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Effect of Size of Exposure on Parameter Estimates and Correlations

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Background and Motivation

- Most mortality models are designed for large population size, e.g. England and Wales
- Actuaries are interested in modelling relatively much smaller population, e.g. Pension scheme
- Investigate how population size affects the accuracy of parameter estimates
- Mortality model, e.g. M7

\[
\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_x^2) + \gamma_t^{(4)}
\]

- Sampling variation in death number, \( D(t, x) \), leads to noise in \( \kappa_t^{(i)} \), hence in drift \( \mu_{\kappa} \), variance \( \sigma_{\kappa}^2 \) of random walk
Method, $\theta = (\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \gamma^{(4)})$

- $E^{EW}, D^{EW} \rightarrow \hat{\theta}^{EW}$
- $\hat{m}^{EW} \rightarrow E^{w} = w \times E^{EW}$
- $D^{w} \sim \text{Poi}(\hat{m}^{EW} E^{w}) \rightarrow D^{w,j}$
- $\bar{\theta}^{w,j}$

- England and Wales (EW), Male, age 50-89, year 1961-2011
- $w = 1, 0.1, 0.01$
- $j = 1, \ldots, 1000$, independent scenarios
- Fit M7 to $D^{w,j}, E^{w}$ of each scenario
Simulated Crude Death Rate \( m^w = \frac{D^w}{E^w} \)

Figure: The simulated death rate (upper) and its log-variance (lower), at year 1981 (left), age 70 (right)
Parameter Estimates

Estimates of $\kappa^{(1)}$

Estimates of $\kappa^{(2)}$

Estimates of $\kappa^{(3)}$

Estimates of $\gamma^{(4)}$

$w = 0.01$  $w = 0.1$

$w = 1$  England and Wales

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Uncertainty of Parameter Estimations

![Graphs showing log-St Dev of κ and γ for different years and birth years with different weights (w = 0.01, w = 0.1, w = 1).]

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Correlation of $\widehat{\theta}^{w,j} - \widehat{\theta}^{EW}$

- $X^w = (\epsilon_1^{(1),w} \ldots \epsilon_{n_y}^{(1),w}, \epsilon_1^{(2),w} \ldots \epsilon_{n_y}^{(2),w}, \epsilon_1^{(3),w} \ldots \epsilon_{n_y}^{(3),w}, \epsilon_1^{(4),w} \ldots \epsilon_{n_y+n_a-1}^{(4),w})$

- $\epsilon_t^{(1,2,3),w} = \tilde{\kappa}_t^{(1,2,3),w} - \tilde{\kappa}_t^{(1,2,3),EW}$

- $\epsilon_t^{(4),w} = \tilde{\gamma}_t^{(4),w} - \tilde{\gamma}_t^{(4),EW}$

- $\epsilon_{t-x} = \tilde{\gamma}_{t-x} - \tilde{\gamma}_{t-x}$
Colour palette for PowerPoint presentations

Dark blue: R176 G52 B88
Gold: R217 G171 B22
Mid blue: R64 G150 B184
Secondary colour palette
Light grey: R220 G221 B217
Pea green: R121 G163 B42
Forest green: R0 G132 B82
Bottle green: R177 G179 B162
Cyan: R0 G156 B200
Light blue: R124 G179 B225
Violet: R128 G118 B207
Purple: R143 G70 B147
Fuscia: R233 G69 B140
Red: R200 G30 B69
Orange: R238 G116 29

Parameter Projection

Estimated and Projected $\kappa^{(1)}$

Estimated and Projected $\kappa^{(2)}$

Estimated and Projected $\kappa^{(3)}$

Estimated and Projected $\gamma^{(4)}$

$w = 0.01$  $w = 0.1$  $w = 1$

England & Wales exposure

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Projected Mortality Rate and Survival Index

Log Projected Mortality Rates

Projected Survival Index

\[ w = 0.01 \quad w = 0.1 \quad w = 1 \quad EW \]

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Drift of Multivariate Random Walk \( \mu^{(i)}_w = \frac{1}{n_y} \sum_{t=1}^{n_y} \Delta \kappa_t^{(i),w} \)

<table>
<thead>
<tr>
<th>Std Dev(( \mu^{(i)}_w ))</th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
<th>( i = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w = 1 )</td>
<td>0.0000842</td>
<td>0.0000086</td>
<td>0.00000113</td>
</tr>
<tr>
<td>( w = 0.01 )</td>
<td>0.0008289</td>
<td>0.0000844</td>
<td>0.0001095</td>
</tr>
</tbody>
</table>

\[ \times 10 \]

- The higher the \( w \), the lower the standard deviation of drift.
- \( w \) has no significant effect on the mean of the drifts
- No significant linear correlation between drifts for all \( w \)
- The correlation generally decays as \( w \) decreases
Co-Variance Matrix of Multivariate Random Walk, $V^W$

- $V_{i,j}^W = E \left[ (\Delta \kappa_t^{(i),w} - \mu_w^{(i)}) (\Delta \kappa_t^{(j),w} - \mu_w^{(j)}) \right]$, where $i, j = 1, 2, 3$

- $V_{i,j}^{EW} = \begin{pmatrix} 6.80 \times 10^{-4} & 2.19 \times 10^{-5} & 4.95 \times 10^{-7} \\ 2.19 \times 10^{-5} & 1.31 \times 10^{-6} & 3.22 \times 10^{-8} \\ 4.95 \times 10^{-7} & 3.22 \times 10^{-8} & 3.33 \times 10^{-9} \end{pmatrix}$

- $E[V_{i,j}^1] = \begin{pmatrix} 6.93 \times 10^{-4} & 2.15 \times 10^{-5} & 5.43 \times 10^{-7} \\ 2.15 \times 10^{-5} & 1.43 \times 10^{-6} & 2.99 \times 10^{-8} \\ 5.43 \times 10^{-7} & 2.99 \times 10^{-8} & 4.33 \times 10^{-9} \end{pmatrix}$

- $E[V_{i,j}^{0.01}] = \begin{pmatrix} 19.1 \times 10^{-4} & -1.69 \times 10^{-5} & 4.84 \times 10^{-6} \\ -1.69 \times 10^{-5} & 1.28 \times 10^{-5} & -1.98 \times 10^{-7} \\ 4.84 \times 10^{-6} & -1.98 \times 10^{-7} & 1.01 \times 10^{-7} \end{pmatrix}$

- The mean of $V_{i,j}^W$ shifts up from the original $V_{i,j}^{EW}$ as $w$ decrease
- Lower $w$ also results in higher standard deviation to the co-variance matrix
AR(1) model for the Cohort Effect $\gamma_{c}^{(4),w}$

- $\gamma_{c}^{(4),w} = \mu_{\gamma}^{w} + \delta_{1}^{w} c + \alpha_{\gamma}^{w} \gamma_{c-1}^{(4),w} + \xi_{c}^{w}$

Density of $\alpha_{\gamma}^{w}$

Density of $\mu_{\gamma}^{w}$

Density of $\delta_{1}^{w}$

Density of St. Dev of $\xi_{c}^{w}$
Conclusions

• The accuracy of the parameter estimates depends significantly on the population size

• Hence smaller population results in greater uncertainty in projections

• Forecasting needs to allow for small population bias in parameter estimates