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BINARY CALCULATION

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1. The development of this paper involves the consideration of certain arithmetical, mechanical, and photo-electrical technicalities, but the ultimate aim is a very simple one and it may therefore be advisable to state it at once. The ultimate aim is to persuade the whole civilized world to abandon decimal numeration and to use octonal numeration in its place; to discontinue counting in tens and to count in eights instead.

2. However, it seems unlikely that the whole civilized world will be persuaded to complete this change during the next twelve months, having previously declined similar invitations. Therefore the more immediate aim is the adoption of octonal numeration for scientific and business purposes, for the great mass of figures recorded and manipulated for the benefit only of the scientific and business man, the few final results required for presentation to the layman being transformed into the denary scale of notation from the octonary by means of conversion tables, or otherwise.

HISTORICAL

3. In order to dispel at once any anticipation that this paper aspires to novelty, except in mere detail, it may as well be said that what is being contended for is in effect a return to the arithmetical methods of about 5000 years ago when, it seems, arithmetic first came into existence. In the British Museum is the Rhind papyrus, one of the most ancient mathematical treatises known, a manuscript written by a scribe named Ahmes about 1650 B.C., but believed to be founded upon a still older treatise. It is entitled "Directions for obtaining knowledge of all dark things", and is a collection of problems in arithmetic and geometry, with the answers, and its conciseness should appeal to a certain modern school of thought, for frequently the method by which the answer was obtained is not stated.

4. There seems no doubt that the denary system owes its origin to the fact that we have ten digits on our two hands; so fully is this accepted that the fact that the Greenlanders count in twenties, as do the barefooted Mexicans, is regarded as evidence of their tropical origin. Not unnaturally the denary system was preceded by the quinary, the five digits of one hand only, but in truth it is only in the last few thousand years that man has learned to count in either fives or tens. By the Statute of Shrewsbury, in the seventh century, the qualification for a witness in a court of law was an ability to count up to nine (*The Story of Arithmetic*—Cunnington). Long before he was able to recognize such quantities man had been impressed by the quality of two-ness, owing to his possessing two legs, arms, ears and eyes. Hence a scale of notation with the radix 2; counting was originally binary. The evidence for this is plentiful. The binary system is still found in Australia among tribes ethnologically the oldest. A pure pair system still occurs in many Papuan languages of the Torres Straits and the adjacent coast of New Guinea; in Africa it is practised by the Bushmen; in South America it is found among the ethnologically oldest tribes (*Encyc. Brit.*).

5. The civilization to which Ahmes belonged had abandoned the binary scale, and the difficulties involved in multiplication were very great, as can be readily imagined. Those who wish to experience these difficulties should multiply two such numbers as LXXXIX by XXXVIII in Roman numerals, assuming no knowledge of the multiplication tables. The difficulties of multiplication are further illustrated by the number of different ways proposed at various times for carrying out the process, five methods of multiplication being given by the Hindoo mathematician, Bhaskara (twelfth century), and Pacioli (end of fifteenth century) gave eight. To these may be added the ancient method of repeated doubling as developed with considerable skill by the Egyptians before 1650 B.C., and invariably used by Ahmes, in conjunction with an occasional multiplication by 10.

Using M to represent the multiplicand, the method is as follows:

Multiplicand	...	M
Double ...	...	2M
Redouble ...	...	4M
Redouble ...	...	8M

By adding the first line and the last two we obtain 13M. In other words Ahmes multiplied by the binary system, although without the convenience of binary numeration, which had by then been superseded.

6. Radices other than 2, 5, 10 and 20 have been considered from time to time; the result has been summarized thus: If the radix is too small, the number of figures to be used is too large. If the radix is too large, the number of symbols required is too large. 8, 10 and 12 are medium radices, either 8 or 12 being preferable to 10. "The octonary system, founded upon the number eight, most completely presents the qualities which are desired in a system of notation" (T. F. Brownell, *Pop. Science Mo.* Vol. XIII, p. 427). "No doubt, an octaval system of numeration, with its possible subdivision 8, 4, 2, 1, would have been originally better; but there is no sufficient reason for a change now" (*Science*, Vol. IV, p. 415). Just over a hundred years ago (1834) Lappenberg's *A History of England under the Anglo-Saxon Kings* appeared. The following is quoted from Benjamin Thorpe's translation published in 1845: "Of greater importance for historical investigation would be the knowledge of the numeral system in use among the Saxons. I am inclined to the belief that the octonary, on account of its facility of division, was the one followed" (Vol. I, p. 82). *Ben trovato!* When the searcher lights by chance upon this, in so unexpected a quarter, he is tempted to search no further, but instead to say with Horace: "*Hic est aut nusquam quod quærimus.*"

7. What is proposed in this paper, then, is a reversion to Ahmes' method of multiplication, a method which is believed to have existed since 3000 B.C., so going back 5000 years to the binary system of multiplication. It is, however, proposed to incorporate:

(1) Octonary numeration, believed to have been used 1500 years ago by our Anglo-Saxon forefathers;

(2) The symbol for zero and the consequent convenience of positional value, developed by the Maya about 2000 years ago;

(3) A more recent and far less wonderful invention, the selenium photo-electric cell.

The first of these will facilitate the keeping of records, the second makes for convenience, and the third for rapidity, but the process in which they are suggested as aids is essentially that of five thousand years ago.

THE BINARY SCALE

8. An experienced adder can cast not merely one column at a time but two, and in exceptional cases three. In casting three columns at a time he may be regarded as working not in a pure scale of ten but in a hybrid scale of notation 1000, subdivided for convenient nomenclature into the denary scale. It is now proposed to use a similar modification of the binary scale. Figures would be tabulated in s.n. (scale of notation) 8, would be transformed on sight or mechanically into s.n. 2 for the purpose of multiplication, the results of which as actually expressed in s.n. 2 would then be read off at sight in s.n. 8. Thus the two numbers already given in Roman numerals, namely 89 and 38, would be expressed as 131 and 46 respectively in s.n. 8, but would be transformed on sight to 1,011,001 and 100,110 respectively in s.n. 2. These last two numbers can be multiplied together by anyone who knows that one times one is one, and who can count, and could therefore be dealt with by a child before he had learnt his multiplication tables, or by Ahmes who never knew any. The multiplication can also be effected by a machine, virtually with the speed of light; in fact by using a light ray.

9. Amongst others Leibnitz interested himself in s.n. 2 and has come (surely erroneously?) to be credited in many dictionaries (but not the *S.O.E.D.*) with having "invented" binary arithmetic. In every Chinese temple of importance hangs a mysterious scroll, consisting apparently of three rows of lines of different lengths. This is the Cora or binary system of counting, attributed to Fohi (twenty-third century B.C.), the founder of the Empire (*The Story of Arithmetic*—Cunnington). No doubt at some time or other it has occurred to everyone who has used an arithmometer that the mechanism would be much simpler, and the speed vastly greater, if we could calculate in s.n. 2 instead of in the denary scale. But always it has seemed that some missing link robbed this idea of its practicability. Something else is required, some other idea to

wed with this one, before anything fruitful can result. The difficulty, of course, was to imagine in which direction to look for this second ingredient; what form might it be expected to take? Or rather what forms, for it seems it can take two forms at least. Before proceeding with the present subject it will be chronologically proper and otherwise appropriate to refer to a suggestion formulated in another quarter.

10. Four years ago a certain M. Valtat of France visualized the missing link as something to enable us to pass by a mechanical process from s.n. 10 to s.n. 2. So far as I understand his proposal he was not so much concerned with applying any new principle *to the actual calculating machine but instead to use the principle* of existing arithmometers, or more correctly the combined principles of the two different standard types of arithmometer. It was, then, to the problem of conversion that he directed his attention, and he has suggested an ingenious machine for this purpose. It appears that he has in mind the conversion of multiplicand and multiplier from s.n. 10 only immediately prior to the multiplication, and the immediate reconversion of the product from s.n. 2 to s.n. 10.

11. Working along a different line of approach, the present writer, on the contrary, was visualizing the recording and compilation of figures in some other scale of notation than 10 over a period of time, possibly over a period of years, and for this purpose s.n. 2 would be exasperating and a source of much error. Seven significant figures in s.n. 10 may require as many as twenty-three in s.n. 2, and anyone who has had the least practical experience can visualize the danger of transposition and other integral error in compiling and dealing with line after line of figures, as many as twenty-three to the line, consisting only of units and cyphers. Thus the author was looking in an entirely different direction for the missing link, and he now submits that it has been discovered in s.n. 8. As soon as it is thought of it becomes obvious, and it would be difficult to understand why it was not previously found, but that one did not know what one was looking for. S.n. 8 converts into s.n. 2, and the latter reconverts into s.n. 8, on sight. Any clerk can master the conversion in two minutes, for he has only

to commit the following table to memory:

S.n. 8	S.n. 2
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

12. Numbers in s.n. 2 written in groups of three (which happily is already our custom in s.n. 10) can be read off at sight in s.n. 8. The reverse process is as easy, but here time can be saved by designing a simple machine by which, when a number in s.n. 8 is tapped out on the keys, the s.n. 2 equivalent at once results, either in printed or punched-hole form, or otherwise. Seven significant figures in s.n. 10 require only eight in s.n. 8, and the integers range from 0 to 7, giving groups indistinguishable from the kind of numbers we are already competent to compile and memorize, except that 8 and 9 are never seen. Any clerk who has once memorized the eight self-evident equalities above will be able to read off at sight in s.n. 8 results thrown out of a machine in s.n. 2. Those results might be given on paper ruled with vertical lines upon which 0 is not represented, and 1 is represented by piercing a hole or printing a dot or other mark. For example if the machine provides

| •| • | • •|• |•   •|• • |• • •| |

this represents in s.n. 2:

0 0 1, 0 1 0, 0 1 1, 1 0 0, 1 0 1, 1 1 0, 1 1 1, 0 0 0

but the clerk would ignore this and read it at sight in s.n. 8 as 12,345,670.

13. So it is definitely part of the present submission, in order to facilitate arithmetical work of all kinds at any stage, that records and factors should be kept year after year in s.n. 8, every calculation except such simple ones as the mere addition of two numbers being performed in s.n. 2, the conversion from 8 to 2 and back to 8 being as easy as the conversion from type to script when we copy with pen and ink from a printed book. It should be clearly under-

stood that this paper visualizes sums assured and premiums, for example, tabulated in s.n. 8 throughout the lifetime of the policy, and monetary functions in s.n. 8 which have never been and will never be in any other scale from the time they are first calculated and tabulated—calculated as s.n. 8 numbers (by s.n. 2 process) from s.n. 8 mortality and interest functions, of which the former at least need never appear in s.n. 10 at any time. One could not recommend s.n. 2 for such a purpose; one can and does recommend s.n. 8, and it is an integral part of the present plea for using s.n. 2 for calculations that we use s.n. 8 for records and for functions. So far as the present writer is concerned s.n. 8 is the hitherto missing link.

14. However, he is not content to employ existing methods, nor even a combination of known methods, for mechanical calculation, firstly because he has been searching for greater speed, and secondly because all along it was his aim to produce a machine which would function automatically when once set in operation and provide a large number of products before again requiring any human attention. The multiplicand and the multiplier are to be supplied to the machine in s.n. 2, but the conversion from s.n. 8 will be automatic. A machine to be used for occasional calculations would be provided with keys figured from 1 to 7, the numbers to be multiplied would be set by depressing selected keys, and each key would automatically control three figures in s.n. 2. A machine to be used for extensive work would be supplied with multiplicands and multipliers in successive lines on two rolls of paper, represented by punched holes in s.n. 2, but the preparation of the punched rolls would have been effected on another machine by pressing keys corresponding to the numbers in s.n. 8.

15. Before outlining the method by which the light-ray machine would work it will be advisable first to describe a simple, inexpensive form of machine, which may be all that is required for many purposes, the action of which is purely mechanical, and which is capable of multiplying the equivalent of two s.n. 10 numbers each of seven significant figures, printing or indicating by punched holes the results on a third roll of paper, at a speed of about 4000 products per hour. This machine, if actuated by a small electric motor, could be left unattended when once started, and would

continue in operation until the rolls were exhausted, when it would stop itself.

16. In order to make clear the underlying principle of the machine we will first consider an ordinary long multiplication. Suppose that 21,303 is to be multiplied by 13,212. The arithmetical process is:

```
    21303
    13212
    -----
    42606
   21303
  42606
 63909
21303
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to complete which we have only to perform an addition sum.

Now consider the following scheme for performing the same multiplication:

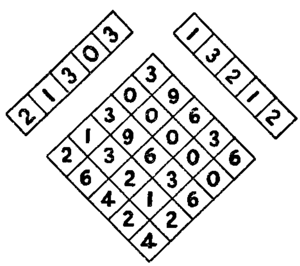

Diagram I.

Once again we have only to add to arrive at the product, indeed we have precisely the same process carried out in a diamond form, a rearrangement which is mechanically convenient, the rows to be added being diagonal to the diamond.

MECHANICAL MULTIPLYING MACHINE

17. In Diagram II, A is a mechanical device corresponding to the diamond. Every square in A is a potential hole, but each hole is doubly closed, first by a series of strips, each covering one row running downward from left to right, and second by a similar

series running downward from right to left. B is the multiplicand applied either by pressing keys in s.n. 8, each of which controls three rows of holes, or by a row of holes in a roll of paper in s.n. 2. As soon as B is in position it controls the opening of rows of holes

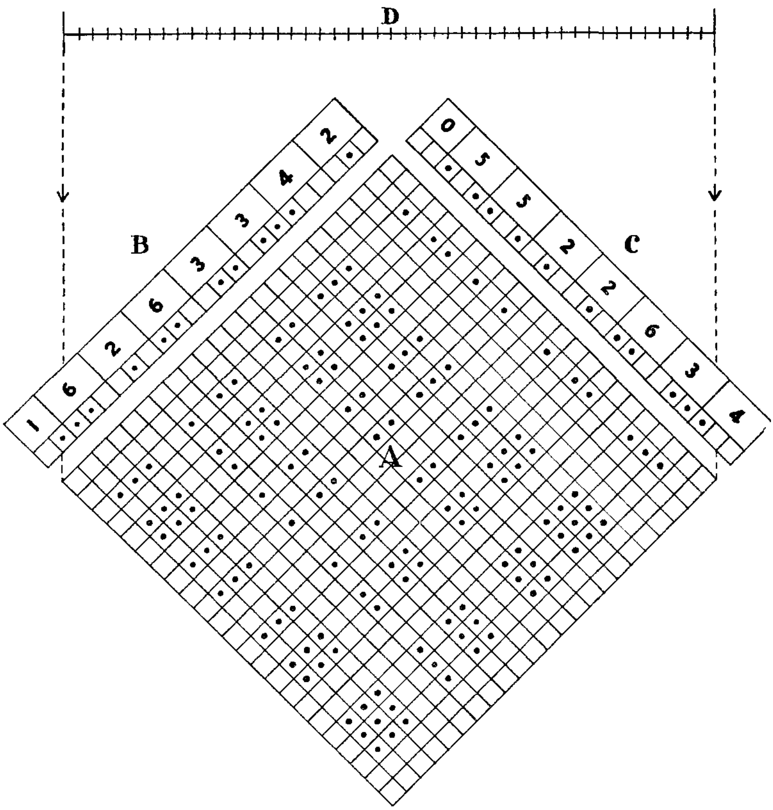

Diagram II.

running downward from left to right, the row opening wherever there is a hole in the multiplicand and remaining shut when there is no hole. C is the multiplier which similarly controls the opening of rows running downward from right to left. Thus as soon as B and C are simultaneously brought into position, holes will open in every square in which the process of multiplication requires that one unit should appear. To arrive at the product it only remains to sum the holes in rows diagonal to the diamond, the addition to be effected in s.n. 2. For this purpose a box rises from underneath the diamond A. The box contains needles mounted on springs, one needle in the position of every potential hole in A, so that wherever there is a hole a needle emerges, and where there is no hole the needle cannot emerge but instead is forced back against the compression of its spring.

18. Now we have to count the emerging needles in s.n. 2 in rows running straight up and down the diagram, i.e. diagonal to the diamond A. D is a shaft on which star-wheels are mounted in such a way that when D is moved across A from the top of the diagram to the bottom, that is in a direction at right angles to its own length, one star-wheel coincides with each row of needles, but owing to the diamond lay-out the star-wheels will only be operated in the odd rows and the even rows alternately. As will presently appear, this permits transmission to the left of the carrying stroke without synchronizing with an adding stroke, and permits this transmission to be dealt with as it arises, intermittently between the adding process, so that when D has passed completely over A it already has upon it, without any further process, a mechanical presentation of the sum of the multiplication represented physically by needles emerging vertically through A, i.e. a mechanical presentation of the product of the multiplicand B and the multiplier C.

19. Moreover, it should be noticed that we are not, in s.n. 2, concerned with counting in the ordinary sense of the word; the "counting" is so simple that it may be regarded in a new light. We are concerned only to determine between odd and even; not so much to count the units as to discover whether, with due allowance for units brought forward, the total in each row is an odd figure or an even figure. If it is odd we know at once that it is "one", if it is even we know that it must be "nought". Consequently the star-wheels carried by the shaft D can contain any number of points providing that number is even; we shall not wish to know how far a star-wheel has revolved, but only whether it has received an even number or an odd number of movements. Thus if we have six points on the star-wheel three of them will be distinguishable as odd and the other three, spaced intermittently, as

even. Before the shaft begins to move the star-wheels will be set in a position which signifies even, i.e. 0, and when the shaft reaches the opposite side of the diamond we have only to record, mechanically, photographically, or otherwise, which of the star-wheels are even and which are odd.

20. Between every adjacent pair of star-wheels is a mechanism the purpose of which is to transmit a carrying unit to the star-wheel on its left whenever the star-wheel on its right has completed two movements, i.e. whenever it has changed from odd to even. This mechanism will not be described, partly because it may take many forms. Suffice to say that it may be mounted partly on shaft D and partly on the diamond A, or it may be partly upon one or more additional shafts carried behind shaft D or above it, and in any case parallel to it. The transmission of carrying units has always been the bugbear of calculating machine design, and in s.n. 2 where one carrying unit is required for every two units added this bugbear reaches its maximum intensity and calls for the designing of new methods.

21. Suppose the rows of needles and the corresponding star-wheels to be numbered from right to left from 1 upwards. It will be seen that at the most the $(n-1)$th row may have to transmit

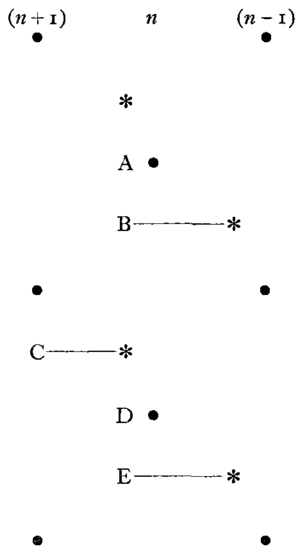

Diagram III.

$(n-1)$ carrying units to the nth row, i.e. never more than the number of holes in the row (one less, in fact, on the right-hand side of the diamond). Thus it is possible to complete the carrying process by providing one carrying position between each pair of adjacent needles in each row, so placed that the position selected will be in line with the position in the next row to the left at which the carrying units are to be received, such positions being similarly one between each adjacent pair of needles in the nth row. For example, see Diagram III, where a short section of the $(n+1)$th, the nth, and the $(n-1)$th rows are shown. Let us for the purpose of the example select a position three-quarters of the way between one needle and the next below it, as that at which the carrying unit is to be transferred to the next column to the left, as indicated by the asterisks, the dots as before representing holes from which needles may emerge.

22. It will be clear that the function of the mechanism between any two adjacent rows must be (*a*) to receive a carrying unit whenever the nth star-wheel turns from odd to even; (*b*) *to hold this carrying unit in suspense*; (*c*) to transfer the carrying unit to the $(n+1)$th row as soon as the mechanism reaches the point marked with an asterisk. Thus from the nth row the carrying mechanism may receive a carrying unit either at A because the needle in the nth column has just turned the nth star-wheel from odd to even, or at B because the nth star-wheel has reached that point in the odd position and the $(n-1)$th carrying mechanism has at that point discharged a carrying unit from the asterisk in the $(n-1)$th column; *but cannot however be required to receive a carrying unit both at* A *and at* B. It will discharge the carrying unit to the $(n+1)$th star-wheel at C, having in the meantime held it in suspense. The carrying mechanism may be so constructed that it can never hold more than one unit in abeyance at a time, the lay-out not requiring a greater capacity. If it has picked up a carrying unit either at A or B it must discharge it at C, and it cannot receive another carrying unit until it reaches D or perhaps E. If it does receive a carrying unit at D it cannot receive one at E. Such then is the principle upon which the carrying mechanism is designed, and it will be sufficient here to add that a number of different forms have in fact been designed, of which two have been actually constructed.

23. The design or designs which we finally select will be chosen partly with a view to economical construction, partly with a view to sturdiness, but mainly with a view to accurate operation at a maximum speed. At this point we may summarize the complete cycle of operation:

(1) The multiplicand and multiplier being already in position, rows of holes open downward from left to right corresponding to holes in the multiplicand, and similarly downward from right to left corresponding to holes in the multiplier, and the diamond A will then show clear holes only where two rows of holes cross.

(2) Needles rise up whenever there are clear holes in A. The multiplication may now be considered complete, i.e. the emerging needles give a physical representation of it, but the product is not yet available because we still have to sum the needles in rows and record the addition in s.n. 2.

(3) Shaft D passes across A, and the star-wheels on it, one to each row, count the needles. Between each pair of star-wheels is a carrying mechanism which operates intermediately so that when D has completed its transit of A, the star-wheels show the product in physical form.

(4) At the end of the transit of D the star-wheels operate a mechanism which records the product, as for example by printing or punching it upon a roll, and simultaneously the needles are withdrawn below A.

(5) D retraces its passage across A to the starting position, the star-wheels are returned to zero and the carrying mechanism to the non-operating position. Simultaneously the multiplicand and multiplier rolls are moved forward one place, closing the holes in A, so bringing the next multiplicand and multiplier into position.

24. A further development would be to arrange the diamond round the outside of a cylinder so that D could pass continuously round and round it (or alternatively it could revolve in one continuous direction under a stationary D), in order to save some of the time of the return stroke (although some of that time is also required for other purposes) and to avoid constructing a reciprocal movement. This would increase the complexity of the mechanism for opening the rows of holes and for protruding needles through

those holes which open. Even without employing a rotary movement it is considered that the machine can be made to work at a speed of about one operation per second. Of other possible developments it is not necessary to say anything here—"*facile est inventis addere*".

25. It is simple to use known methods so that besides recording the products their cumulative total is formed. Except for special purposes it will not be necessary to record cumulative totals item by item, after each product has been added, but this can also be provided if required. At the other extreme, if all we require is the sum of a large number of products, the printing or punching mechanism can be put out of action, and merely a total accumulated. For many practical purposes the multipliers will be factors used over and over again. The punched rolls will be prepared once and for all on specially thin material, such even as thin metal. Moreover certain other special series of multipliers would be kept always ready for instant use, such as various coefficients, the natural numbers from 1 upwards, Σn, i.e. 1, 3, 6, 10, etc., $\Sigma^2 n$, i.e. 1, 4, 10, 20, etc. It has already been noticed that the machine can be made to record a column of products p, and at the same time a column of Σp. If the latter is run through as a multiplicand sheet with the machine set for addition, we can arrive simultaneously at a printed or punched record of Σ^2 and Σ^3, and addition of the latter column will at once give us Σ^4. It is hardly necessary to point out the many purposes for which this is required.

26. The machine operates so speedily that it can also be employed for ordinary addition. The figures to be totalled would be fed through as a multiplicand, no multiplier sheet being required, but instead a lever on the machine would hold the unit open at some convenient position in the multiplier row, the other rows remaining permanently closed. The punching process would take longer than the whole work of adding on an adding machine (operating in s.n. 8 by the same methods as the present s.n. 10 machines) but when a column of figures already exists in s.n. 2, it can readily be added as described above. A special adding machine could be made in which some twenty or more amounts could be added by each sweep of the shaft D, if it was thought worth while, embodying the same principle in binary counting as

is involved in the multiplying machine. Such a machine would work automatically without attention and would total over a thousand amounts per minute. A simple variation of it would supplant the irksome toil involved in summation graduation.

THE LIGHT-RAY MACHINE

27. It is not intended to describe this faster machine fully, but rather to indicate its operation by contrast with the machine already described. The vertical needle is replaced by a momentary light ray directed on a selenium photo-electric cell. This momentary illumination is sufficient to cause a momentary and feeble flow of electric current; these impulses, acting through a relay, can be counted with extraordinary rapidity. We have only to ensure that but one impulse at a time is received at any "place" and further that it does not synchronize with a "carrying" impulse. No "holes" other than those in the multiplicand and multiplier sheets are required; we do not even have to provide the small amount of time required to open the holes in the diamond. The machine might be of very different design, in fact, from that already described, and the multipliers might follow one another on a long ribbon passing continuously through the machine without stopping, the multiplicand being as before and being turned up one line after each product is completed, simultaneously with the printing of (*a*) the product and (*b*) the cumulative total. To avoid time delay in clearing the product it would probably be found desirable to have two product-receivers alternated. The clearing process can simultaneously act as a transfer to the cumulative total.

28. It appears by consideration of other modern equipment, such as the synchronized alternating-current clock, and the new high-speed cinematographic camera, that on a conservative estimate the light-ray machine might deal with five to ten products every second, say between 15,000 and 40,000 products per hour, providing a printed, punched, or photographed record in s.n. 2 of every product and, if required, every cumulative total. A somewhat more complicated machine, but just as rapid, would provide these two columns in s.n. 8, but this hardly seems worth the extra mechanism. Apart from the increased speed of the light-ray machine it will be appreciated that it is self-operating and can be

left unattended. If it is provided with all the necessary material on the morning of 1st January, the whole of the valuation will be completed within 15 minutes or so. Indeed when the actuary arrives there will still be time to complete the valuation over again on eight or nine other bases before he goes out to lunch, and yet have ready the figures necessary for the Annual General Meeting in the afternoon. "*Contigimus portum quo mihi cursus erat!*"

CONCLUSION

29. When once functions have been made available in s.n. 8, calculation in s.n. 2 will go a long way towards removing the need for approximate methods for joint-life and other complex values; or alternatively will enable us to test such approximate methods both easily and thoroughly. For those calculations the result of which are not at once, if ever, intended for public consumption there can, it is submitted, be still less difficulty in using octonary numeration and binary arithmetic. Research workers have sometimes been appalled by the volume of multiplication which has faced them; now the most insatiable need no longer suffer the slightest qualm. Curve fitters in particular can have nothing to lose by working in one scale of notation rather than another. Moreover, at a time when graphical representation is so much employed, not only for scientific purposes but for business use as well, it is interesting to notice that we arrive at the same graph from figures expressed in whatever notation as readily from one as from another, providing we employ paper ruled and subdivided appropriately. Punched card systems are readily adaptable to use in the binary multiplying machine. Indeed binary numeration may be regarded as the most logical for any punched card system, since this system depends essentially upon punching or not punching a hole—the same alternative in effect as the choice between odd and even, between 1 and 0, as in the binary scale. Thus a quantity expressed by seven significant figures in s.n. 10 occupies a space of $9 \times 7 = 63$ possible hole-positions, and the same quantity can be expressed by 23 figures in s.n. 2 and requires only that number of hole-positions, a card-area saving of about five-eighths. Finally it may be pointed out that octonalization of English currency presents no greater difficulty than its decimalization. In s.n. 8 £·1 repre-

sents half-a-crown. For £·01 we might revive the term "groat" (its value corresponding to the present $3\frac{3}{4}d.$) and by dividing this new groat into four pence we should have a new halfpenny, £·001, differing very slightly only from the value of our present halfpenny.

30. The author concludes by expressing the hope that octonal numeration, which he likes to believe was in use in this country 1500 years ago, and which has since been considered from time to time, will be sympathetically reconsidered in the light of the simplified and extremely rapid arithmetical operation in the binary scale to which it so readily leads. The author has become more and more convinced that any difficulty in employing the octonary scale is more apparent than real. Any intelligent clerk could add together two amounts expressed in this scale; indeed there would be little difficulty in acquiring the art of casting any number of amounts less than the number which justify using a machine. Already we use s.n. 12 in effect when casting a pence column, and to cast a column in s.n. 8 would seem on the whole to be somewhat easier. In examining results, except those intended for presentation to the public, it is not thought that there would be any difficulty in accustoming ourselves to s.n. 8. Almost entirely figures may be regarded as relative. A column of figures showing, for example, the new business written in 1935 by the life offices in this country would be just about as informative if expressed in s.n. 8 as in s.n. 10. When we examine figures expressed in s.n. 10 we do so by relation to the subconscious knowledge that 5 lies halfway between 0 and 10, and so on in proportion; and just as for the pence column we have to remember that the halfway mark is 6 instead of 5, so for octonary work we have merely to accustom ourselves to the fact that the halfway mark is 4. At least for scientific and business purposes it is asked that the suggestion to employ octonal numeration be not hastily dismissed merely because at first sight it may seem one to which it would be awkward to accustom ourselves; as Terence said:

"*Haec dum incipias, gravia sunt,*
Dumque ignores ubi cognoris, facilia."

ABSTRACT OF THE DISCUSSION

Mr E. William Phillips in submitting his paper said that he had brought to the meeting that evening only so much of the binary calculator as seemed to him to be novel. He had not brought a roll with punched holes, where a hole represented 1 and no hole 0, because he assumed that everyone was familiar with an even more complicated method of representing numbers by punched holes. He had not brought more than a sample of the apparatus to show how the punched holes would set up the product and how the sum of the product would be represented by punching further holes, as he took it for granted that that part of the machine was fairly easy to understand. In any case all that part of the machine represented merely standard practice.

To sum a possible product in s.n. 2 and to deal with the carry units intermittently was, however, a new process so far as he knew, and he had had that much of the machine constructed and had satisfied himself that it would work. It was fitted at the moment not with the actual moving pins which would come through the holes but with dummy pins to represent the multiplication shown in Diagram II. He had, however, decided to provide for the summing of only 30 of the 45 columns because that form of contracted multiplication provided 8 or 9 significant s.n. 10 figures in the converted answer.

The Institute would probably not consider itself a body before which it was appropriate to discuss machine design. He thought the members would be interested in the machine design only to the point of satisfying themselves that the machine was a practicable and feasible proposition. He had satisfied himself on this point and he hoped that by what he had brought to the meeting he would satisfy his audience. The actual form in which the machine would ultimately be made from time to time was not a matter with which he would ask them to concern themselves; all that he had tried to do was to design a machine and construct it up to a point where he could prove that it would work. That it did work had been shown at three o'clock on the previous, and to him historical, Saturday afternoon when the multiplication shown in Diagram II was set up and the correct answer obtained.

He had begun by wondering whether it would be worth while to convert from s.n. 10 to s.n. 8 in order to calculate in s.n. 2, and the more he had dwelt on the subject the more he had begun to wonder whether it would not be better to do a great deal of the work which fell to the actuary, and especially pure research work, in s.n. 8 throughout, so eliminating the trouble of conversion. As the possibilities opened up by binary calculation loomed larger, so the relative importance of the conversion shrank; that was one reason why he had not discussed at any length whether conversion could best be performed by conversion tables or in some other way.

Mr F. H. Wales said that he felt a certain amount of disappointment in the form taken by the paper; from the title originally chosen, "multiplication", he had expected at least a survey of the methods at present in use for the multiplication of one number by another. There were in use at the present time four main methods of multiplication, two of which were strictly accurate and two approximate. Accurate results could be obtained by the laborious method of long multiplication, taught to children in their first lessons in arithmetic, and by the use of the common arithmometer or calculating machine, involving very little mental effort for the operator. Approximations to accuracy were possible by the use of logarithms and, arising out of this, by the application of the slide-rule. Perhaps as a fifth method mention might be made of prepared multiplication tables such as *Crelle*. His knowledge of Mr Phillips' ingenuity had encouraged him to hope that some entirely new method had been invented, but the author appeared to recognize only the mechanical method, though his approach to the problem would be novel to most people.

The paper reminded him of a certain mythical figure, the giant Procrustes. Procrustes was the possessor of an iron bed, on which he invited his visitors to lie; if they proved too long for the bed he would lop off their legs to make them fit, and if too short he stretched them to the necessary length. Mr Phillips had invented a very ingenious calculating machine, but unfortunately the denary scale of notation did not fit it, and so he proposed that the scale should be cut down by lopping off the figures 8 and 9 and working in the octonary scale.

He was appalled by the thought of such a change. Mr Phillips admitted that it would be impossible to get the world to change to octonary notation, and therefore suggested that offices should continue to express sums assured, premiums, reserves and so on in the denary scale in prospectuses, reports and other documents intended for public consumption. For internal use, however, they were to use the octonary scale, and to convert published figures to this by means of conversion tables, since it was not possible to convert at sight from s.n. 10 to s.n. 8 by a simple rule such as could be used when changing from any scale to a scale based on an integral power or root of its radix. He felt that they would be in the same position as the people of a country whose rulers decided that for all official purposes some language other than the one which they learned as children was to be used; there was the new language to be learned. Indeed, it would be a worse position; the words would be the same, but would have meanings different from those formerly appropriate. Given a set of numbers, how was it possible to know in what scale they were expressed, unless they were preceded or followed by s.n. 10 or s.n. 8 as the case might be? It would probably be better to invent new figures for use in connection with the octonary scale.

The octonary scale was certainly worth consideration, however, as being one of the scales whose radix was an integral power of *two*. The number two was fundamental as Mr Phillips pointed out, and as students of biology would recognize. Although he did not attach much importance to the point, it was of interest to note that the Stock Exchange made some use of binary methods, as the only fractions which it appeared to recognize in quoting prices were halves, quarters, eighths, sixteenths, thirty-seconds and sixty-fourths, which became ·4, ·2, ·1, ·04, ·02 and ·01 in octonals. Incidentally the last fraction, one-sixty-fourth, was Mr Phillips' "groat".

The binary scale lent itself to mechanical methods, since but one figure was used, of course with the cipher to keep it in its proper place. The large number of significant figures required to represent comparatively small numbers was, however, a disadvantage. In order to reduce his numbers to manageable proportions, Mr Phillips proposed a radix which was a power of 2. He might have chosen 4, 8, 16, 32 or even a higher radix, but the higher the radix chosen, the more symbols would be required for figures. He felt that if Mr Phillips had proved his case for the octonary scale—which he was not prepared to admit—the radix 16 would merit consideration. Admittedly it involved the invention of six new figures. If these were called α, β, γ, δ, ϵ, and ζ, it was possible to convert to s.n. 2 by extending the table given by the author on page 192. It was necessary to group the figures in fours, and for the conversion of numbers up to 7 to put an extra 0 in front of the figures in the table; from 8 to ζ the table was repeated, putting 1 in front instead of 0. There was a little more to learn with 16 as a radix, but he felt that anyone capable of thinking outside the denary scale should have little more difficulty with 16 than with 8. He did not advocate the use of Greek letters to represent the additional figures, as they were already used in mathematics for other purposes; new figures would be preferable.

Reverting to the octonary scale, tables of reciprocals would be necessary, for he was afraid that Mr Phillips' machine would not be suitable for division by either of the processes at present possible with an arithmometer. New tables of logarithms would also be required; 8 would appear to be the logical base for the table, and, as an exercise in the use of s.n. 8, he had calculated $\log_e 8$ in s.n. 10 $(=2{\cdot}0794)$ and converted it by continuous multiplication. The result was that $\log_e 8=2{\cdot}0505$ approx. in s.n. 8 or 010· 000, 101, 000, 101 in s.n. 2.

Among other tables which actuaries would require were mortality, sickness and interest tables, but he shuddered to think of the work involved in their construction, even with the aid of Mr Phillips' machine.

He agreed with the author that, were octonal notation adopted, our coinage would require modification; half-a-crown and its multiples were the only fractions of a pound which could conveniently be octonalized, for all other integral sums of shillings and pence led to recurring octonals. He wondered how Mr Phillips proposed to deal with decimal coinages such as the dollar.

It was of some interest to note that Ahmes' method of multiplication, as demonstrated in paragraph 5 of the paper, was considerably simplified by writing 13 on the same line as M, half 13, ignoring fractions, on the next line, half again on the next line, and so on. By crossing out every line where an even number appeared in the multiplier column and adding the figures remaining in the multiplicand column the product was arrived at. This method had been used, he believed, by Russian peasants, and merely involved the conversion of the multiplier into the binary scale and multiplication by the necessary powers of 2. He drew attention to this point as it had occurred to him that it might be possible to make a machine which would use this method of multiplication, though obviously it could not operate as rapidly as Mr Phillips' machine.

Other things being equal—he would emphasize that qualification—the fewer the revolutions required in a machine the faster would be the machine in multiplication operations. Probably the commonest forms of calculating machines in use at the present time made use of the principle invented by W. T. Odhner of a gearwheel with a variable number of teeth. This type of machine, at any rate if operated electrically, would necessitate revolutions equal in number to the sum of the digits in the multiplier, and the carriage would move along one place for each significant figure. It made use of the principle of continued addition. In contrast to this type, there were machines involving Bollée's method, based on the ordinary multiplication table. This type of machine should prove faster in operation than the Odhner type, as only one revolution was necessary for each digit in the multiplier, the carriage moving along one place for each significant figure. Mr Phillips' machine required but one revolution whatever the size of the multiplier, and the carriage was fixed; therefore, given its data in suitable form, it should unquestionably be the fastest of the three types. Its disadvantage was that it could not operate in the denary scale, and this should, in his submission, be sufficient to condemn it. He could not conceive that anybody would require products at the rate of 40,000, or even 4,000 per hour; such a revolutionary change as the octonary scale should not be imposed upon mankind in general for the sake of a few individuals.

Even with his machine available, he questioned whether Mr Phillips could produce his valuation results early on the morning of 1st January, unless of course he closed his year in September or October. As was well known, the actual multiplications were only a part of the work involved in valuation, and the collection and arrangement of the data presented quite a major problem, although the mechanical sorting machines now available considerably shortened the time of preparation.

The idea of using Jacquard's invention of punched cards to feed the data to a calculating machine was employed by Babbage when constructing his unfinished analytical engine. He agreed with Mr Phillips that the punched card system could very usefully be employed at the present time in calculations involving a column of factors which was required over and over again. He felt, however, that it should be possible to adapt machines of the Bollée type so that they could be fed automatically. The fact that seven significant figures in s.n. 10 required 63 possible hole positions did not disturb him unduly—it was not he but the machine which read the figures—and the waste of space was not very great.

He found the reading of Mr Phillips' papers stimulating, and the present paper had introduced to him some very interesting literature of an historical nature, in which he had noticed a statement made by Napier which he felt would have formed a fitting text for Mr Phillips, and which he proposed in conclusion to quote. In introducing his system of logarithms to the world, Napier said: "Seeing there is nothing that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which beside the tedious expense of time are for the most part subject to slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove these hindrances."

Dr L. J. Comrie (a visitor) said he had been very interested in what Mr Phillips had said, partly because it fell to his lot to do a great deal of multiplication; in the Nautical Almanac Office they had millions of multiplications a year. He did not wish to damp the author's ardour, because he admired his multiplication machine, and he realized that the scale of 8 had merely been a vehicle for getting numbers into this undoubtedly clever multiplication machine, but he felt convinced that neither Mr Phillips nor anyone else would ever persuade actuaries to use the scale of 8.

He thought it was a first requirement in any calculating machine, and one on which he always insisted whenever he had the opportunity, that the data or questions should be given to the machine in a natural form and that the answer should be given by the machine in a natural form. What happened in between was no concern whatever of the operator; that could be left to the machine. If Mr Phillips could take a number in the familiar form, in s.n. 10, convert it to s.n. 2, multiply it with astonishing rapidity and convert it back to s.n. 10, so that it could be handled in the familiar form, he would certainly have achieved something worth while.

He would like to say a few words about his own attitude towards multiplication in bulk. In arithmetic there were four fundamental operations—addition, subtraction, multiplication and division. There were some universal machines, such as the Brunsviga, Monroe and others, which could be used to perform any of these processes; but where the volume of work was extensive, specialist machines were desirable. There was a group particularly adapted to addition and subtraction, and the feature distinguishing those machines from the others was that they would record whatever numbers were given to them as well as the answers produced, so that there was a definite, permanent record.

But in the process of multiplication by the use of the arithmometer or crank-type machines it was necessary to spend a considerable amount of time in giving the machine its questions, and a considerable amount of time in recording, and it was just in the recording stage that errors crept in. The setting and the handle-turning process also were fairly accurate; if the errors made were analysed, more would be found in the record of the answer than in either of the other stages, and the reason was that it was a slow process. It might perhaps seem astonishing that a slow process could be inaccurate and a fast one accurate, but that was true, because in the former case the mind had time, between looking at a number and recording it, to transpose figures or to become confused when certain figures were doubled. Members were perhaps familiar with the Powers and Hollerith punching machines, where the operator was trained to look at a document and punch figures at the rate of three, four, five, and in some cases even six a second; there, where the action was reflex, the process was very accurate, and a good punch operator would punch figures at the rate of three or four a second more accurately than most people could write them at the rate of one a second.

The weakness therefore lay in the recording. If the machine were speeded up, the whole process was not accelerated, because from 80 % to 90 %, or even 95 %, of the time was spent in putting the multiplicand and multiplier on the machine and in recording the answer; only the remaining 5 % to 20 % was spent in multiplying, and by doubling the speed of the machine only a small fraction of the total time taken was halved. In order, therefore, to obtain the maximum benefit from a fast machine it was necessary to be able to put questions to it at a much greater rate and to take down the answers at a much greater rate.

The machines already existing did not satisfactorily fulfil those requirements. He thought that perhaps the first recording machine was the Moon-Hopkins, now called the Burroughs typewriter accounting machine. That was a direct multiplying machine, by which he meant that it worked by the multiplication table and not by repeated addition. It recorded one factor only—the multiplicand—and also the product, and as the multiplier was not recorded, it was not what he sought.

The next development was the "United" machine, which appeared in America, at St Louis, about 1927 or 1928. This machine contained two keyboards, similar to that of any ordinary adding machine; one factor could be put on one keyboard and one on the other, and, by depressing a motor-bar, in three seconds, no matter what numbers had been set, the product was formed, and both factors and the product were recorded on a tape. The sum of a number of products could also be obtained, because there was a separate adding mechanism for that purpose. Unfortunately that machine was first manufactured just before the depression in America, and so did not flourish. He had seen one three years ago in America. It was now being taken up by an influential company, however, and would shortly appear on the English market as the multiplying component of a card-punching machine. It was also a direct multiplying machine.

The next effort to produce something that would record was that of an engineer named Cordt in Leipzig, who exhibited a machine at the Berlin Exhibitions of 1931 and 1934. He took two machines that already existed, the Archimedes, of the arithmometer type, and the Astra multi-register adding machine, and combined them. Naturally the combination had all the defects of a home-made, put-together apparatus; it was not a machine designed in the first place as a multiplying and recording machine. It served certain purposes, and public utility work in particular; nevertheless it was not sufficiently attractive to induce any agent in this country to take it up, so that England had not seen it.

The next development was perhaps the Hollerith multiplying punch, which again was a direct multiplying machine. It was fed by means of cards on which holes representing multiplier and multiplicand were punched. The product was formed by direct multiplication and recorded on the card in the form of further holes. The capacity was eight figures by eight, and the time taken with a full-size multiplier and multiplicand was five seconds. The speaker had recently carried out three-quarters of a million multiplications with that machine in six months, or 120,000 a month, which allowed just five seconds for each, taking an ordinary working day. The feature that induced his office to use it was the fact that it recorded its products, and he had been trying very hard to make the company concerned realize that it was not merely a machine for putting holes in cards so that the cards might be used in tabulators and sorters but that it was a multiplying machine, and that the market really needed a machine for rapid multiplication and that would leave a record of all the factors concerned. He hoped that they were beginning to see his point of view.

There also existed the Rheinmetall machine, used for invoicing, and first shown at the Berlin Exhibition in September 1934. Two numbers, usually price and quantity, could be typed by ordinary typewriter keys, and the product automatically typed by the same keys.

If Mr Phillips could produce a machine, working with the selenium cell, that was much faster than any other machine, and with a capacity of one multiplication a second, various other problems would arise. One was the collection of enough work to keep the machine occupied; actuaries could answer that question much better than he could. He solved the problem in his own case by collecting three-quarters of a million multiplications, getting the cards ready, hiring the machine for six months and then sending it back, so that he did not have to pay for any idle moments; but it was not everyone who could do that. As machines became more and more expensive it became more and more important to prevent them from being idle, and for that purpose the card or tape-reading system was useful. It was generally acknowledged to be a very expensive proposition to lay a cable from England to New York, and, in order fully to utilize such a cable, messages were prepared by punching tapes, and several tapes were fed together at very high speed through the transmitter, so that the cable was never idle.

Mr Phillips, in conversation before the meeting, mentioned that he contemplated his machine would deal with the problem tackled by Babbage with his difference-engine. Over a hundred years ago Babbage started building a difference-engine; it was never finished, but as much as was completed was now in the Science Museum at South Kensington. It was intended that that machine should work from sixth finite differences, and ever since Babbage's efforts came to naught, as a result of the refusal of the British Government to subsidize the machine beyond £17,000, people had desired a machine for that purpose. Scheutz made one which was used in the Registrar-General's office in this country for a time, and actually was used in the construction of some mortality tables. Only two of those machines were made. It might be news to some people that the Babbage problem had now been solved, and that it was possible to walk into a shop not far from Staple Inn and purchase a machine that would integrate from finite differences up to the sixth. That machine was the National accounting machine, made and sold by the National Cash Register Company. It was simply a multi-register machine with six adding mechanisms and a keyboard; from the keyboard a number could be put into any combination of the six registers. A number in any register could be transferred to any combination of the remaining registers. It was simplicity itself to enter a sixth difference and to build up the fifth, fourth, third, second and first differences and the function. Everything *that went into the machine and every result produced is printed.* Apart from the setting, the time taken for integration from a sixth difference was about six or seven seconds; it took a little more than a second for each difference produced. That machine would also facilitate what was not mentioned in connection with Babbage's machine: the tabulation of the successive differences of a function up to the fifth. It took only a few seconds to set eight, ten or twelve figures, twelve being the maximum, and it took about one second for each difference.

He would conclude as he had started, by urging Mr Phillips to go directly from natural numbers with his clever multiplication machine and come back to natural numbers. That was, he was sure, the form in which people would insist on having their data and answers.

Mr A. H. Rowell said that on reading the paper his main feeling was that it must have a practical value, because the author was essentially practical, and could not be criticized in that respect. He wondered, however, how much of the author's obvious enthusiasm arose from a genuine desire to inflict the scale of notation which he advocated on the business world, and how much from the manifest enjoyment which he obtained from his position as an inventor. It was necessary only to look at the piece of mechanism which the author had designed, to see what enjoyment he must have had and how much more was probably in store for him.

But, although the author was a practical man, it would seem that he had probably under-estimated the difficulty which most people would find in passing from one scale of notation to another. The author had probably become accustomed to it, but others would be much slower in becoming acclimatized. He had mentioned that point to an accountant who thought he had met an actual case of the octonary system in practical use. In auditing the accounts of one of the largest breweries in the country, he said that it was near enough, when estimating caskage, to express one-eighth of a cask as being equivalent to ·1, and a quarter as ·2, and so on, and he thought that perhaps that gave rise to the well-known phrase of "one over the eight".

Unless he had misunderstood Mr Phillips, he could foresee difficulties which to an accountant in an ordinary life office might be unendurable. Mr Phillips suggested that the premiums in the actuarial department should be recorded in the octonary scale, and that the same premiums in the accountants' department should be in the denary scale. In his own office, it was sufficiently difficult at present to reconcile the premiums in the two departments.

He felt that the members owed Mr Phillips a real debt of gratitude for having stimulated their minds on the matter. If they could allow the author to lead them, the future might be very interesting and in some respects much simplified; but, even if they and their children after them became most devoted adherents of the octonary cause, they would find themselves in an intolerable state of confusion when they turned to the past and tried to read the writings of those denary days.

Mr H. L. Trachtenberg agreed that a more elementary treatment of the subject might be preferred. The statement that a child could multiply two numbers in the binary scale on learning that one times one was one ignored the fact that the child would have to learn that one plus one was nought with a one carried over.

He did not agree with Mr Wales that the letters alpha, beta, gamma, etc., would not be suitable for the scale of 16; mathematical science was full of symbols which had different meanings in different connections without there being any confusion. He believed that the scale of 16 would be an excellent one, because it would bring the figures to a smaller compass, whereas the octonary scale would increase the compass. The "intelligent clerk" referred to by the author would have no more difficulty in dealing with s.n. 16 than with s.n. 8 once he had mastered it, but no matter how ingenious any machine which was put forward might be, there remained the fact that the intelligent clerk had to remember something.

He agreed with Dr Comrie that the octonary scale must go, and that anything that went in and came out of the machine must do so in the denary scale. It did not matter what happened in the machine, but it was not possible to adopt an entirely new system of everyday reckoning.

Mr T. W. Chick (a visitor) expressed his gratitude for the privilege of being able to see Mr Phillips' machine and to study his paper. A point had occurred to him concerning the suggested speed of the Light-ray machine. A machine was always limited in its speed of operation by the maximum permissible speed of its weakest part, and by questions of balance, inertia, etc. He wondered whether there was any danger in the case of Mr Phillips' machine of the suggested speed of 40,000 multiplications per hour being unobtainable through these mechanical considerations. While it would be quite possible to photograph a record of the products at that speed, it seemed doubtful whether they could be printed or punched at that speed, for he understood that the maximum rate at which punched records could be prepared at present was between 19,000 and 20,000 records per hour. Perhaps Dr Comrie would confirm or correct that figure. He suggested that the quickest method of all might be to record the factors photographically in s.n. 2, then "read" them by means of a reflected light ray and photo-electric cell, perform the multiplication, and then record the product in the same manner and notation as the factors; or, if punched records were preferred, it might even be possible to record the hole positions photographically and to produce the holes by some chemical action during the developing process. In general, however, he felt that much of the time saved in the actual process of multiplying by the binary system would be nullified by the time required to prepare the binary representations of the given factors and, when necessary, to prepare the products in whatever scale of notation might be required.

With regard to the commercial value of Mr Phillips' invention, he felt that if the method was applicable to insurance problems as a special means of surmounting a particularly tedious process involved therein, then it would appear that such a machine would justify its existence, for service with efficiency was the principal aim of every organized concern and it mattered not to the outside public what intermediate processes were used in arriving at figures intended for public consumption if those processes were understandable to those who used them and resulted in a saving of time and expense. On these grounds he felt that Mr Phillips should be accorded every moral and material support in the developing of his interpretation of the binary process in insurance and like problems.

He was very much afraid it would be hopeless to try to convert the world to the usage of octonary notation. It could not be done in a lifetime, and there were other problems of greater importance awaiting solution. Even at present, with the majority of the world's inhabitants using the denary system, it seemed quite impossible to persuade the minority who used the duodecimal and other non-decimal systems to abandon them in favour of the denary system throughout all commercial and scientific work. Such a change would not involve the abolition of any digits and there would be little confusion after it had been effected; but to convert the world to the octonary system, requiring the abandonment of the figures "8" and "9", would be striking a crushing blow at one of the foundation stones of present civilization. The recording of time, for example, would be affected; the 12 and 24 hour system would have to be abandoned, for 8 o'clock and 9 o'clock would become "10" o'clock and "11" o'clock in the new notation and every clock in the world would have to be replaced or altered. History would have to be rewritten because of the recording of dates in the denary notation; otherwise it would become a subject written in another time language. Having to learn another language, he said, was quite sufficient obstacle nowadays in the passing of examinations, and to have to learn another time language, to say nothing about having to relearn arithmetic in another system of notation would be too great a demand. But he felt that Mr Phillips appreciated this aspect of the matter in all its significance and that he must not therefore be interpreted as being a serious advocate of such a revolutionary change.

It appeared to him that the ideal scale of notation should be based upon a radix which by its very nature or algebraical properties symbolized something of the multi-dimensional universe, in the measuring of length, of area and of cubical capacity. He felt that the most convenient radix (representing length) should be such that its algebraic square (representing area) and its cube (representing capacity) should bear some useful relationship to those of its multiples and sub-multiples which were likely to be most used such as two, four, etc. and a half, a quarter, etc. He thought that the number which most nearly fulfilled that condition was "two", for "two" was numerically equal to the half of its square, a quarter of its cube, an eighth part of its fourth power and so on, and these relationships might conceivably be far more useful than the corresponding relationships for other numbers in the denary, duodecimal or other notations. But, as Mr Phillips had pointed out, if "two" were the chosen "radix" then the number of figures to be used was too large; in that case it seemed only logical to fall back upon one of its multiples, say "4" or "8", and thus reduce the number of figures required to more manageable proportions.

In passing, he would like to mention that a certain American concern employed a system of twelves for all interdepartmental accounts, because its products were dispatched in neat rectangular crates of "three by four" dimensions with great convenience in assembling orders. Admittedly, what was being discussed was "8" as a basis and not "12", but he mentioned that as a point of interest to those who were studying these matters exhaustively.

Mr G. Green said that the request made of him to close the discussion on Mr Phillips' interesting paper must have been due to some confusion on the subject of notation. At the present time, as was well known, the subject of international actuarial notation was under examination, and it had fallen to his lot to be somewhat concerned with that matter, but the question of notation as put forward in the present paper was not the same thing at all, and he could not claim any special knowledge of it.

As usual, Mr Phillips had presented a very interesting paper, though one which caused to some minds very disturbing reflections. The Fat Boy in the *Pickwick Papers* was usually asleep, but on one occasion he remained awake and saw something which he afterwards described, saying "I wants to make your flesh creep". He did not like to compare Mr Phillips with the Fat Boy, because he doubted whether Mr Phillips was ever asleep; but he had very little doubt that Mr Phillips, when writing his paper, expected to encounter a good deal of scepticism and opposition.

Since the normal process of multiplication and the scale of ten became settled, he thought he was right in saying that there had been only two major innovations. The first was that introduced by Napier, the logarithmic method, and Napier had been stimulated to turn his mind to that question because of the very great difficulties which the astronomers of that day found in carrying on their work. No doubt in some quarters that idea met with a good deal of opposition, but Napier found an enthusiast in Henry Briggs, who carried his work to perfection, and there could of course be no doubt that that introduction meant an immense gain to the scientific life of the world. The other innovation was the introduction of machinery. Many years ago in Staple Inn Hall a competition took place between an actuary with a book of logarithmic tables and another actuary armed with a new-fangled machine—the arithmometer—and the latter won easily. He believed that one of them was Woolhouse, and he thought it was he who upheld the log tables, but he was open to correction on that point.

Throughout all those years, however, one thing had been common to both ordinary life and scientific life, namely the scale of notation. He doubted very much whether it would be feasible to carry out the author's suggestion that a difference should be introduced on the scientific side in that regard. If they could train a clerk to think naturally in the scale of 8, and to keep the more or less permanent records of the office in that scale, there would still be very great difficulties. There would be, as pointed out in Hall and Knight's *Algebra*, if not elsewhere, the difficulty of language. In the scale of ten there were units, tens, hundreds, thousands, and so on, but there were at present no corresponding words in the scale of eight. The present year contained 366 days, and in the scale of eight that would be 556, but this could not be expressed as five hundred and fifty and six. Then there were the decimal point and percentages; 5% became $3\cdot\dot{1}46\dot{3}$ per 64, which sounded rather horrible.

On the other hand, in the ninth chapter of Chrystal's *Algebra* would be found a point which had been referred to by the last speaker, that in certain directions work was better and more easily done in other scales. The example given in Chrystal's *Algebra* was that of mensuration, where, owing to the fact that the foot contained 12 inches, it was often easier to work in duodecimals than in the ordinary decimal scale. He remembered being told in his boyhood by his father that the apprentices at a hat manufactory had, as one of their first jobs, to learn to work in duodecimals. There was no need, therefore, to exaggerate the difficulty of training clerks to work in s.n. 8.

The most serious difficulty in the work of an office would undoubtedly be the fact that there was no clear distinction between those records which could be usefully kept in s.n. 8 and those which would have to be in s.n. 10. In a valuation the bulk of the work consisted in collecting the data and getting them into the form required for multiplication. The suggested system would not help in that direction at all, and in most offices he thought the actual multiplications did not take sufficiently long to make it a matter of tremendous moment, nor was it often necessary to go into extreme refinement in curve fitting. Most graduations performed in actuarial practice were more or less rough and ready, because the matters in question were not of sufficient importance to call for a very long and elaborate set of calculations.

One point which interested him very much in reading the paper was this. He appreciated fully that the scale of 8 was necessary to the author's system. When s.n. 8 was converted into the binary scale, all the arrangements of 0 and 1 three at a time were required, and *were all used.* That last feature was essential. He had tried to see why it was not feasible to start from s.n. 10 and omit the conversion into s.n. 8. Expressed in the binary scale, s.n. 10 required sometimes an arrangement of four figures, 0 and 1, but it did not require all the possible arrangements. 1,010 in s.n. 2 equalled 12 in s.n. 10, but 12 was not a single figure, and that fact, he believed, would bring the author's whole system to naught. So far as the necessity of using s.n. 8 was concerned, he thought that the author had made his point quite clear.

He was afraid that he knew very little about the actual machine. The Institute had been very fortunate that evening in having such an entertaining and instructive account from Dr Comrie. Whether or not the paper led to a finished machine being produced, it had the advantage of stimulating thought, and it contained a very interesting historical account of the operation of multiplication. It was a paper which must appeal to those with a mechanical turn of mind, and, in spite of all that had been said, he thought it was a paper which might possibly bear valuable fruit some day.

The President (Mr C. R. V. Coutts), in proposing a hearty vote of thanks to Mr Phillips for his paper, said he was sure it was useful for the Institute to have such a stimulating paper from time to time, quite apart from there being any immediate practical possibilities. He sympathized with Mr Phillips in the fact that progress had been hampered all through existence by the scale of 10. It was most unfortunate that, just because there were five fingers on each hand, all calculations had been based throughout history on that scale, but from a practical point of view he imagined that there was no more chance of altering that than of altering what was at one time called the narrow gauge of 4 ft. $8\frac{1}{2}$ in. on the railways to the much more satisfactory 6-ft. gauge which died in the year 1892.

He thought that perhaps the difficulty of using a scale of 8 had been somewhat exaggerated. It seemed to have been assumed that if, as Mr Phillips suggested, the scale of 8 were adopted for certain official purposes, it must be used for all other purposes, and that it would be necessary to alter the calendar and other measurements. He thought that justice had scarcely been afforded to the author in that respect. Many members would remember learning to convert shillings and pence into decimals of a pound. The process quite easily became automatic, and it did not follow that a customer in a shop, instead of asking for something costing 16*s*. 8*d*., would say "I want something costing £$0{\cdot}833\dot{3}$".

He thoroughly agreed that from the point of view of the actuary there was nothing in the numerical work of making a valuation; the main work, as everyone knew, lay in collecting the data, and the valuation, even without the aid of a machine, could be completed in the course of a morning; but when Dr Comrie, in his extremely interesting and valuable account of the different machines which had been invented, said that he performed so many thousand multiplications a day, he must have given his audience the impression that his was a horrible existence.

He would like to say in conclusion that he hoped that the author would pursue his idea and would not be discouraged by any cold water which had been thrown on it that evening.

Mr E. William Phillips, in reply, stated that he did not contemplate that it would be possible to abolish the present system of counting in tens; but the question of using s.n. 8 for convenience in recording and s.n. 2 for convenience in calculating seemed to him to be an entirely different one. If in this country all scales proceeded in multiples of ten it would have been easier for him to understand the surprise that had greeted his suggestion that for certain purposes the scale of 8 might be used. Even in America the decimal system was not universal; although it applied to currency there it did not apply to weights and measures. When it was said in the discussion that it would not be possible to use scale 8 because there were 24 hours in a day and 60 minutes in an hour, the only reply was that neither 24 nor 60 had anything to do with 10, and that for hours and minutes and days and months scale 8 could be used as easily as scale 10. Shillings and pence were commonly converted into decimals and they could equally well be converted into octonals. So long as there were 24 hours in the day, and 60 minutes in the hour, and 1760 yards in the mile, and 20 shillings in the pound, and 12 pence in the shilling, and 4 farthings in the penny, there was no single scale of notation which would prevent difficulties arising. A decimal system did not exist in this country but only decimal numeration, and in many cases octonal numeration could be as readily used, and in some cases more readily.

As some foundation for the belief that counting used to be octonal in this country it was surely curious that there remained as survivals of those old days 8 furlongs in the mile, 8 drams in the fluid ounce, 8 gallons in the bushel, 8 bushels in the quarter, 16 grams in the ounce, 16 ounces in the pound, and 8 stone in the hundredweight, not to mention the 8-hour day; and a right-angle was divided into 8 points—of $11\frac{1}{4}$ degrees each. Mr Rowell had referred to the dividing up of a cask, but had omitted to mention that there were 8 gills in a quart, and 8 pints in a gallon.

He could not agree with Mr Wales that a scale of 16 should be adopted; in his view it was necessary that the figures which were to be read off and written down by the clerks should be familiar figures. Except that the digits 8 and 9 would never be used, quantities expressed in s.n. 8 would appear quite normal, whereas if they were expressed in s.n. 16 they would look very strange indeed. Mr Wales had mentioned the "Russian peasant" method of multiplication under which a binary process of calculation was carried out with the inconvenience of decimal numeration, just as in Egypt 5000 years ago. A friend who had just returned from Abyssinia had informed him that that was how it was done there to this day, and that was how he had seen it done in Moscow in the previous year; this 5000-year old method still existed in certain parts of the world.

Dr Comrie had referred to Babbage. That point was dealt with in paragraph 25 of the paper where he spoke of forming a Σ column, taking the result sheet out at the end and putting it back into the machine as an operation sheet to form a Σ^2 column, and so on indefinitely. That was a difference-engine, not performing simultaneously, it was true, but column after column, the point being that the record of the first summation was punched automatically, ready to be put back into the machine. The Institute was fortunate in having no fewer than thirteen items by Dr Comrie in its library, and he had been very interested in the work Dr Comrie had done in adapting standard machines to special purposes, but he did not consider that the Babbage problem had been entirely solved, as no existing machine was both self-operating and self-recording. Self-recording could be obtained by hand operation, or by using the existing punched-card machines it was possible to approach a little nearer to what Babbage was trying to do, providing the tabulator was coupled to a punch.

He had always regarded it as essential that the binary calculator should operate from punched holes, and self-record its results in punched holes, because he did not think there was any future for calculating machinery which was not self-operating and self-recording in this way.

He had been doing a good deal of conversion from s.n. 10 to s.n. 8 and back again, and it was really quite simple. Legendre had suggested 100 years ago a method of dividing repeatedly by 64, with a prepared table for the conversion of the remainders which could, of course, never exceed 63. Alternatively conversion tables might be used. Whatever might be the position in other professions, actuaries had to do a tremendous number of calculations in which they were concerned only with a limited number of figures at the end of the reckoning, the large mass of the figures representing merely intermediate steps. It was only final figures that had to be reconverted and that was quickly done. If an enormous mass of figures were being handled, every one of which had to be printed in a book for the use of the layman, he would never for a moment advocate conversion into s.n. 8 to secure the benefit of binary calculation, as this would be offset by the labour of converting all the results into s.n. 10. But when enormous masses of figures were not to be used by the layman but were to be tabulated merely as factors in further calculations, they could be calculated in s.n. 2 and recorded in s.n. 8, which was the form in which they would be required for further use; he thought it was an entirely unjustifiable criticism first to insist on converting them into s.n. 10 and then complain of the trouble of this conversion and the subsequent reconversion into s.n. 8. Why should they ever be converted at all?

He had been asked about the punching speed and he would say that for many of the processes for which he imagined a binary calculating machine could be used, the punched roll which was put into it would have been prepared upon the machine itself. If there were two punched rolls another roll would be punched containing the answers, and then the answers in turn became the operators. He agreed with Dr Comrie that it was necessary to start with tens and finish with tens so far as the lay public was concerned, and would always be so necessary; but he did not think they should debar themselves from changing from one scale to another whenever it was convenient, just as it was permissible to convert natural numbers into logarithms. He thought instead that the criterion that a practical arithmetician would apply was that he would put his figures into whatever form best suited the calculations in hand. The conversion into s.n. 8 should be considered as analogous to a conversion into logarithms, but with the advantage of a facility not possessed with logarithms in that no reconversion was necessary before a number of quantities could be added together. That perhaps partly answered those who had quite rightly pointed out that multiplication was only part of the process of the life office valuation.

Reference had been made to the Scheutz machine, which was on the lines of the Babbage Difference-Engine, but which was actually brought to a condition where it worked successfully. There was an account of that machine in the library of the Institute, with specimens of the tables calculated and typeset by the machine.

Finally he would like to thank the President for encouraging him to follow up his ideas. He was not yet convinced that for certain purposes binary calculation would not be practicable; he had never regarded the binary machine as a commercial proposition, for the present at least it was obviously something which would have only a limited use and which would be made to order only as and when required. But he claimed that it was a machine which would do certain things, and do those things very quickly and automatically, and in certain cases enable complicated calculations to be performed which otherwise would have been regarded as impracticable. He was also convinced that those who had once had a little experience with the machine would not find the difficulties so great as they seemed to imagine. As the President had pointed out, every clerk had to be familiar with calculations in shillings and pence, and if they could add in twelves why should they not add in eights? Elsewhere many clerks were already familiar with scales which in effect involved counting in eights, and it was necessary to count in sixteens when dealing with rupees and annas.

Meanwhile the discussion had been very valuable to him; it had set him thinking on certain lines and helped him to realize more clearly than before the limitations of the idea, and—on the other hand, if he might venture to say so—the conditions under which it had possibilities.

The author, in further reply, has written as follows:

Upon reflection I feel that I dismissed too abruptly Mr Wales's suggestion that s.n. 16 would be preferable to s.n. 8; not that my preference for the latter has abated, but because by his suggestion he has repaired my omission to point out that the relationship between s.n. 2 and s.n. 8 is merely one example of the general relationship between any two scales of which one is an integral power of the other. This example Mr Green explained in an interesting descriptive way, although the student to whom the relationship is not self-evident may perhaps be more precisely assisted by noticing the identity:

$$\ldots + 2^8 + 2^7 + 2^6 + \ldots + 2^2 + 2 + 1 = \ldots + 8^2\,(2^2+2+1) + 8\,(2^2+2+1) + 2^2 + 2 + 1$$

and the equation which results if 0 is substituted for any or all but one of the terms on the left. Mr Green's exposition is analogous to the known and easily provable fact that with a series of weights 1, 2, 4, etc., viz. all the powers of 2 up to 2^{n-1}, it is possible in *one way only* to weigh every integral amount up to (2^n-1) and in doing so every combination of the weights one or more at a time will be exhausted, i.e.

$$^nC_1 + {}^nC_2 + \ldots + {}^nC_n = (1+1)^n - 1 = 2^n - 1,$$

where the minus 1 represents $^nC_0$. This was pointed out by Tartaglia as long ago as 1556; at least one example of s.n. 2 before Leibnitz. The same principle is involved in the set of cards which are used to determine instantly the age of any person who selects all the cards on which his age appears. (See *Mathematical Recreations*, W. W. R. Ball, p. 260.)

The sedecimal scale suggested by Mr Wales was advocated in 1927 by Mr John L. Hodgson, B.Sc., M.I.Mech.E., and again in 1929 in *The Time Journey of Dr Barton*, from which (p. 48) the following footnote is quoted:

"In this system there would be sixteen digits instead of ten. The advantage of the sedecimal system over the decimal system is that things such as lengths, weights and electrical resistances can be more easily divided up into halves, quarters, eighths, sixteenths, and so on, than into tenths and hundredths. It therefore follows that eight, sixteen or thirty-two are, from the point of view of those who have to measure and to weigh (and that is most of us), more suitable radices for our numbers than ten. Also, by adding unity to any series of sedecimal weights or electrical resistances the whole series up to any given weight or resistance can be checked against that weight or resistance. Thus, $1+1+2=4$, $1+1+2+4=8$ and $1+1+2+4+8=16$. And lastly, the radix sixteen is to be preferred to the radix eight, as it allows a greater number of things to be enumerated by means of a given number of digits.

"In a letter to the present editor Mr Bernard Shaw has suggested names for the additional numbers required by the sedecimal system as follows: ....eight, nine, pi, pip, toc, chac, ep, poap, ten, eleven.... eighteen, nineteen, piteen, pipteen, tocteen, chacteen, epteen, poapteen, twenty....twenty-pi....twenty-poap, thirty...."

Binary fractions were referred to also in the discussion and it is worth noticing that although the ancient Egyptians used a denary notation for whole numbers they had a special series of binary fractions for measuring quantities of grain, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, and $\frac{1}{64}$, represented by "Horus eye" notation. Moreover many natives of India, although long acquainted with denary notation, are still ignorant of its application to fractions and use instead binary fractions at least as far as $\frac{1}{16}$.

Mr C. H. Reid, I.C.S. (retd.), a member of the Institute, has kindly informed me that in the wilds of India the native counts on his fingers, using the thumb *of the same hand* with which to count them; hence the radix 4:

4 cowries	= 1	ganda (a cowrie is a shell)
4 gandas	= 1	pice (the commonest Indian coin)
4 pice	= 1	anna
4 annas	= 1	suki (local term, not universal)
4 sukis	= 1	rupee
4 rupees	= 1	chik (now a European colloquialism, but probably with a vernacular basis)
4 chiks	= 1	gold mohur (the standard coin of Moghul times)

Recently Professor D'Arcy W. Thompson has stated: "I fancy that the historian of mathematics has still a vast deal to do in the *comparative* study of Egyptian and other early arithmetics" (*Nature*, Vol. cxv, p. 902). A year or two later Professor Archibald provided a descriptive bibliography of Egyptian mathematics containing about 240 items!—of which less than thirty are stated to be unimportant. There has been an extensive although not recent literature upon binary calculation since Leibniz drew attention to this subject (*Histoire de l'académie des sciences de Paris*, 1703, pp. 58–63). Legendre used binary numeration for calculating high powers (*Essai sur la théorie des nombres*, Paris 1798, p. 229). De Lagny proposed a new system of logarithms on the plan of binary arithmetic which he found shorter and more easy and natural than the common logarithms (*L'arithmétique nouvelle*, Rochefort 1703). Jacob Bernoulli solved the problem of calculating π in the binary system. These references and ten or twelve others are given by Professor Archibald in an article upon binary arithmetic which should form the starting point for anyone who wishes to study the subject (*American Mathematical Monthly*, Vol. xxv, 1918, p. 139). The most recent reference is to an article upon fractions, decimals, subtraction, division, etc., in the binary system by E. Collingdon (*Journal de mathématiques élémentaires*, De Longchamps 1897, tome 21, pp. 101–6, 126–31, 148–51, and 171–4). Professor Archibald discusses the "Russian peasant" method of multiplication and (turning to the lighter side) gives an exposition of "Nim, a game with a complete mathematical theory". For this the binary scale is used, and it will be recognized also as necessary for working out the theory of the "Tower of Hanoi" described by De Parville (*La Nature*, 1884, pt. 1, pp. 285–6) which has recently reappeared as a puzzle under the name "Pyramid Patience", and the "Chinese Rings" still well known but apparently first described by Cardan in 1550 in *De Subtilitate* (see *Mathematical Recreations*, Ball, pp. 228–34).

The following note has been received from **Mr D. P. Misra** since the Discussion:

"It seems difficult to believe that the Anglo-Saxons used octonal notation either in speaking or in writing. The historical works referred to have not brought forward any strong evidence to support this point. If that notation had been used by the Anglo-Saxons they would certainly have called 16, 24, etc., respectively di-eight, tri-eight, etc., or other names equivalent to these in the octonal scale. Similarly what we now know as 23 would have been called by the Saxons "di-eight-seven" and written as 27, but even if several numbers and units were proved to have been used and named in the octonal scale we should not be justified in concluding that all the numbers in use were so based. A modern example is found in India where 8 pice make one duwanni (a 2 anna piece) and 8 duwanni make 1 rupee. Neither this fact nor the fact that 4 pice make 1 anna and so on would justify a statement that India uses the octonal scale or the scale of 4. In Vedas and other Sanskrit works, words meaning tri-sept (**त्रिषप्त**), tri-eight (**त्र्यष्ट**), di-sept (**द्विसप्त**), etc., are undoubtedly found, but it would not be wise or necessary to infer from this that the septenary or the octonal scale was used in earlier days in India.

The scale advocated by Mr Phillips is attractive, but the practical inconvenience in its adoption would be very great. It would be difficult enough if the octonal scale were to be used throughout in writing, thinking and speaking and would require generations before it could be universally adopted, and to think in one scale and write and speak in another scale would be adding still further difficulties. It is true that despite the advantage which the centesimal scale of measuring angles may have over the sexagesimal scale, the latter is the scale commonly used. This is perhaps on account of the extensive re-calculation of tables which would be necessary in a change over to the former scale, but this only reinforces the arguments against the adoption of the suggestion which Mr Phillips advocates."