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Testing for a Common Cohort Effect in Two Populations

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Introduction

Research Questions

- Do the male and female share the same cohort effect?
- Parameter estimates: what is the impact of population size?
- How accurate is the χ^2 approximation in the Likelihood Ratio Test

Model: M7, $m(t,x) = -\log[1-q(t,x)]$

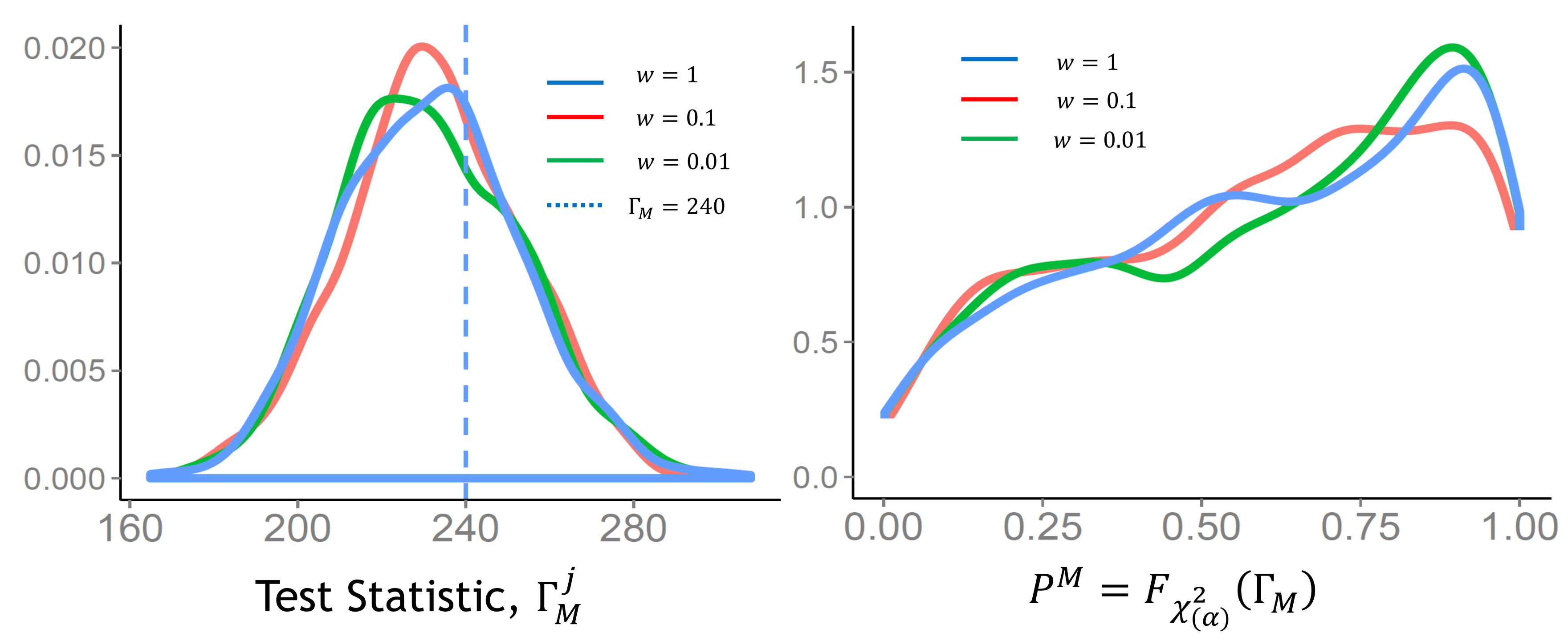
$$\text{Logit } q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(3)}\left((x-\bar{x})^2 - \hat{\sigma}_x^2\right) + \gamma_{t-x}^{(4)}$$

Data: $E_M^{EW}(t,x)$ and $E_F^{EW}(t,x)$, the exposures and corresponding deaths $D_M^{EW}(t,x)$ and $D_F^{EW}(t,x)$, of England and Wales(EW), Males and Females respectively, age 50-89 last birthday, years 1961-2011

Likelihood Ratio Test (LRT) for Systematic Parameter Difference

Test: Systematic parameter difference of different populations

- $\theta = (\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}, \gamma_{t-x}^{(4)})$
- $\hat{\theta}_M^{EW}$, MLEs given E_M^{EW}, D_M^{EW}
- Simulate death, $D_M^{w,j}$ for scenarios $j = 1,2,\ldots,k$, where $D_M^w \sim \text{Poi}(\hat{m}_M w E_M^{EW})$
- Weight, $w = 1, 0.1, 0.01$
- $D_M^{w,j} \to \hat{\theta}_{M,w}^j$, the set of MLEs of $D_M^{w,j}$, wE_M^{EW}
- **Test: $H_0: \theta_w^j = \hat{\theta}_M^{EW}$ vs $H_1: \theta_w^j \neq \hat{\theta}_M^{EW}$**
- Based on the log-likelihood function $l(\theta, D, E)$, calculate the test statistic for LRT $\Gamma_M^j = -2(l_0^j - l_1^j)$, where
$$l_0^j = l(\hat{\theta}_M^{EW}, D_M^{w,j}, wE_M^{EW});\ l_1^j = l(\hat{\theta}_{M,w}^j, D_M^{w,j}, wE_M^{EW})$$
- Under H_0, simulated test statistic has approximately a χ_α^2 distribution, where $\alpha = 3n_{year} + n_{cohort} - 3 = 240$.
- Define $P_j^M = F_{\chi^2_{(\alpha)}}(\Gamma_M^j)$, if χ^2 approximation is accurate, then $P_1^M, \ldots, P_k^M$ should be iid uniform.

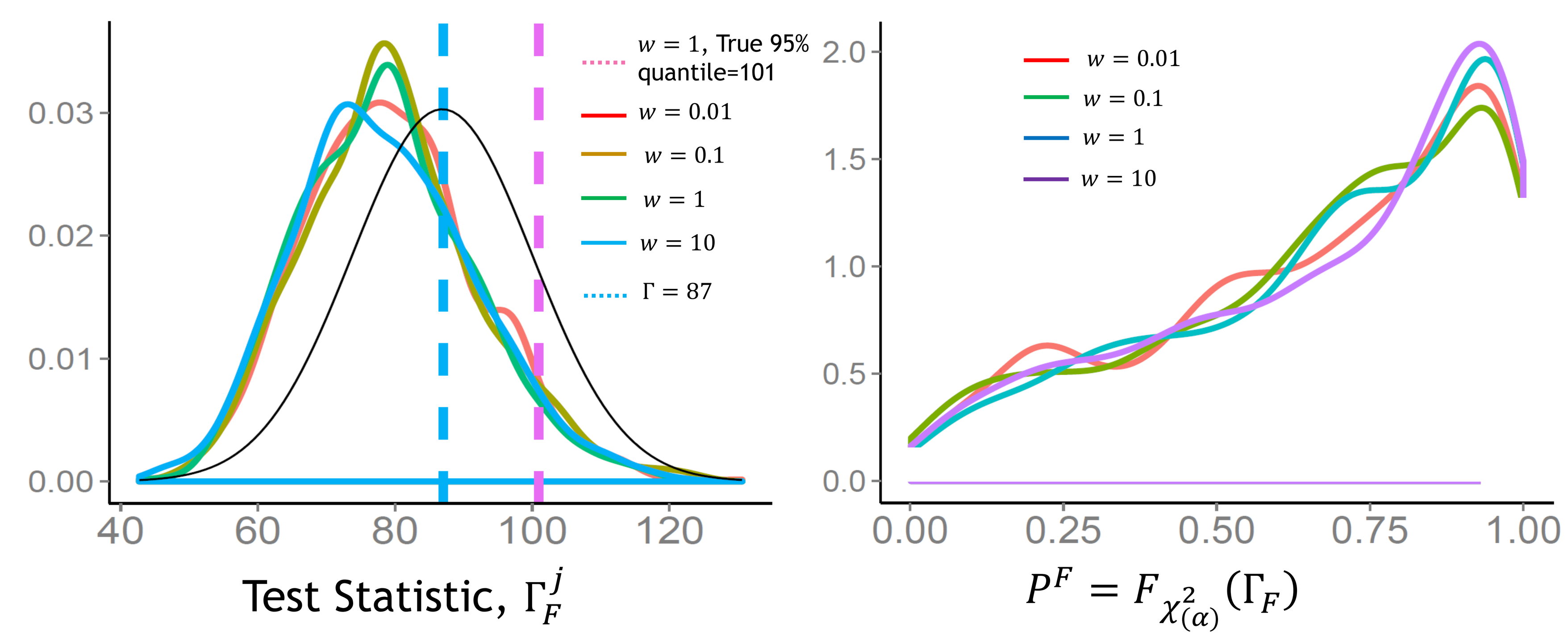


- The test statistic is not centered around 240, nor is the χ^2 p-values uniform.
- Changing the weight has no obvious effect on the distributions of test statistic and p-values

Test if Males and Females Share the Same Cohort Effect

- H_0: $\gamma_{F,w}^{(4),j} = \hat{\gamma}_{M,EW}^{(4)}$; H_1: $\gamma_{F,w}^{(4),j} \neq \hat{\gamma}_{M,EW}^{(4)}$. Now only "87" constraints.
- $\hat{\theta}_F^{EW}$, MLEs of $E_F^{EW}(t,x), D_F^{EW}(t,x)$
- $\hat{\theta}_{F'}^{EW}$, MLEs of $E_F^{EW}(t,x), D_F^{EW}(t,x)$, when fixing $\gamma_{F,EW}^{(4)} = \hat{\gamma}_{M,EW}^{(4)}$
- $\hat{m}_{F'}$, the estimated death rate, given $\hat{\theta}_{F'}^{EW}$
- $D_F^w \sim \text{Poi}(\hat{m}_{F'} E_F^w)$, where $E_F^w = w \times E_F^{EW}$
- Simulated death $D_F^{w,j}$, for scenarios $j = 1, \ldots, k$
- $D_F^{w,j} \to \hat{\theta}_{F,w}^j$, $\hat{\theta}_{F',w}^j$, before and after fixing $\gamma_{F,w}^{(4),j} = \hat{\gamma}_{M,EW}^{(4)}$
- **Test: H_0: $\gamma_{F,w}^{(4),j} = \hat{\gamma}_{M,EW}^{(4)}$; H_1: $\gamma_{F,w}^{(4),j} \neq \hat{\gamma}_{M,EW}^{(4)}$**
- Calculate the test statistic $\Gamma_F^j = -2(l_{0,F}^{j,w} - l_{1,F}^{j,w})$, where
$$l_{0,F}^{j,w} = l\left(\hat{\theta}_{F',w}^j, D_F^{w,j}, E_F^w\right);\ l_{1,F}^{j,w} = l(\hat{\theta}_{F,w}^j, D_F^{w,j}, E_F^w)$$

- Under H_0, test statistic has approximately a χ_{87}^2 distribution
- Define $P_j^F = F_{\chi^2_{(\alpha)}}(\Gamma_F^j)$, if χ^2 approximation is accurate, then $P_1^F, \ldots, P_k^F$ should be iid uniform.



$w \times E_F^{EW}$	Proportion
10	1.9%
1	1.3%
0.1	2.3%
0.01	1.7%

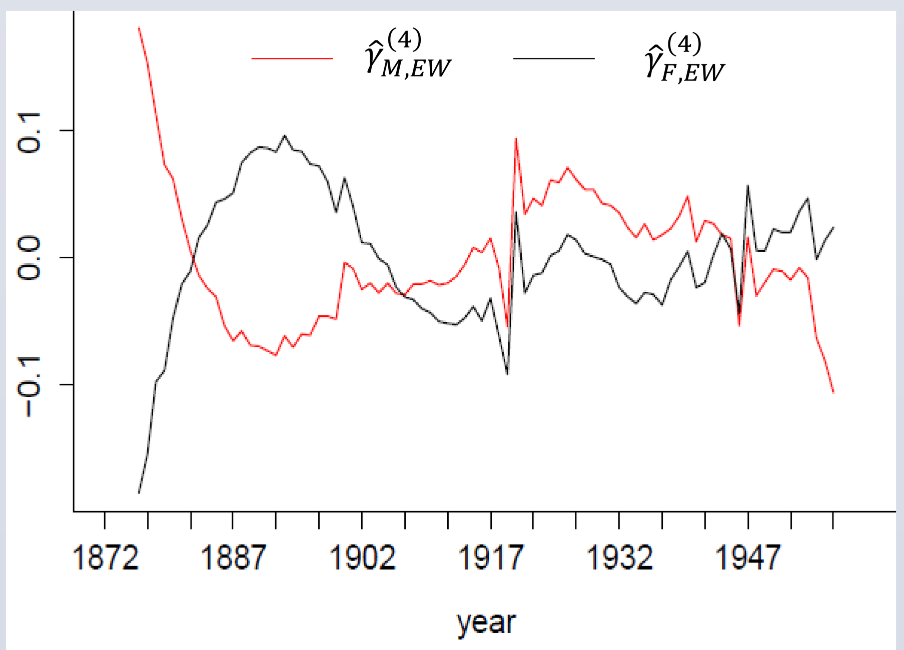
Table: The simulated rejected probability of the 1000 scenarios under χ_{87}^2 approximation.

- The simulated rejected probability for each w are significantly smaller than 5%.
- The distribution of test statistics(S_w) is not significantly influenced by the population size
- χ^2 approximation is not accurate and it is more appropriate to calculate the p-values based on a bootstrap approximation.

LRT with Bootstrap Simulation

Test: Females have same cohort effect as Males

- **Test: H_0: $\gamma_F^{(4)} = \hat{\gamma}_{M,EW}^{(4)}$ vs H_1: $\gamma_F^{(4)} \neq \hat{\gamma}_{M,EW}^{(4)}$**
- $P_{95} = 101$, the 95% quantile of the bootstrapped S_1, with $k = 1000$ scenarios, used as critical value
- Test statistic $\Gamma = 6311$
- Reject H_0 if $\Gamma > 101$



Conclusion: Females do not share the same cohort effect

Conclusions

For the underlying data and the M7 model

- The distribution of test statistic is biased from χ^2, thus it is not appropriate to work out the p-values based on χ^2 assumption
- Although decreasing the size of exposure brings in more uncertainty to the parameter estimates and projection, its sole impact on the LRT test statistic is not significant.
- Strong empirical evidence shows that males and females have different cohort effect. Additionally, further tests also reveal significant difference between period effect.