Institute and Faculty of Actuaries

‘ENID Loading’ – we finally cracked it and your life just got easier ...!

Jerome Kirk (PwC) and Yuriy Krvavych (PwC)

21 June 2016

Motivation

Under the upcoming Solvency II insurers are (indirectly!) required to allow for all possible events when setting their technical provisions, including those *'that may not have been historically realised'* (EIOPA/CEIOPS and Lloyd's guidance), i.e.

- *Binary Events* or *Events Not In Data (ENID)*

There are many ENID loading approaches, and it's up to insurers to decide which one suits best.

'Lloyd's Approximations': Lloyd's Technical Provisions Guidance (2011) recommends using the *Truncated Statistical Approach*, however does not provide explicit analytical formulae for calculating the uplift. The industry has developed **two particular analytical approximations** of the reserve mean load **assuming log-normality** of reserve risk profile.

This research

1. discusses the importance of ENID loading in managing reserve uncertainty;
2. examines the quality of Lloyd's Approximations; and
3. proposes a new distribution-free approach to estimating ENID load

Institute and Faculty of Actuaries

- The role of ENID loading – background -

21 June 2016

My part in all of this ... cast your mind back to 2009

- **Enid** Mary Blyton was still a children's author
- Actuaries were in a flat spin over the pure evil that was "binary event" (and risk margin)
- Lloyd's were putting together the TP guidance
    - with a specific view to proportionality and practical application
- I saw the problem as follows:
    - We need to do something
    - Surely it can't be a big number
    - No need to get hung up or spurious
    - Must be simple, transparent and explainable
    - Put something out and know that methods will develop

Luckily I had recently read the CAS paper "Yep, We're Skewed"

- And we included: In reality there are many possible approaches to allowing for binary events. Three are highlighted below:
  1. Use history as a guide
  2. Estimate vulnerability to a range of current threats - 'scenario' approach
  3. Uplift reserve to allow for limited range of understanding

"It is proposed that, unless further developments are made, method 3 is used and reserves are explicitly uplifted at a Solvency II line of business level to allow for binary events."

The **"Lloyd's approximation"** was born

- Indicative results based on market data were 3-5%. In July we added:
  - *"In all cases the method and allowances for binary events should be well documented"*
- Then waited for the "further developments" ... 6 years later there have been developments!

ENIDs – why are they important?

- You might argue *"... Surely, there is a bigger number (BE) to worry about. Why ENIDs?"* ... but at the end of the day you do want to know what is a reasonable best estimate of your expected cost of future payments ... this is the fundamental of reserving and you want to do a proper job.
- Most historic reserving issues have come from ENID type of situations – e.g. take Lloyd's in the late 80s and 90s. So, given we know they do happen, you do need to focus on them.
- Missing a bit off "because its difficult" is no excuse......if your wedding cake wasn't iced because that part was difficult but you were still paying a lot of money for the service you wouldn't be pleased.......now think of boards and reserving actuaries!!!
- You do need to do it for regulatory purposes.
- You might want a back pocket conversation that will make anyone leave you alone at a party (maybe except the GIRO dinner).
- In summary: **the ENID loading is simply used to make reserve uncertainty more informative!** Ideally, whilst allowing for parameter uncertainty, the ENID loading should also propagate an informative choice of parametric distribution.

Institute and Faculty of Actuaries

- Lloyd’s Approximations -

21 June 2016

Background

The **Truncated Statistical Distribution approach** defines the reserve uplift factor as

the ratio of the 'true mean' to the 'mean only including realistically foreseeable events'

- **'realistically foreseeable events'** - loss events with a return period of up to Y years
- the true (untruncated) reserve value is X and its distribution has a parametric form F_X
- the reserve values based on the 'realistically foreseeable events' are drawn from the truncated reserve distribution, i.e. X given $X \leq F_X^{-1}(p)$, where $p = 1 - 1/Y$ (one-sided truncation focusing on adverse reserve outcomes only, but one could also use two-sided truncation to allow for unforeseeable reserve releases)
- **information available to the actuarial function:** mean (**Central Estimate**) and variability (**CoV**) of reserve and the degree of asymmetry of reserve releases/strengthening (**Skewness**) – all based on realistically foreseeable events.

Lloyd's Approximations

Two so-called 'Lloyd's Approximations' of the load of reserve mean, $\mathbf{Mean\_Load}$, assuming **log-normality** of the true reserve distribution (here $\mathbf{Mean\_Load}$ is simply **uplift factor** minus 1):

Lloyd's Approximation 1:

$$\boxed{\frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1} \tag{1}$$

Lloyd's Approximation 2:

$$\boxed{\frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1} \tag{2}$$

where $\mathrm{CoV_{tr}}$ is the coefficient of variation of the reserve based on the truncated set of loss data representing realistically foreseeable events with the return period of up to $Y = 1/(1-p)$ years.

- The two approximations, whilst being similar, could produce noticeably different results (both uplifts differ by factor p)!
- Quality of approximations?

Quick derivation from first principles

Step 1. The k-th non-central moment of $X_{obs} = \{X|X \leq b\}$ $= \{X \leq F_X^{-1}(p)\}$ is the k-th truncated non-central moment of log-normally distributed reserve X which is equal to:

$$\boxed{m_{tr}^{(k)} = \mathbb{E}[X_{obs}^k] = \mathbb{E}[X^k|X \leq b] = m^{(k)} \cdot \frac{\Phi\left(\Phi^{-1}(p) - k\sigma\right)}{p}} \tag{3}$$

Equation 3 is used to derive the load for reserve Mean and CoV.

From here,

$$\boxed{\textbf{Mean\_Load} = \frac{m^{(1)}}{m_{tr}^{(1)}} - 1 = \frac{1}{\alpha} - 1},$$

where $\alpha = \frac{\Phi\left(\Phi^{-1}(p)-\sigma\right)}{p} \in (0,1)$ and tends to 1 as p goes to 1.

Demystifying Lloyd's Approximations (2)

Step 2. By assuming $\mathrm{CoV} \approx \mathrm{CoV}_{\mathrm{tr}}$ (crude assumption!!!), we get the estimate of shape parameter σ of log-normal reserve risk profile, $\widehat{\sigma} = \sqrt{\ln\left(\mathrm{CoV}_{\mathrm{tr}}{}^2 + 1\right)}$, and use it further to approximate $\mathbf{Mean\_Load}$:

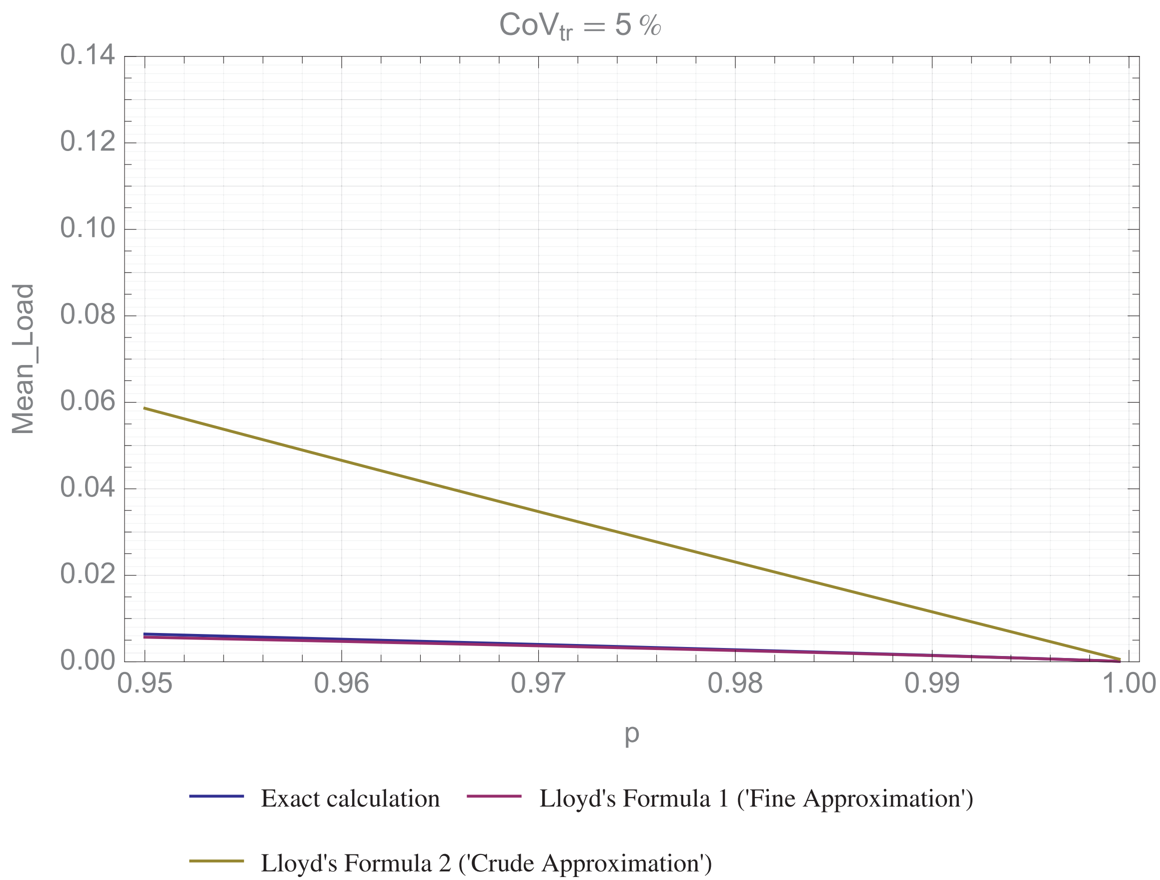
$$\begin{aligned}
\textbf{Approximation 1} &= \frac{1}{\alpha} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \widehat{\sigma}\right)} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV}_{\mathrm{tr}}{}^2 + 1\right)}\right)} - 1.
\end{aligned}$$

The second approximation is then obtained from **Approximation 1** by further assuming $p \approx 1$, in which case we have a crude approximation

$$\textbf{Approximation 2} = \frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV}_{\mathrm{tr}}{}^2 + 1\right)}\right)} - 1.$$

The true (exact) value of ENID load is between the two approximations, but Approximation 1 is generally of much better quality than Approximation 2.

Functional relationship between $\mathbf{Mean\_Load}$ and p under varying value of $\mathrm{CoV}_{\mathrm{tr}}$.



Lloyd's Approximations – concluding remarks (1)

Institute and Faculty of Actuaries

- Know what Lloyd's Approximation you are dealing with!
  - Approximation 2 is !very! conservative and could overstate the ENID load by 65% to 350% for reserve risk profiles with CoV of 10% to 35% and observable on a set of data with return period of up to 20 years; whereas
  - Approximation 1, whilst generally being of better quality, could understate the ENID load by at least 25% for reserve risk profiles with CoV above 35%
- Do they allow for parameter uncertainty due to limited historical data?
  - Yes, as the Truncation Distribution Approach explicitly does it. However, there is a secondary uncertainty associated with the choice of truncation point or parameter p.
  - Hard to choose p? Please use Bayesian Inference to combine Scenario Analysis and Expert Judgement.
- Do they allow for model uncertainty?
  - No, not really. **Although 'log-normality' choice might still be appropriate for certain reserving classes!**

Moving beyond the 'log-normality bubble'

- Log-normality does not cover the whole range of practically feasible reserve risk profiles, i.e. for $\mathrm{CoV}_X \leq 50\%$ and disproportionally higher(lower) skewness
  - e.g. how about $\mathrm{CoV}_X = 20\%$ and skewness γ_X of 1.0 (or 0.4) or equivalently 5 (or 2), when expressed per unit of CoV_X (i.e. Skewness-to-CoV (SC) ratio)?
  - with log-normality we could only achieve the SC ratio in the range from 3 to 3.25 for $\mathrm{CoV}_X \leq 50\%$
- Is there a way of estimating ENID load using only the reserve risk profile's characteristics like CoV and Skewness without knowing the parametric structure of reserve distribution? Yes, the new approach presented in the next slides addresses this!

Institute and Faculty of Actuaries

- Distribution-free approximation -

21 June 2016

... rather inspired by the idea of being able to solve any world's problem on the back of a cocktail napkin

The art of 'guesstimation' ...

... **Problem 1:** *how far does a football player travel during the course of a 90-minute game?*

Answer: ≈ 20km

... **Problem 2:** *if all French baguettes sold in Paris last year were placed end-to-end, what distance would they cover?*

Answer: ???

Single Shape Parameter distributions (1)

- Most two-parameter distributions commonly used in insurance for reserving and loss modelling are of a special type:
  - their *scale* and *shape* parameters are separated
  - the shape of the distribution is fully explained by its shape parameter;
  - or equivalently, any higher-order statistic like skewness, kurtosis, etc. is fully explained by CoV
- This class of distributions is called **Single Shape Parameter (SSP) distributions**.
- **Examples of SSP distributions include:** *Gamma, Inverse-Gaussian (Wild), Log-Normal, Dagum, Suzuki, Exponentiated-Exponential (Verhulst), Inverse-Gamma (Vinci), Birnbaum-Saunders, Exponentiated-Fréchet and Log-Logistic.*
- Not all two-parameter distributions are of SSP type - e.g. Log-Gamma distribution (i.e. $\mathrm{Exp}\left[\mathrm{Gamma}(\alpha, \beta)\right]$).

Single Shape Parameter distributions (2)

The SSP distributions can be split into three main categories:

- **Moderately skewed distributions ($1.5 < SC \leq 3$)**
  - *Gamma*: $SC = 2$;
  - *Inverse-Gaussian (Wald)*: $SC = 3$;
- **Significantly skewed distributions ($3 < SC < 4$)**
  - *Log-Normal*: $SC = 3 + \mathrm{CoV}^2 \in (3, 3.25),\ \mathrm{CoV} < 50\%$;
  - *Suzuki*;
  - *Exponentiated-Exponential (Verhulst)*;
  - *Dagum*;
- **Extremely skewed distributions ($SC > 4$)**
  - *Inverse-Gamma (Vinci)*: $SC = \frac{4}{1-\mathrm{CoV}^2} > 4,\ \mathrm{CoV} < 100\%$;
  - *Birnbaum-Saunders*;
  - *Log-Logistic*;
  - *Exponentiated-Fréchet*.

SSP characterisation of reserves

SSP characterisation of reserves using the four main SSP distributions: *Gamma*, *I-Gaussian*, *Log-Normal* **and** *I-Gamma*

Table 1: *Differentiation of reserve risk profile by type of reserve class.*

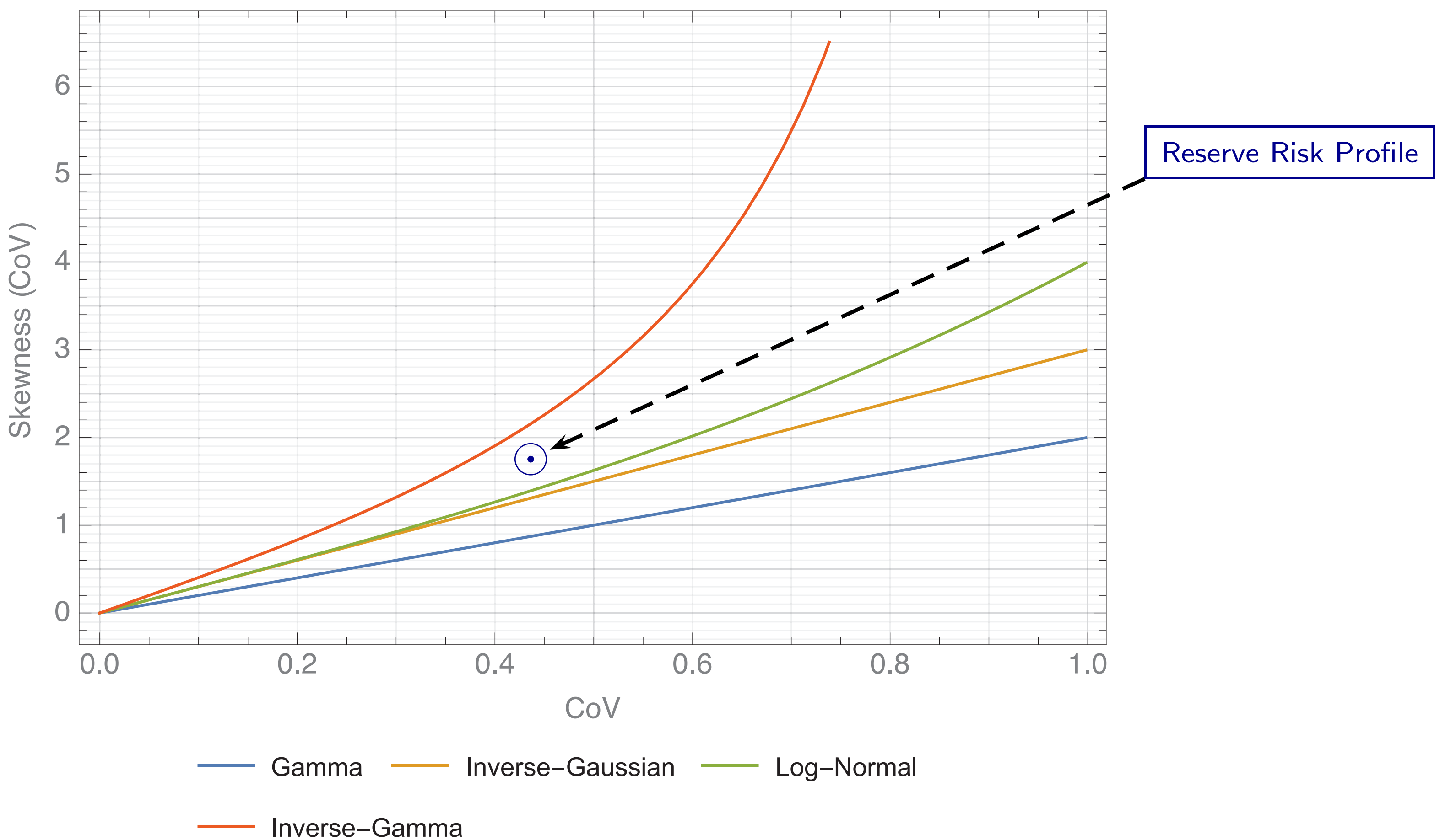
Type of reserving class				
Duration	**CoV range**	**Skewness (SC ratio)**	**Parametric distribution(s)**	**Example of reserving class**
Short tail	10%-12%	1.9 to 2.1	Gamma	Motor (ex Bodily Injury)
Short tail	12%-16%	2.0 to 3.0	Gamma, Inverse-Gaussian (Wald)	Home
Short tail	10%-16%	2.9 to 3.1	Inverse-Gaussian (Wald), Log-Normal	Comm Property/Fire, Comm Accident
Long tail	12%-25%	3.0 to 3.5	Log-Normal	Motor Bodily Injury, Marine
Long tail	18%-50%	3.0 to 4.0	Log-Normal, Inverse-Gamma (Vinci)	Workers Comp, Prof Liab, Comm Liab
Long tail	25%-70%	> 4	Inverse-Gamma (Vinci)	Asbestos and other long tail books

In reality, your reserve risk profile is unlikely to follow any of the four SSP distributions exactly.

Locating any reserve risk profile ...

... in a system of CoV–SC coordinates with respect to the four main SSP distributions: *Gamma*, *I-Gaussian*, *Log-Normal* and *I-Gamma*

Figure 1: *Skewness as a function of CoV for the four parametric distributions.*



Setting up the problem ...

- **Key assumption:** the true unknown distribution of reserve X is assumed to be of SSP type, and thus the unknown true value of skewness γ_X is defined by $\mathrm{CoV}_X = \mathrm{CoV}$

$$\gamma_X = \gamma\left(\mathrm{CoV}\right) = SC\left(\mathrm{CoV}\right) \cdot \mathrm{CoV}. \tag{4}$$

- **Goal:** find the way of estimating $\mathbf{Mean\_Load} = \frac{m}{m_{tr}} - 1$ using only the following information
  1. truncation point p;
  2. CoV_{tr} – 'observable' volatility of reserve X; and
  3. SC ratio of reserve distribution (one of the key characteristics of a SSP type of distribution)

- **Final approximations are tabulated:** by CoV_{tr}, SC and p. Estimate adjustments can be applied if needed?

The idea behind the distribution-free approximation

The main theory was outlined at LMAG in February 2016. This presentation focuses on practical aspects only, hence the **high level key steps** (for more technical details please refer to Appendix and/or the paper):

1. Decompose the reserve risk profile X into its smaller copy $\widetilde{X} = \frac{X-m}{s}$ and true value of BE, m, and volatility CoV, i.e. $X = m\left(1 + \mathrm{CoV} \cdot \widetilde{X}\right)$, or equivalently $\mathrm{VaR}_p\left(X\right) = m\left(1 + \mathrm{CoV} \cdot \mathrm{VaR}_p\left(\widetilde{X}\right)\right)$.
2. Use 1. to derive the truncated first two moments m_{tr} and CoV_{tr}, and express ENID loading $\frac{m}{m_{tr}}$ and CoV_{tr} as analytical functions of true volatility CoV.
3. Numerically invert CoV from $\mathrm{CoV}_{tr} = \mathrm{CoV}_{tr}\left(\mathrm{CoV}\right)$ and use this value to get the ENID loading estimate $\frac{m}{m_{tr}}\left(\mathrm{CoV}\right)$.

Tabulated results

Key outputs: ENID load approximations can be pre-computed and tabulated by CoV_{tr}, SC and p at any desirable resolution.

Sample output of **Distribution-Free (DF) Estimates:**

Table 24: **Mean_Load** *approximation (in %) under* $\mathrm{CoV}_{\mathrm{tr}} = 30\%$.

SC =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
2.0	4.415	4.001	3.585	3.169	2.749	2.327	1.899	1.463	1.014
2.2	4.492	4.069	3.646	3.221	2.794	2.364	1.928	1.485	1.030
2.4	4.570	4.138	3.706	3.273	2.838	2.400	1.957	1.507	1.045
2.6	4.647	4.206	3.765	3.324	2.881	2.436	1.986	1.529	1.060
2.8	4.724	4.274	3.825	3.375	2.925	2.472	2.015	1.550	1.074
3.0	4.802	4.342	3.884	3.426	2.968	2.507	2.043	1.572	1.089
3.2	4.879	4.410	3.943	3.477	3.010	2.542	2.070	1.592	1.103
3.4	4.957	4.479	4.002	3.527	3.053	2.577	2.098	1.613	1.117
3.6	5.035	4.547	4.061	3.578	3.095	2.611	2.125	1.634	1.131
3.8	5.113	4.615	4.120	3.628	3.137	2.646	2.152	1.654	1.145
4.0	5.191	4.683	4.179	3.678	3.179	2.680	2.179	1.674	1.158
4.2	5.269	4.751	4.237	3.727	3.220	2.714	2.206	1.694	1.172
4.4	5.346	4.818	4.295	3.777	3.261	2.747	2.233	1.714	1.186
4.6	5.422	4.884	4.352	3.825	3.302	2.781	2.259	1.734	1.199
4.8	5.497	4.949	4.409	3.873	3.342	2.814	2.285	1.754	1.213
5.0	5.568	5.012	4.463	3.920	3.382	2.846	2.311	1.773	1.227
5.2	5.636	5.073	4.516	3.966	3.421	2.878	2.337	1.793	1.240

Additional outputs: Correction factors $\mathbf{f} = \dfrac{\textbf{Exact Value}}{\textbf{DF Estimate}}$ for the four main parametric SSP distributions.

Quality of the distribution-free approximation ...

... **is generally good** across all SSP distributions in the range between Gamma and Inverse-Gamma.

Distribution-Free approximation is better than Lloyd's Approximation 1, when assuming log-normality.

Log-Normal vs. Distribution-Free estimates

ENID load estimates for practical values of $\mathrm{CoV}_{\mathrm{tr}}$ in the range between 10% **and** 35%: comparison between Log-Normal and Distribution-Free estimates.

Log-Normal [$p = 99.5\%$]	Log-Normal [$p = 95\%$]	Distribution-Free [$p = 95\%$]
$< 0.75\%$	1.35% to 5.9%	1.30% to 7.8%

So we are guessing most of you assume a Log-Normal distribution for most of your reserve risk modelling and assume you can "foresee" to a 1-in-200 ... this would imply an ENID load of no more than 0.75% ... how does that compare to your selections, when assuming log-normality?

For a typical observable reserve volatility of 15%, assuming log-normality, to get 3% of ENID load you need to foresee only to a 1-in-12 and for a 5% load to a 1-in-8.

Practical implementation

- Numerical computations used in ENID load approximation are coded (R, Matlab or Wolfram M). Pre-computed and tabulated ENID load approximations can then be further utilised in developing a 'reference calculation tool'.

- This can be easily implemented in Excel using grid search and interpolation, ... or even developed and deployed as an app (e.g. Wolfram).

For a given reserve risk profile with CoV_{tr}, SC and p

- read (grid search) the tabulated approximation;
- compute the correction factor by
  - *locating the reserve risk profile with respect to known SSP distributions* – by comparing its SC ratio to those of the known parametric SSP distributions;
  - *interpolate the correction factor* between the parametric distributions adjacent to the reserve risk profile; and
- use it to adjust the initial distribution-free approximation of ENID load.

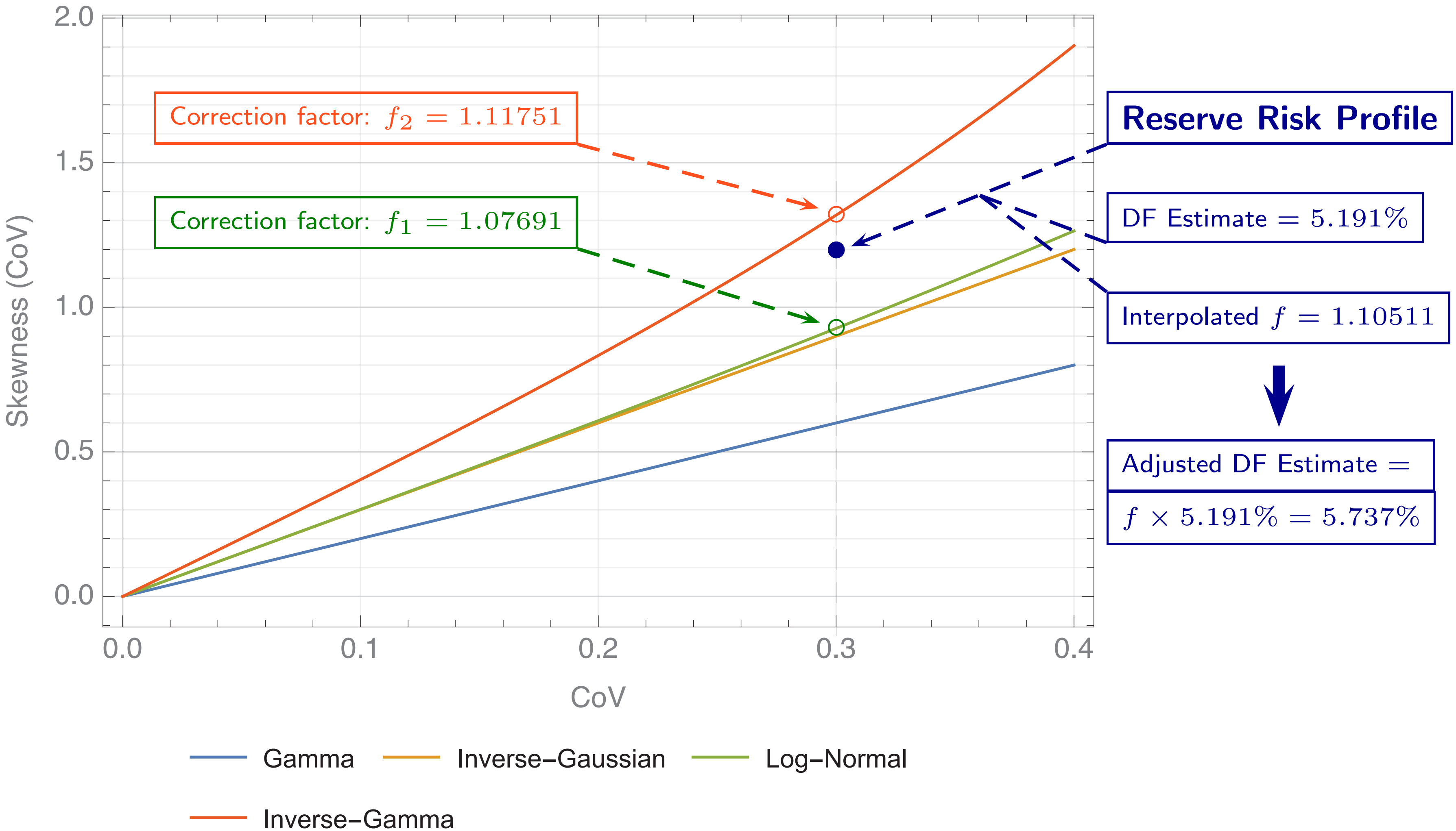
Practical example

- Consider a reserve X with
  - truncated (observable) volatility of $\mathrm{CoV}_{\mathrm{tr}}(X) = 30\%$;
  - implied SC ratio of 4; and
  - $p = 0.95$, i.e. assuming the reserve is formed based on the loss events with the return period of up to 20 years.
- The given reserve risk profile is confined between **Log-Normal** and **Inverse-Gamma** distributions, as for the given level of $\mathrm{CoV}_{\mathrm{tr}}(X)$ at 30%:

$$SC_{\mathrm{Log-Normal}} = 3+0.3^2 = \mathbf{3.09} < \boxed{SC_X = \mathbf{4}} < SC_{\mathrm{Inv-Gamma}} = \frac{4}{1-0.3^2} \approx \mathbf{4.4}.$$

Practical example (contd): locating the profile

... in a system of CoV–SC coordinates with respect to the four main SSP distributions: *Gamma*, *I-Gaussian*, *Log-Normal* and *I-Gamma*



Institute and Faculty of Actuaries

- Conclusions -

So what does this mean for you?

Key takeaway points

- importantly you can estimate an ENID load with more confidence
- you can worry less about explaining it - link it to parameterisation of the capital model and say this falls out ... also helps with consistency
- will help to create a better documentation and form an expert judgement perspective ... note you do still need to justify the point of "foreseeable" ... but this is not change
- should help in validation / regulatory aspects of your Technical Provisions
- and finally / most importantly it confirms that Lloyd's original estimates were sensible! ...

References

KRVAVYCH, Y. Binary Events Loading for Solvency II Technical Provisions: Practical Approximations. *ASTIN Colloquium*, Sydney, 2015. [Can be downloaded from SSRN].

FLEMING, K. Yep, We Are Skewed. *Variance* vol. 2, issue 2, 2008, pp. 179 – 183. [Can be downloaded from here].

Technical Provisions under Solvency II: Detailed Guidance. LLOYD'S, 2011.

ADAM, J. et al. *Guestimation: solving the world's problems on the back of a cocktail napkin*. Princeton University Press, 2008.

Thank You

Contact

J. Kirk and Y. Krvavych
PricewaterhouseCoopers LLP
Actuarial Services

7 More London Riverside
London SE1 2RT

J. Kirk: jerome.kirk@uk.pwc.com
Y. Krvavych: yuriy.krvavych@uk.pwc.com
www.pwc.co.uk

Questions

Comments

Expressions of individual views by members of the Institute and Faculty of Actuaries and its staff are encouraged.

The views expressed in this presentation are those of the presenter.

Institute and Faculty of Actuaries

Institute and Faculty of Actuaries

- Appendix -

21 June 2016

Getting the distribution-free approximation (1)

Key steps:

1. $X = m\left(1 + \mathrm{CoV}\cdot\widetilde{X}\right)$, where $\widetilde{X} = \frac{X-m}{s}$, and also

$$\mathrm{VaR}_p\left(X\right) = m\left(1 + \mathrm{CoV}\cdot\mathrm{VaR}_p\left(\widetilde{X}\right)\right)$$

2. **The truncated mean** of reserve X is then

$$\begin{aligned} m_{tr} &= \mathbb{E}[X \,|\, X \le \mathrm{VaR}_p\left(X\right)] \\ &= m\left(1 + \mathrm{CoV}\cdot\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \le \mathrm{VaR}_p\left(\widetilde{X}\right)\right]\right), \end{aligned}$$

from where the ENID uplift factor for reserve mean is defined as

$$\boxed{\frac{m}{m_{tr}} = \frac{1}{1 + \mathrm{CoV}\cdot\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \le \mathrm{VaR}_p\left(\widetilde{X}\right)\right]}} \qquad (5)$$

Getting the distribution-free approximation (2)

3. **The truncated variance** of X is calculated as follows

$$\begin{aligned} s_{tr}^2 &= \mathbb{E}\left[(X - m_{tr})^2 \,|\, X \leq \mathrm{VaR}_p\,(X)\right] \\ &= m^2\,\mathrm{CoV}^2\left(\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]\right), \end{aligned} \tag{6}$$

where $\widetilde{b} = \mathrm{VaR}_p\left(\widetilde{X}\right)$.

By combining Equation 6 and Equation 5 we obtain the following formula for truncated variance:

$$\boxed{\mathrm{CoV}_{tr}^2 = \frac{\mathrm{CoV}^2 \cdot \left(\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]\right)}{\left(1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]\right)^2}} \tag{7}$$

Getting the distribution-free approximation (3)

4. $\widetilde{X}$ **is assumed to be approximated by a Fleishman quadratic polynomial** of a standard normal random variable $Z \sim \mathcal{N}(0,1)$:

$$\widetilde{X} \overset{d}{\approx} P_2(Z) = a_1 Z + a_2(Z^2 - 1), \tag{8}$$

where the Fleishman coefficients a_1 and a_2 are calibrated so that $P_2(Z)$ has unit variance and its skewness is equal to γ – skewness of X:

$$\begin{cases} 1 & = & a_1^2 + 2a_2^2, \\ \gamma(\mathrm{CoV}) & = & 6a_1^2 a_2 + 8a_2^3 \end{cases} \tag{9}$$

5. The p-quantile of random variable $\widetilde{X}$ **is assumed to be approximated by the Normal Power approximation**:

$$\widetilde{b} = \mathrm{VaR}_p\left(\widetilde{X}\right) \approx z_p + \gamma(\mathrm{CoV}) \cdot \frac{z_p^2 - 1}{6}, \tag{10}$$

where $z_p = \mathrm{VaR}_p\left(Z\right)$.

Getting the distribution-free approximation (4)

Institute and Faculty of Actuaries

Further analytical transformations:

6. Calibrate Fleishman coefficients a_1 and a_2, and express them as functions of CoV;

7. Express the n-th truncated moment $\mathbb{E}\left[\widetilde{X}^n \,|\, \widetilde{X} \leq \widetilde{b}\right]$ for $n = 1, 2$ as a function of CoV by:

   - finding the equivalent probability condition of $\left\{\widetilde{X} \leq \widetilde{b}\right\}$ defined through the random variable Z – this will come in the following form of $\{c \leq Z \leq d\}$ with c and d being functions of CoV; and then

   - calculating the n-th truncated moment of the standard normal random variable Z, i.e. $\mathbb{E}\left[Z^n \,|\, c \leq Z \leq d\right]$ for $n = 1, ..., 4$.

Let us further denote the n-th truncated moment of Z by

$$I_n = \mathbb{E}\left[Z^n \,|\, c \le Z \le d\right], \, n \ge 0. \tag{11}$$

Then using the Fleishman approximation of $\widetilde{X}$ in (8), we obtain

$$\begin{aligned}
\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \le \widetilde{b}\right] &= a_2 I_2 + a_1 I_1 - a_2 I_0, \\
\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \le \widetilde{b}\right] &= a_2^2 I_4 + 2a_1 a_2 I_3 + \left(1 - 4a_2^2\right) I_2 \\
&\quad -2a_1 a_2 I_1 + a_2^2 I_0.
\end{aligned}$$

The n-th truncated moment of Z in (11) can be computed iteratively using the following formula

$$\begin{cases}
I_n = -\frac{d^{n-1}\varphi(d) - c^{n-1}\varphi(c)}{\Phi(d)-\Phi(c)} + (n-1) I_{n-2}, \text{ with} \\
I_1 = -\frac{\varphi(d)-\varphi(c)}{\Phi(d)-\Phi(c)}, \text{ and } I_0 = 1.
\end{cases}$$

Getting the final approximation

We conclude that both $\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]$ and $\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \leq \widetilde{b}\right]$ are analytical functions of CoV, as

- I_n is a function of c and d;
- both c and d are functions of a_1, a_2 and $\widetilde{b}$; and finally
- a_1, a_2 and $\widetilde{b}$ are analytical functions of $\gamma(\mathrm{CoV})$.

The following equation is then numerically solved for CoV:

$$\boxed{\mathrm{CoV}_{tr}^2 = \frac{\mathrm{CoV}^2 \cdot \left(\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]\right)}{\left(1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right]\right)^2}} \tag{12}$$

The derived ultimate estimate $\widehat{\mathrm{CoV}}$ is then used to estimate the ENID load.