Actuarial Research Centre

Institute and Faculty of Actuaries

Actuarial Research Centre (ARC)

PhD studentship output

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Multi-population mortality models

Vasil Enchev

joint work with

Torsten Kleinow and Andrew Cairns

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HERIOT WATT UNIVERSITY

Actuarial Research Centre

Motivation and objectives

- Multi-population mortality models at older ages
- Research is pretty limited in comparison to the single-population mortality models
- Comparison
  - Fitting solution for the models
  - Identifiability problems correction
  - Joint mortality scenario forecast
- Rank the models value

The models

Model 0 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + \beta_{(x,i)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

The Li and Lee model (2005)

$m_{(x,t,i)}$	Crude mortality rates (deaths over exposure)	$x \times t \times i$
$\alpha_{(x,i)}$	Determines the general mortality shape	$x \times i$
$K_{(t)}$	Explains the evolution of the global mortality over time	$t \times 1$
$B_{(x)}$	Shows which rates change more rapidly in responde to $K_{(t)}$	$x \times 1$
$\kappa_{(t,i)}$	Explains the country specific evolution of the mortality over time	$t \times i$
$\beta_{(x,i)}$	Shows which rates change more rapidly in responde to $\kappa_{(t,i)}$	$x \times i$

The models

Model 0 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + \beta_{(x,i)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

Model 1 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + \beta_{(x)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow \beta_{(x)}$

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↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow \beta_{(x)}$

Model 2 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + B_{(x)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow B_{(x)}$

$m_{(x,t,i)}$	Crude mortality rates (deaths over exposure)	$x \times t \times i$
$\alpha_{(x,i)}$	Determines the general mortality shape	$x \times i$
$K_{(t)}$	Explains the evolution of the global mortality over time	$t \times 1$
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The models

Model 0 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + \beta_{(x,i)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

Model 1 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + \beta_{(x)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow \beta_{(x)}$

Model 2 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}K_{(t)} + B_{(x)}\kappa_{(t,i)} + \varepsilon_{(x,t,i)}$

↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow B_{(x)}$

The Kleinow model (2015)

Model 3 $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + \beta^1_{(x)}\kappa^1_{(t,i)} + \beta^2_{(x)}\kappa^2_{(t,i)} + \varepsilon_{(x,t,i)}$

Country specific parameter: $K_{(t)} \rightsquigarrow \kappa^1_{(t,i)}$ ↲

↳ Common parameter: $\beta_{(x,i)} \rightsquigarrow \beta^2_{(x)}$

$m_{(x,t,i)}$	Crude mortality rates (deaths over exposure)	$x \times t \times i$
$\alpha_{(x,i)}$	Determines the general mortality shape	$x \times i$
$\kappa^1_{(t,i)}$	Explains the evolution of the global mortality over time	$t \times 1$
$\beta^1_{(x)}$	Shows which rates change more rapidly in responde to $\kappa^1_{(t,i)}$	$x \times 1$
$\kappa^2_{(t,i)}$	Explains the country specific evolution of the mortality over time	$t \times i$
$\beta^2_{(x)}$	Shows which rates change more rapidly in responde to $\kappa^2_{(t,i)}$	$x \times 1$

Identifiability problems

- The models are undetermined
  - Parametrisation I
    Model 0
    $\log[m_{(x,t,i)}] = \tilde{\alpha}_{(x,i)} + \tilde{B}_{(x)}\tilde{K}_{(t)} + \tilde{\beta}_{(x,i)}\tilde{\kappa}_{(t,i)} + \varepsilon_{(x,t,i)}$

$$\widetilde{\alpha}_{(x,i)} = \alpha_{(x,i)} + aB_{(x)} + c\beta_{(x,i)} \qquad \widetilde{B}_{(x)} = \frac{B_{(x)}}{b}$$
$$\widetilde{\beta}_{(x,i)} = \frac{\beta_{(x,i)}}{d} \qquad \widetilde{K}_{(t)} = b(K_{(t)} - a)$$
$$\widetilde{\kappa}_{(t,i)} = d(\kappa_{(t,i)} - c)$$

  - Parametrisation II
    Model 2
    $\log[m_{(x,t,i)}] = \alpha_{(x,i)} + B_{(x)}[\underbrace{K_{(t)} + C_{(t)}}_{\widetilde{K}_{(t)}} + \underbrace{\kappa_{(t,i)} - C_{(t)}}_{\widetilde{\kappa}_{(t,i)}}] + \varepsilon_{(x,t,i)}$

Parameter constraints

	Model 0	Model 1	Model 2	Model 3
Common constraints				
	$\sum_x B_{(x)} = 1$	$\sum_x B_{(x)} = 1$	$\sum_x B_{(x)} = 1$	$\sum_x \beta^1_{(x)} = 1$
	$\sum_t K_{(t)} = 0$	$\sum_t K_{(t)} = 0$	$\sum_t K_{(t)} = 0$	$\sum_x \beta^2_{(x)} = 1$
	-	$\sum_x \beta_{(x)} = 1$	-	-
Country specific constraints				
For each i:				
	$\sum_x \beta_{(x,i)} = 1$	-	-	$\sum_t \kappa^1_{(t,i)} = 0$
	$\sum_t \kappa_{(t,i)} = 0$	$\sum_t \kappa_{(t,i)} = 0$	$\sum_t \kappa_{(t,i)} = 0$	$\sum_t \kappa^2_{(t,i)} = 0$
Time specific constraints				
For each t:				
	-	$\sum_i \kappa_{(t,i)} = 0$	$\sum_i \kappa_{(t,i)} = 0$	$\sum_i \kappa^2_{(t,i)} = 0$
	-	Quasi identifiability constraint	True identifiability constraint	Quasi identifiability constraint

Parameters estimation

- The used countries data

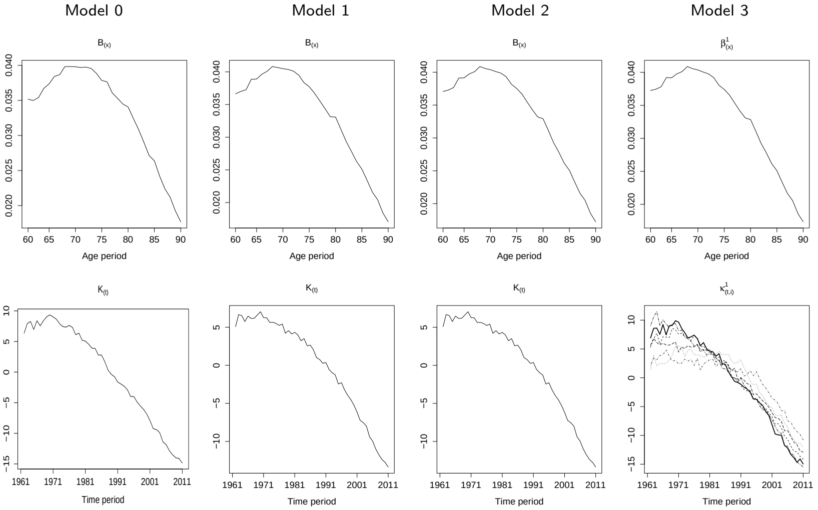
		Exposure to risk at the age of 60 in 2010	
i	Country	Male	Female
1	Austria	47023.17	49526.5
2	Belgium	65344.67	66434.33
3	Czech Republic	71575.69	71575.69
4	Denmark	34420.17	35132.33
5	Sweden	59759.83	59742.67
6	Switzerland	46527.67	47078.67

- Single step maximum likelihood estimation

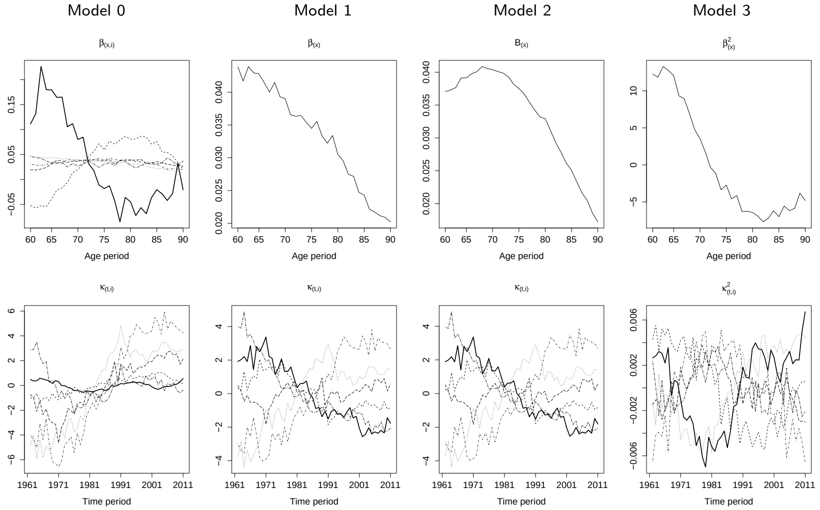
$$l = \sum_{x,t,i} \left[D_{(x,t,i)} \log(m_{(x,t,i)}) - E_{(x,t,i)} m_{(x,t,i)} \right] + \mathbf{C}_{(x,t,i)}$$

- Optimizing l with the Newton-Raphson method

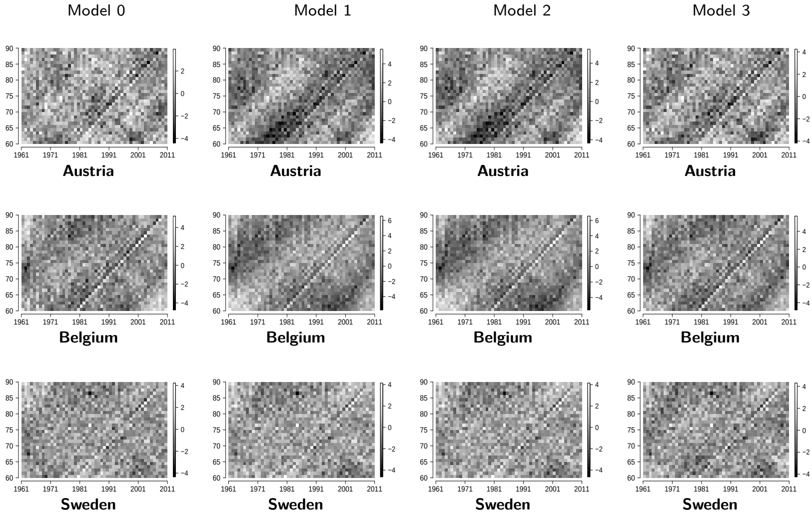
Common parameters plots



Country specific parameters plots



Standardized residuals plots



Comparison of the models

- Information criterion - ranking the models

BIC value $= -2[\log - \text{likelihood value}] + \log(N)k^{effective}$

	N	k	k^{effective}	log-likelihood	BIC	rank
Model 0	9000	740	726	-46716.43	86822.64	(2)
Model 1	9000	590	531	-48396.82	91958.89	(3)
Model 2	9000	560	502	-48477.21	92383.72	(4)
Model 3	9000	840	776	-46369.96	85674.45	(1)

- Estimation time - iterations count

	Model 0	**Model 1**	**Model 2**	**Model 3**
Number of iterations	200+	10+	5+	40+

Forecasting joint mortality rates

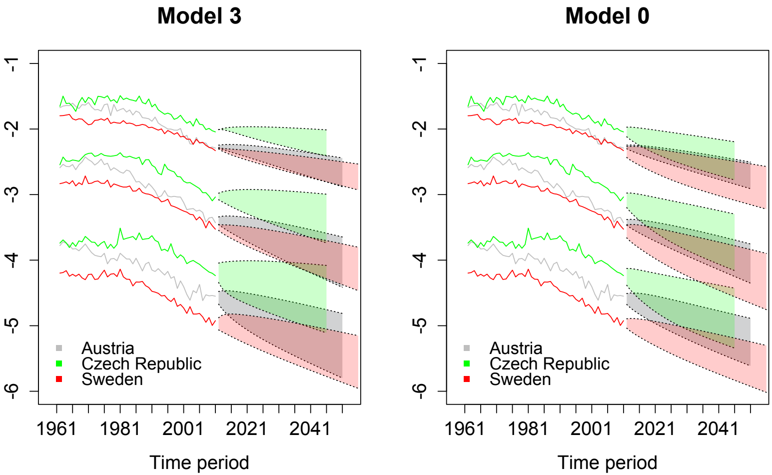
- Choosing time series process
  - Dependent on the parameters shape
  - Constraints over the parameters
  - In sample residuals correlation ratios
- Common parameters

Model 0	**Model 1**	**Model 2**	**Model 3**
Random walk with drift	Random walk with drift	Random walk with drift	Multi variate random walk with common drift
$K_{(t)} = d + K_{(t-1)} + \sigma Z_{(t)}$	$K_{(t)} = d + K_{(t-1)} + \sigma Z_{(t)}$	$K_{(t)} = d + K_{(t-1)} + \sigma Z_{(t)}$	$\kappa^1_{(t,i)} = d_c + \kappa^1_{(t-1,i)} + CZ_{(t)}$

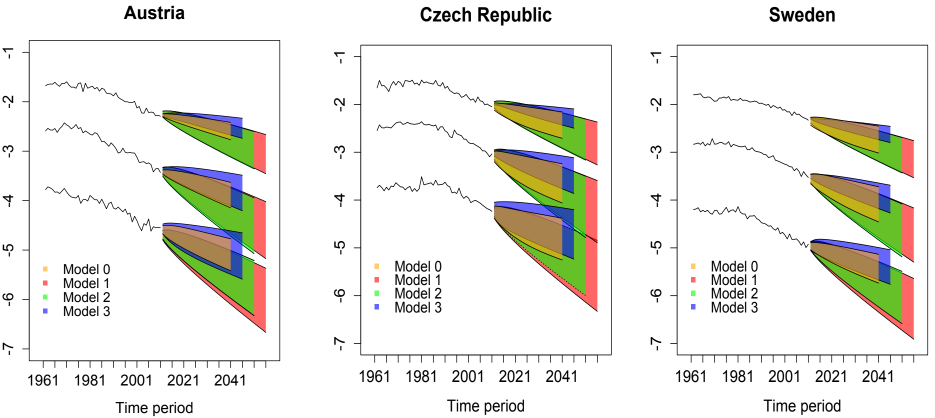
- Country specific parameters

Model 0	**Model 1**	**Model 2**	**Model 3**
First order vector autoregression process	Multi variate random walk with drift	Multi variate random walk with drift	First order vector autoregression process (Φ is diagonal matrix)
$\kappa_{(t,i)} = \Phi \kappa_{(t-1,i)} + CZ_{(t)}$	$\kappa_{(t,i)} = d + \kappa_{(t-1,i)} + CZ_{(t)}$	$\kappa_{(t,i)} = d + \kappa_{(t-1,i)} + CZ_{(t)}$	$\kappa^2_{(t,i)} = \Phi \kappa^2_{(t-1,i)} + CZ_{(t)}$

Forecasting results



Forecasting results



Conclusion

- Comparison of multi-population models
- Reduced parameter number
- Models that are estimated faster
- Generation of joint scenarios for future mortality rates