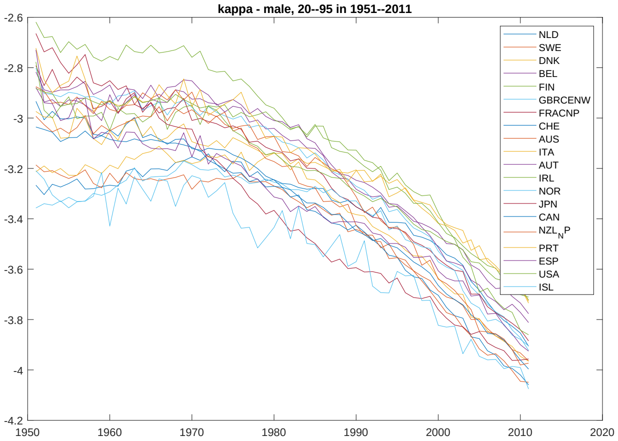

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Time Series Model for Period Effects

We propose the following model for the dynamics of the period effects κ_c for any population $c \in \mathcal{C}$:

$$\begin{aligned} \kappa_{t+1,c} - \kappa_{t,c} &= \mu_c + \zeta_c(\kappa_{t,c} - \kappa_{t-1,c}) + \lambda_c(m_t - \kappa_{t,c}) + \sigma_c Z_{t+1,c} \\ Z_{t,c} &= \rho_c W_t + \sqrt{1-\rho_c^2} W_{t,c} \qquad \left(Corr(Z_{t,c_1}, Z_{t,c_2}) = \rho_{c_1}\rho_{c_2} \right) \end{aligned}$$

where $\zeta_c, \rho_c \in (-1,1)$, $\lambda_c \in [0,1)$, $\sigma_c > 0$ and $\{W_t, W_{t,c}\}_{c \in \mathcal{C}, t \in \mathcal{T}}$ are independent and identically distributed random variables with a standard normal distribution.

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"Reversion" is to the minimum period effect at time $t \in \mathcal{T}$ as

$$m_t := \min_{c\in\mathcal{C}} \kappa_{t,c}$$

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Time Series Model for Period Effects

Special case, $\mu_c, \zeta_c, \rho_c = 0$

$$\begin{aligned} \kappa_{t+1,c} - \kappa_{t,c} &= \lambda_c(m_t - \kappa_{t,c}) + \sigma_c Z_{t+1,c} \\ m_t &:= \min_{c \in \mathcal{C}} \kappa_{t,c} \end{aligned}$$

with $Z_{t,c}$ iid normal.

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- The minimum process m_t, and therefore, all $\kappa_{t,c}$ processes have a downward drift (despite $\mu_c = 0$):
- $$\mathbb{P}(m_{t+1} \leq a \mid \{\kappa_{t,c}\}_{c \in \mathcal{C}}) = 1 - \prod_{c \in \mathcal{C}} \Phi\left(\frac{(1-\lambda_c)\kappa_{t,c} - a + \lambda_c m_t}{\sigma_c}\right).$$

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- Setting $a = m_t$ we obtain

$$\mathbb{P}(m_{t+1} \leq m_t \mid \{\kappa_{t,c}\}_{c \in \mathcal{C}}) = 1 - \prod_{c \in \mathcal{C}} \Phi\left(\frac{(1-\lambda_c)(\kappa_{t,c} - m_t)}{\sigma_c}\right) > \frac{1}{2}$$

Φ is the $N(0,1)$ distribution function.

Time Series Model for Period Effects

- Conditional probability for the minimum to decrease

$$\mathbb{P}(m_{t+1} \leq m_t \mid \{\kappa_{t,c}\}_{c\in\mathcal{C}}) = 1 - \prod_{c\in\mathcal{C}} \Phi\left(\frac{(1-\lambda_c)(\kappa_{t,c} - m_t)}{\sigma_c}\right) > \frac{1}{2}$$

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  Φ is the $N(0,1)$ distribution function.
- The larger σ_c the larger the probability of m_t decreasing
- The smaller the differences between countries the larger the probability of m_t decreasing

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Time Series Model for Period Effects

- Conditional probability for the minimum to decrease

$$\mathbb{P}(m_{t+1} \leq m_t \mid \{\kappa_{t,c}\}_{c\in\mathcal{C}}) = 1 - \prod_{c\in\mathcal{C}} \Phi\left(\frac{(1-\lambda_c)(\kappa_{t,c} - m_t)}{\sigma_c}\right) > \frac{1}{2}$$

  Φ is the $N(0,1)$ distribution function.
- The larger σ_c the larger the probability of m_t decreasing
- The smaller the differences between countries the larger the probability of m_t decreasing
- The more countries the larger the probability of m_t decreasing

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Time Series Model for Period Effect - Co-integration

$$\kappa_{t+1,c} - \kappa_{t,c} \quad = \quad \mu_c + \zeta_c(\kappa_{t,c} - \kappa_{t-1,c}) + \lambda_c(m_t - \kappa_{t,c}) + \sigma_c Z_{t+1,c}$$

Individual components $\kappa_{t,c}$ are not stationary but they turn out to be co-integrated.

If all processes $\kappa_{t,c}$ (for all c) have a common minimum reversion parameter λ and a common drift μ and there is no autoregressive term, so $\mu_c = \mu$, $\lambda_c = \lambda > 0$ and $\zeta_c = 0$ for all $c \in \mathcal{C}$, then the processes $\{\kappa_{\cdot,c}\}_{c\in\mathcal{C}}$ are co-integrated.

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Time Series Model for Period Effect - Co-integration, proof

$$\kappa_{t+1,c} - \kappa_{t,c} \quad = \quad \mu_c + \zeta_c(\kappa_{t,c} - \kappa_{t-1,c}) + \lambda_c(m_t - \kappa_{t,c}) + \sigma_c Z_{t+1,c}$$

Fix a $c^* \in \mathcal{C}$ and define $\tilde{\kappa}_{t,c} := \kappa_{t,c} - \kappa_{t,c^*}$ for any $c \in \mathcal{C}$. We then find for any $c \neq c^*$

$$\begin{aligned} \tilde{\kappa}_{t,c} &= (1-\lambda)(\kappa_{t-1,c} - \kappa_{t-1,c^*}) + \tilde{Z}_t \\ &= (1-\lambda)\tilde{\kappa}_{t-1,c} + \tilde{Z}_t, \qquad \tilde{Z}_t = \sigma_c Z_{t,c} - \sigma_{c^*} Z_{t,c^*}. \end{aligned}$$

Since $0 < \lambda \leq 1$ we obtain that $\tilde{\kappa}_{t,c}$ is a stationary AR(1) process for all $c \neq c^*$.

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Since $0 < \lambda \leq 1$ we obtain that $\tilde{\kappa}_{t,c}$ is a stationary AR(1) process for all $c \neq c^*$.

Furthermore, we find that

$m_t = \min_{c\in\mathcal{C}}(\kappa_{t,c^*} + \tilde{\kappa}_{t,c}) \;=\; \kappa_{t,c^*} + \min_{c\in\mathcal{C}} \tilde{\kappa}_{t,c}$ and therefore

$$\begin{aligned} \Delta\kappa_{t+1,c^*} := \kappa_{t+1,c^*} - \kappa_{t,c^*} &= \mu_{c^*} + \lambda(m_t - \kappa_{t,c^*}) + \sigma_{c^*} Z_{t+1,c^*} \\ &= \lambda \min_{c\in\mathcal{C}} \tilde{\kappa}_{t,c} + \mu_{c^*} + \sigma_{c^*} Z_{t+1,c^*}. \end{aligned}$$

The first term in the last expression is a minimum over stationary processes and the other terms are stationary too, hence $\Delta\kappa_{t,c^*}$ is stationary.

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Time Series Model - two-dimensional model

We take $C = 2$, $\lambda_1 = \lambda_2 := \lambda$, $\mu_1 = \mu_2 = \mu$ and $\text{Corr}(Z_{t+1,1}, Z_{t+1,2}) = \rho$.
And we assume that there is no AR term, that is, $\zeta_1 = \zeta_2 = 0$

$$\kappa_{t+1,1} - \kappa_{t,1} = \lambda(\min_{c\in\{1,2\}} \kappa_{t,c} - \kappa_{t,1}) + \sigma_1 Z_{t+1,1} + \mu, \quad (3)$$

$$\kappa_{t+1,2} - \kappa_{t,2} = \lambda(\min_{c\in\{1,2\}} \kappa_{t,c} - \kappa_{t,2}) + \sigma_2 Z_{t+1,2} + \mu. \quad (4)$$

We define

$$m_t = \min_{c\in\{1,2\}} \kappa_{t,c} \quad \text{and} \quad M_t = \max_{c\in\{1,2\}} \kappa_{t,c}.$$

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Time Series Model - two-dimensional model

$$\kappa_{t+1,1} - \kappa_{t,1} = \lambda(\min_{c\in\{1,2\}} \kappa_{t,c} - \kappa_{t,1}) + \sigma_1 Z_{t+1,1} + \mu, \quad (5)$$

$$\kappa_{t+1,2} - \kappa_{t,2} = \lambda(\min_{c\in\{1,2\}} \kappa_{t,c} - \kappa_{t,2}) + \sigma_2 Z_{t+1,2} + \mu. \quad (6)$$

We then obtain

$$\lim_{t\to\infty} \mathbb{E}[M_{t+1} - M_t] = \lim_{t\to\infty} \mathbb{E}[m_{t+1} - m_t] = \mu - s\sqrt{\tfrac{\lambda}{2\pi(2-\lambda)}} \quad (7)$$

and

$$\lim_{t\to\infty} \mathbb{E}[M_t - m_t] = s\sqrt{\tfrac{2}{\lambda\pi(2-\lambda)}} \quad (8)$$

with $s = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

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Time Series Model - Extra drift from minimum reversion

Table: Generated drift for different minimum reversion parameters λ according to (7) (second column) and using 10^5 simulations (third to last column), $s = 1$ and $\mu = 0$.

	Exact	Simulation				
λ	$\lvert\mathcal{C}\rvert = 2$	$\lvert\mathcal{C}\rvert = 2$	$\lvert\mathcal{C}\rvert = 3$	$\lvert\mathcal{C}\rvert = 4$	$\lvert\mathcal{C}\rvert = 8$	$\lvert\mathcal{C}\rvert = 16$
0.0125	-0.0447	-0.0448	-0.0671	-0.0817	-0.1129	-0.1401
0.025	-0.0635	-0.0635	-0.0952	-0.1158	-0.1602	-0.1987
0.05	-0.0903	-0.0903	-0.1355	-0.1649	-0.2280	-0.2828
0.1	-0.1294	-0.1294	-0.1942	-0.2362	-0.3266	-0.4052
0.2	-0.1881	-0.1881	-0.2821	-0.3432	-0.4745	-0.5887
0.4	-0.2821	-0.2821	-0.4231	-0.5147	-0.7118	-0.8830

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Time Series Model - Expected range

Table: Expectation of the stationary distribution for $M_t - m_t$ for different minimum reversion parameters λ according to (8) (second column) and using 10^5 simulations (third to last column).

	Exact	Simulation				
λ	$\lvert\mathcal{C}\rvert = 2$	$\lvert\mathcal{C}\rvert = 2$	$\lvert\mathcal{C}\rvert = 3$	$\lvert\mathcal{C}\rvert = 4$	$\lvert\mathcal{C}\rvert = 8$	$\lvert\mathcal{C}\rvert = 16$
0.0125	7.1589	7.1593	10.7386	13.0627	18.0644	22.4106
0.025	5.0781	5.0774	7.6185	9.2656	12.8138	15.8962
0.05	3.6137	3.6132	5.4202	6.5931	9.1178	11.3112
0.1	2.5887	2.5886	3.8831	4.7229	6.5316	8.1031
0.2	1.8806	1.8806	2.8209	3.4313	4.7454	5.8866
0.4	1.4105	1.4104	2.1156	2.5735	3.5591	4.415

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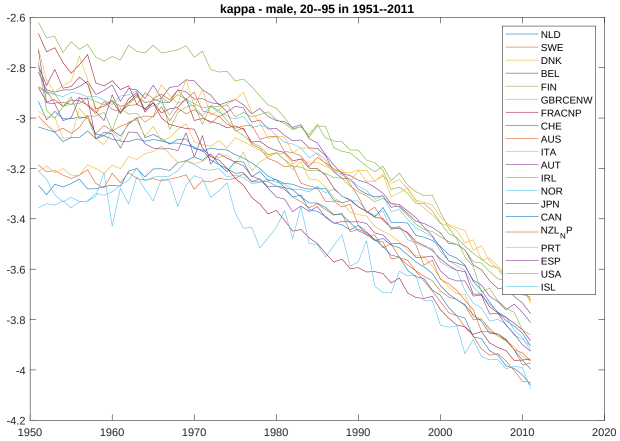
Fitting the model to kappa - just a reminder

$$\log \mu_{xtc} = \alpha_x + \beta_x \kappa_{t,c}$$



Summary Statistics

$$
\begin{aligned}
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\end{aligned}
$$

	Log L	K	BIC	μ	λ	ζ	$\bar{\sigma}$
Drift + AR	3785.59	42	-7274.10	-0.0202	-	-0.3289	0.0334

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KV + AR	3782.30	42	-7267.52	-	0.0355	-0.3263	0.0354
KV + Drift	3735.49	42	-7173.91	-0.0095	0.0187	-	0.0347

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KV + Drift	3735.49	42	-7173.91	-0.0095	0.0187	-	0.0347
KV + Drift + AR	3792.99	43	**-7281.83**	-0.0144	0.0201	-0.3290	0.0334

Summary statistics for males aged 20–95 in years 1951–2011, CAE. K is the number of parameters.

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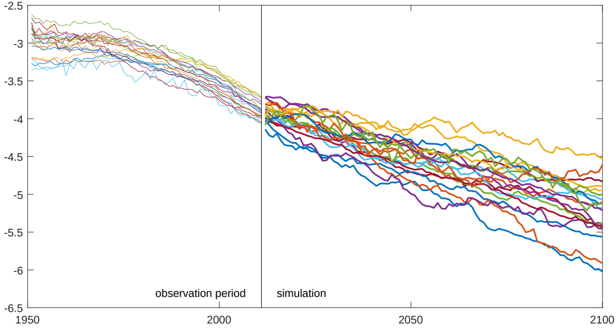
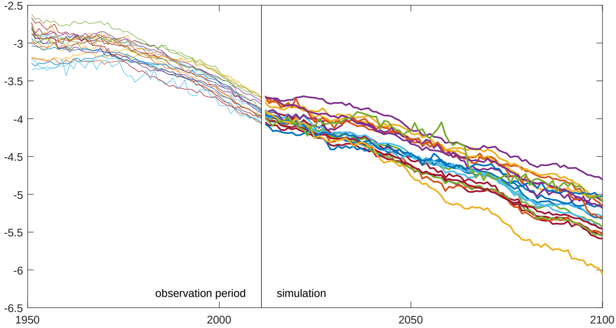
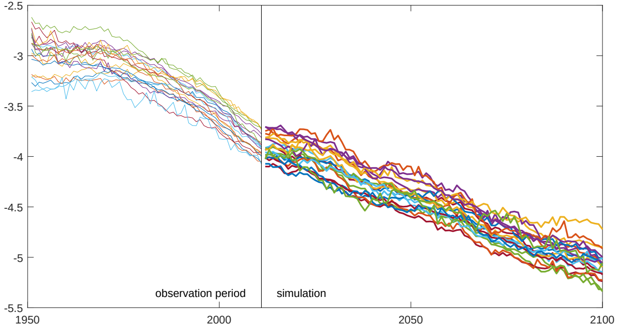
Projections - Random Walk



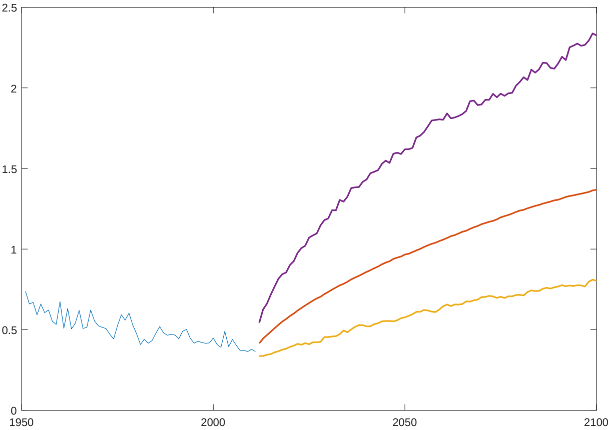
Projections - Random Walk + AR



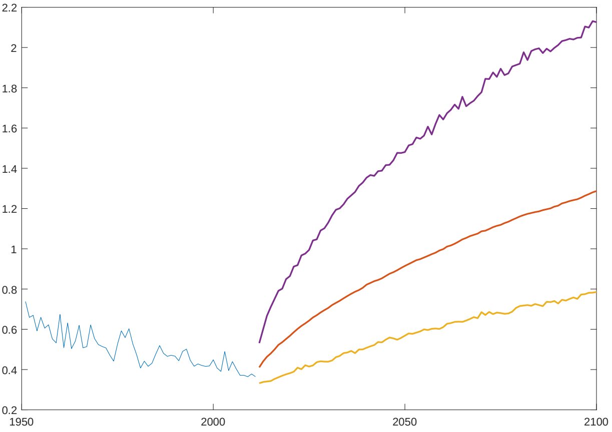
Projections - Random Walk + AR + KV



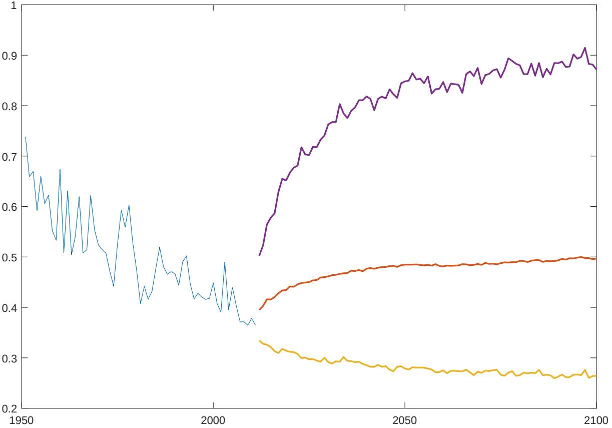
Projections - Random Walk



Projections - Random Walk + AR



Projections - Random Walk + AR + KV



Different dataset

$$\begin{aligned}\kappa_{t+1,c} - \kappa_{t,c} &= \mu + \zeta(\kappa_{t,c} - \kappa_{t-1,c}) + \lambda(m_t - \kappa_{t,c}) + \sigma_c Z_{t,c} \\ Z_{t,c} &= \rho_c W_t + \sqrt{1-\rho_c^2} W_{t,c}\end{aligned}$$

	Log L	K	BIC	μ	λ	ζ	$\bar{\sigma}$
Drift + AR	3605.35	30	-6996.87	-0.0234	-	-0.3447	0.0425
KV + AR	3566.37	30	-6918.91	-	0.0674	-0.2664	0.0439
KV + Drift	3541.84	30	-6869.84	-0.0121	0.0391	-	0.0454
KV + Drift + AR	3610.95	31	**-7000.94**	-0.0194	0.0257	-0.3321	0.0422

Summary statistics for females aged 20–95 in years 1921–2011, CAE, 14 countries

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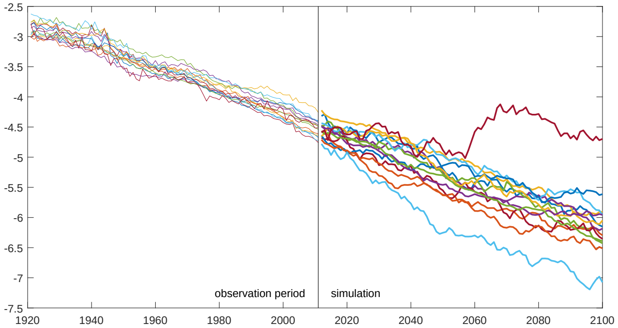
Projections - Random Walk

Female population in 14 countries, aged 20 - 95, years 1921 - 2011



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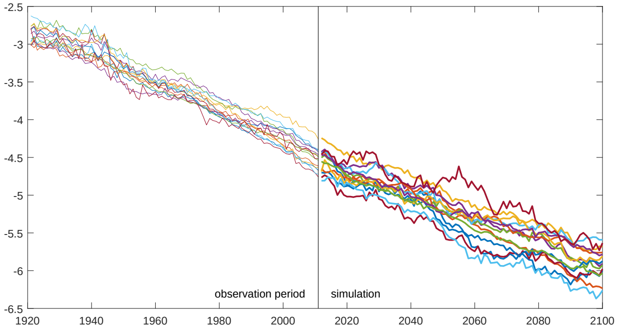
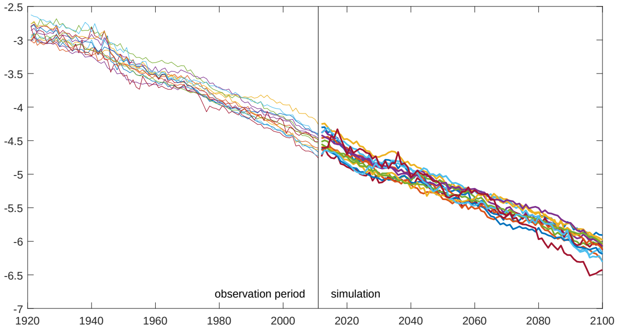
Projections - Random Walk + AR

Female population in 14 countries, aged 20 - 95, years 1921 - 2011



Projections - Random Walk + AR + KV

Female population in 14 countries, aged 20 - 95, years 1921 - 2011



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Conclusions

$$
\begin{aligned}
\kappa_{t+1,c} - \kappa_{t,c} &= \mu + \zeta(\kappa_{t,c} - \kappa_{t-1,c}) + \lambda(m_t - \kappa_{t,c}) + \sigma_c Z_{t,c} \\
Z_{t,c} &= \rho_c W_t + \sqrt{1-\rho_c^2}\, W_{t,c}
\end{aligned}
$$

- simultaneous projections of mortality in multiple populations
- changing improvement rates
- downward trend generated from random innovations
- learning (copying) from other populations
- better fit than model without KV term (for many datasets)
- improved scenario generation

paper on arXiv:
Kleinow, Vellekoop: Minimum reversion in multivariate time series

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