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On the Law of Mortality and the Construction of Annuity Tables. By WILLIAM MATTHEW MAKEHAM, *Esq., of the Church of England Assurance Institution.*

MOST writers on the subject of life annuities have had occasion to lament the paucity of tables available for the performance of calculations involving two or more lives. The late Mr. David Jones has done much to supply this deficiency by the publication of complete sets of tables for two lives at various rates of interest; but, beyond this, it is extremely improbable that, under the present system, any considerable progress will be made, owing to the multiplicity of the different combinations when three or more lives are concerned, and the consequent magnitude of the task involved in the construction of complete sets of tables for such cases.

It is scarcely necessary, I presume, to enlarge to any great extent upon the advantages of a ready and expeditious mode of computing *accurately* the values of annuities on three or more lives, according to a certain predetermined table of mortality, in preference to the usual methods of approximation at present adopted for the purpose of avoiding calculations of formidable length. Although the values deduced by such methods of approximation, in many instances, are, perhaps, as near the *truth* as the values correctly deduced would be, yet it is generally felt that, having assumed a certain table of mortality as the basis of calculation, it is desirable that the results attained should be strictly consistent with that basis—in short, that our *conclusions* should be in accordance with our *premises.* *Granted* that such a rate of mortality and such a rate of interest will obtain, *then* such a sum, and no other, is the value of the given annuity or other contingent benefit. To this very proper regard for logical consistency, which is the foundation of mathematical science, we owe the construction of tables of annuities certain to five or six decimal places; for it cannot be pretended that *any* assumed rate of interest represents the real value of money so exactly as to render such extreme accuracy at all necessary to the abstract justice of the case.

The chief object of the following investigation has, therefore, been to find a formula which should represent with sufficient accuracy the results of observations on the law of mortality; and which, at the same time, should be adapted to facilitate the construction of *complete* sets of tables of annuities involving several lives.

PART I.—*On the Law of Mortality.*

It seems to be generally admitted, that the *theoretical* law of mortality propounded by Mr. Gompertz, although by no means a perfect representation of the *actual* law, at the same time is so nearly borne out by facts, as to render it highly probable that further progress in the investigation will be made in the track thus opened up; in other words, that practical improvements in the construction of mortality tables may be looked for in some modification of Mr. Gompertz's formula.

As the subject is more conveniently treated *logarithmically,* the theoretical law in question may be defined by stating that the logarithms of the probabilities of living over any given period proceed in geometrical progression.

To see how far this theoretical law is supported by experience, let us examine the following data, derived from three of the most approved mortality tables:—

Age.	1. CARLISLE TABLE.			2. EXPERIENCE.			3. GOVERNMENT ANNUITANTS.		
	Log. of Prob. of Living over 20 Years.	Log.^{2} of same.	Differences	Log. of Prob. of Living over 20 Years.	Log.^{2} of same.	Differences	Log. of Prob. of Living over 20 Years.	Log.^{2} of same.	Differences.
	(–)			(–)			(–)		
20	·07918	$\bar{2}$·89862		·07427	$\bar{2}$·87081		·10154	$\bar{1}$·00664	
			+ ·25968			+ ·29936			+ ·17159
40	·14398	$\bar{1}$·15830		·14797	$\bar{1}$·17017		·15074	$\bar{1}$·17823	
			+ ·60690			+ ·63111			+ ·55518
60	·58237	$\bar{1}$·76520		·63282	$\bar{1}$·80128		·54127	$\bar{1}$·73341	

Now, if the Gompertzian theory were *strictly* true, the two terms in the column of differences would be equal; but, instead of this being the case, it appears that in each of the three instances the *second* difference is considerably greater than the *first*, which shows that the logarithms of the probabilities, instead of proceeding in uniform geometrical progression, increase (numerically) in a far greater ratio in the higher than in the lower ages.

But although the three terms in the first column do not obey the law assigned, yet they may be made to do so by the addition of a certain uniform quantity (x) to each term, such quantity being the numerical value of the following expression, in which the three given terms are denoted respectively by the letters a, b, and c:—

$$x=\frac{b^2-ac}{a+c-2b}.$$

The addition of this quantity to the *logarithms* of the probabilities is, of course, equivalent to multiplying the probabilities themselves by the number corresponding to the quantity added, considered as a logarithm; and the definition of the law of mortality becomes—"the probabilities of living, *increased or diminished in a certain constant ratio,* form a series whose logarithms are in geometrical progression."

I proceed to describe the method of deducing the rate of mortality at every age according to the law last defined, and to exhibit the results in comparison with those derived from actual observations, and also with the results deduced by means of Mr. Gompertz's formula.

Let π_x denote the probability of living one year at age x, and $\pi_x^{\overline{n}|}$ the probability of living n years at same age; then the three quantities in the first columns of the foregoing table will be represented by log. $\pi_{20}^{\overline{20}|}$, log. $\pi_{40}^{\overline{20}|}$, and log. $\pi_{60}^{\overline{20}|}$. Further: let a^{20} stand for the quantity by which $\pi_{20}^{\overline{20}|}$, $\pi_{40}^{\overline{20}|}$, and $\pi_{60}^{\overline{20}|}$, are to be multiplied in order that the law of geometrical progression may prevail, and let q^{20} be the common ratio of the three resulting terms. Now,

$$\log.\ \pi_{20}^{\overline{20}|}=\log.\ \pi_{20}+\log.\ \pi_{21}+\ \ldots\ \log.\ \pi_{39}$$
$$\log.\ (a^{20}\pi_{20}^{\overline{20}|})=\log.\ (a\pi_{20})+\log.\ (a\pi_{21})+\ \ldots\ \log.\ (a\pi_{39});$$

and, by the assumed law of mortality,

$$\log.\ (a\pi_{21})=\log.\ (a\pi_{20})\times q$$
$$\log.\ (a\pi_{22})=\log.\ (a\pi_{20})\times q^2$$
$$\&c. \quad = \quad \&c.;$$

whence

$$\log.\,(a^{20}\pi_{20}{}^{\overline{20}|})=\log.\,(a\pi_{20})+\log.\,(a\pi_{20})\times q\;\ldots\ldots+\log.\,(a\pi_{20})\times q^{19}$$

$$=\log.\,(a\pi_{20})\times\frac{q^{20}-1}{q-1}$$

$$\therefore\ \log.\,(a\pi_{20})=\log.\,(a^{20}\pi_{20}{}^{\overline{20}|})\times\frac{q-1}{q^{20}-1};$$

and having found, by the last equation, the first term of the series log. $^2(a\pi_x)$, the successive terms are obtained by the repeated additions of log. q, and the series log. π_x is then deduced by the simple subtraction of log. a from each term of log. $(a\pi_x)$.

It remains now to test the proposed formula by its application to actual observations, for which purpose I select the well-known "Experience" mortality amongst assured lives. In this case, the data for the ages between 20 and 80 is by far the most important in comparison with the rest; first, because the observations on the ages not included between those limits are made upon numbers too small to give much weight to the deductions made from them; and, secondly, because the great mass of the calculations of an Assurance Office will be but slightly affected by errors in estimating the rate of mortality at the excluded ages. For these reasons, the following law of mortality has been deduced entirely from the observations on lives between the ages of 20 and 80, leaving the remaining portions of the table to be constructed on the assumption that the law so deduced may be taken to represent the true rate of mortality—say, from the age of 10 years upwards, to the extremity of human life.

The data derived from the Experience observations gives

$$\log.\,\pi_{20}{}^{\overline{20}|}=-\cdot07427=a,$$

$$\log.\,\pi_{40}{}^{\overline{20}|}=-\cdot14797=b,$$

$$\log.\,\pi_{60}{}^{\overline{20}|}=-\cdot63282=c.$$

Adding to each of these terms the quantity log. a^{20}, deduced from the formula,

$$\log.\,a^{20}=\frac{b^2-ac}{a+c-2b}=\cdot06106,$$

we have

$$\log.\,(a^{20}\pi_{20}{}^{\overline{20}|})=-\cdot01321\quad \log.=\bar{2}\cdot12091$$

$$+\cdot81815=\log.\,q^{20}$$

$$\log.\,(a^{20}\pi_{40}{}^{\overline{20}|})=-\cdot08691\quad \log.=\bar{2}\cdot93906$$

$$+\cdot81815=\text{ do.}$$

$$\log.\,(a^{20}\pi_{60}{}^{\overline{20}|})=-\cdot57176\quad \log.=\bar{1}\cdot75721$$

whence we obtain the following values of the constants used in the process:—

$$
\begin{array}{rl}
q^{20}=6{\cdot}578850 & \log.=\ {\cdot}8181500 \\
q^{20}-1=5{\cdot}578850 & \log.=\ {\cdot}7465447 \\
q=1{\cdot}098772 & \log.=\ {\cdot}0409075 \\
q-1=\ {\cdot}098772 & \log.=\bar{2}{\cdot}9946338 \\
a^{20}=1{\cdot}150959 & \log.=\ {\cdot}06106 \\
a=1{\cdot}007054 & \log.=\ {\cdot}003053 \\
\log.\,(a\pi_{20})=-{\cdot}000233883 & \log.=\bar{4}{\cdot}3689987.
\end{array}
$$

The following is a specimen of the actual process of finding the logarithms of the adjusted probabilities for each age:—

$$
\begin{array}{rll}
 & & (-) \\
\log.^2(a\pi_{20})=\bar{4}{\cdot}3689987 & \log.\,(a\pi_{20})={\cdot}000233883 & \\
{\cdot}0409075 & & -{\cdot}000023100 \\
\hline
\log.^2(a\pi_{21})=\bar{4}{\cdot}4099062 & \log.\,(a\pi_{21})={\cdot}000256983 & \\
{\cdot}0409075 & & -{\cdot}000025384 \\
\hline
\log.^2(a\pi_{22})=\bar{4}{\cdot}4508137 & \log.\,(a\pi_{22})={\cdot}000282367 & \\
{\cdot}0409075 & & -{\cdot}000027889 \\
\hline
\log.^2(a\pi^{23})=\bar{4}{\cdot}4917212 & \log.\,(a\pi_{23})={\cdot}000310256 & \\
{\cdot}0409075 & & -{\cdot}000030645 \\
\hline
\log.^2(a\pi_{24})=\bar{4}{\cdot}5326287 & \log.\,(a\pi_{24})={\cdot}000340901 & \\
\&c. & \&c. &
\end{array}
$$

The differences of the series log. $(a\pi_x)$ are here taken out, and the value of log. π_{20} being found by subtracting log. a from the first term of the series (viz., log. $a\pi_{20}$), the successive addition of the differences gives the several values of log. π_x thus:—

$$
\begin{array}{rl}
\log.\,(a\pi_{20})= & -{\cdot}000233883 \\
-\log.\,a= & -{\cdot}003053000 \\
\hline
 & -{\cdot}003286883=\log.\,\pi_{20} \\
\Delta= & -{\cdot}000023100 \\
\hline
 & -{\cdot}003309983=\log.\,\pi_{21} \\
\&c. & \quad \&c.
\end{array}
$$

I now beg to direct the reader's attention to the following table, showing the annual mortality amongst 1,000 persons entering upon each year of age from 20 to 80; first, from the actual experience of the 17 Offices; secondly, from the Actuaries' Adjusted Table based upon it; thirdly, from the formula herein described; and, lastly, from Mr. Gompertz's own formula.

Annual Mortality per 1,000.

Age.	Experience.	Actuaries' Adjustment.	New Formula.	Gompertz's Formula.	Age.	Experience.	Actuaries' Adjustment.	New Formula.	Gompertz's Formula.
20	12·38	7·29	7·55	3·74	50	16·30	15·94	15·99	21·10
21	10·95	7·38	7·59	3·97	51	17·20	16·90	16·87	22·36
22	5·95	7·46	7·66	4·20	52	18·95	17·95	17·85	23 64
23	8·51	7·56	7·71	4·46	53	18·52	19·09	18·91	25·06
24	6·74	7·67	7·78	4·72	54	18·12	20·31	20·08	26·51
25	7·67	7·77	7·87	5·00	55	24·92	21·66	21·34	28·08
26	7·15	7·89	7·94	5 30	56	24·46	23·13	22·76	29·74
27	8·94	8·01	8·05	5·61	57	22·01	24·68	24·29	31·48
28	7·71	8·14	8·14	5·95	58	23·98	26·39	26·00	33·30
29	5·27	8·28	8·25	6·30	59	30·10	28·25	27·86	35·28
30	7·19	8·42	8·37	6·68	60	30·13	30·34	29·89	37·32
31	7·65	8·58	8·51	7·07	61	32·83	32·61	32·12	39·52
32	6·09	8·75	8·67	7·49	62	31 64	35·12	34·55	41·81
33	9·48	8·92	8·83	7·94	63	35·30	37·84	37·24	44·24
34	8·88	9·10	9 01	8·41	64	48·20	40·83	40·18	46·83
35	10·57	9·29	9·19	8 91	65	45·29	44·08	43·40	49·55
36	9·55	9·48	9·42	9·43	66	47·81	47·61	46·92	52·43
37	10·08	9·69	9·65	10·00	67	44 93	51·47	50·77	55·44
38	9·95	9 91	9·92	10·59	68	57·35	55·63	54·98	58·67
39	9·57	10·13	10·19	11·21	69	62 05	60 09	59·61	62·05
40	11·61	10 36	10·51	11·88	70	75·53	64·93	64·64	65·63
41	10·80	10 61	10·85	12·58	71	70·72	70·16	70·13	69·41
42	10·56	10·89	11 24	13·33	72	67·20	75 80	76·15	73·36
43	10·61	11·25	11·65	14·11	73	79·36	81·88	82·72	77·58
44	11·75	11·70	12 13	14·95	74	91·82	88·47	89·88	82·03
45	12·17	12·21	12·63	15·86	75	100·54	95·56	97·68	86 67
46	10·99	12 84	13·18	16·76	76	102·35	103·18	106·16	91·59
47	13·06	13·52	13·79	17·78	77	102·08	111·47	115·41	96·79
48	16·56	14·26	14·45	18·79	78	134·55	120·44	125·44	102·26
49	14·85	15·06	15·17	19·92	79	134·69	130 06	136·35	108·01

It will be seen, by inspection, that the numbers in the third column follow very fairly the original and adjusted data in the first and second; while the last column, obtained by the application of Mr. Gompertz's formula *unmodified*, exhibits so little conformity with the original data, as to render it totally unfit to be adopted as a substitute.

I proceed, in the next part, to show how the method of construction herein proposed may be made of considerable utility in forming a complete set of annuity tables involving two or more lives.

PART 2.—*On the Construction of Annuity Tables.*

It will be convenient to abandon the logarithmic form hitherto adopted, and pursue the subject with the aid of the characters denoting simple quantities.

The following equations are deduced directly from the assumed law of mortality as defined in the first part.

$$a\pi_n = a\pi_n$$
$$a\pi_{n+1} = (a\pi_n)^q$$
$$a\pi_{n+2} = (a\pi_n)^{q^2}$$
$$\cdots\cdots$$
$$a\pi_{n+r-1} = (a\pi_n)^{q^{r-1}};$$

whence $a\pi_n \times a\pi_{n+1} \times \ldots \times a\pi_{n+r-1} = a^r\pi_n^{\overline{r|}} = (a\pi_n)^{\frac{q^r-1}{q-1}}$.

Let $B_n = (a\pi_n)^{\frac{1}{q-1}}$, and we have $a^r\pi_n^{\overline{r|}} = \frac{B_n^{q^r}}{B_n}$.

consequently, if v^r be the value of £1 (certain) due r years hence, the value of £1 contingent on a life aged n years surviving the term of r years will be $\left(\frac{v}{a}\right)^r \cdot \frac{B_n^{q^r}}{B_n} = \frac{B_n^{q^r}}{B_n} s^r$ (putting $\frac{v}{a} = s$). The value of an annuity, payable in advance, on a life aged n years, will, therefore, be represented by

$$\frac{1}{B_n}(B_n + B_n^q s. + B_n^{q^2}s^2 + B_n^{q^3}s^3 \ldots \text{ ad infin.}) \quad [1];$$

and, similarly, the value of an annuity on two joint lives aged respectively m and n, by

$$\frac{1}{B_m B_n}\left(B_m B_n + (B_m B_n)^q t + (B_m B_n)^{q^2} t^2 + \ldots\right) \quad [2],$$

where $t = \frac{v}{a^2}$.

In seeking for a suitable modification of Mr. Gompertz's formula, it is, of course, highly desirable to avoid introducing any unnecessary intricacy. Now, it will be observed that the additional constant, a, enters in the formula precisely in the same way as the element of *interest*, which may almost in practice be said to form an inseparable part of it; and consequently, that, for all practical purposes, the proposed modification does not alter the *form* of the function deduced by Mr. Gompertz.

If the two lives be of the same age, p, the value of the annuity becomes

$$\frac{1}{(B_p^2)}\left\{(B_p^2) + (B_p^2)^q t + (B_p^2)^{q^2} t^2 + \ldots\right\} \quad [3].$$

Comparing this with the formula [2], it will readily be seen that the value of an annuity on the two lives aged m and n will be the same as the value of an annuity on the two equal lives aged p, provided that $B_m B_n = B_p^2$. The same property, of course, holds good for any number of lives. Thus, the value of an annuity on three joint lives, each aged p, is

$$\frac{1}{(B_p^3)}\left\{(B_p^3) + (B_p^3)^q z + (B_p^3)^{q^2} z^2 + \ldots\right\},$$

which is also the value of an annuity on any other combination of three lives, aged respectively i, k, and l, provided

$$B_i B_k B_l = B_p^3.$$

The property in question, as I shall now proceed to show, gives the power of constructing a table of the *correct* values of annuities *for any given number of lives* (according to the law of mortality before explained), with a considerably less expenditure of time and labour than is required in constructing a complete set of tables for two lives only according to the usual method.

Taking the qth power of each side of the equation $B_n = (a\pi_n)^{\frac{1}{q-1}}$, we have $B_n^q = (a\pi_n)^{\frac{q}{q-1}}$; but $(a\pi_n)^q = a\pi_{n+1}$, wherefore $B_n^q = (a\pi_{n+1})^{\frac{1}{q-1}} = B_{n+1}$, and, generally, $B_n^{q^t} = B_{n+t}$; consequently, $(B_m . B_n)^{q^t} = B_{m+t} B_{n+t}$, and $(B_p^2)^{q^t} = B_{p+t}^2$; from which it appears that if $B_p^2 = B_m B_n$, then $B_{p+t}^2 = B_{m+t} . B_{n+t}$; and, therefore, having found p, the common age equivalent to m and n, the common age equivalent to $(m+t)$ and $(n+t)$ will be $p+t$. Now, let m be the younger of the two ages m and n, and let $p = m+d$, then $p+t = (m+t)+d$; that is, the addition which must be made to the younger age m, to give the equivalent common age p, is the same which must be made to the younger of any other two ages where the difference is the same, viz., $n-m$.

I annex (Table I.) an extract from a table of annuities on two lives of *equal* ages, according to the proposed law of mortality, constructed in the usual way, but having the values of every tenth part of a year's difference in age inserted by interpolation. The latter process is rendered comparatively easy, by the fact that the values of annuities at consecutive ages are nearly in arithmetical progression. The further subdivision of the ages, when necessary, can be performed by the aid of the column of differences.

Before the table so formed can be used for finding the values of annuities on combinations of unequal ages, we must have a table showing the addition, d, to be made to the younger of two ages whose difference is k, in order to give the equivalent common age. Assume the younger age $= o$, then

$$B_d^2 = B_o B_k \;\therefore\; (B_o^{q^d})^2 = B_o (B_o^{q^k}),$$

$$\text{or } B_o^{2q^d} = B_o^{1+q^k}, \text{ whence } 2q^d = 1+q^k,$$

$$q^d = \frac{1+q^k}{2}, \text{ or log. } q \times d = \text{log. } \frac{1+q^k}{2},$$

$$\therefore\ d = \frac{\log. \dfrac{1+q^k}{2}}{\log. q},$$

by which formula the values of d_x in the annexed table have been computed.

In a similar way, it may be shown that, in the case of three lives, if k and l denote the differences between the youngest and the other two ages respectively, in order to find the equivalent common age we must add to the youngest age the quantity $\dfrac{\log. \dfrac{1+q^k+q^l}{3}}{\log. q}$. To calculate the value of this expression for every combination of k and l would be a work of considerable labour, but by means of a table of the values of q_x (*vide* Table II.), the quantity in question may be easily computed in any particular case. The annuity table for three, or indeed any number of lives, would, of course, be found precisely in the same way as the table for two lives, and would require, in its construction, the same amount of labour, and no more.

TABLE I.—*Two Joint Lives* (*Extract*).

Common Age.	Annuity.	Difference	Common Age	Annuity.	Difference
		(−)			(−)
39·1	13·4751	229	39·6	13·3601	232
39·2	13·4522	229	39·7	13 3369	232
39·3	13 4293	230	39 8	13·3137	233
39 4	13·4063	231	39·9	13·2904	233
39·5	13 3832	231	40·	13·2671	234

TABLE II. (*Extract*).

x.	log. q^x.	q^x.	d_x.	x.	log. q^x.	q^x.	d_x.
11	·4499825	2·818270	6·865	16	·6545200	4 513568	10 766
12	·4908900	3 096635	7·612	17	·6954275	4·959381	11·591
13	·5317975	3 402495	8·376	18	·7363350	5·449229	12·429
14	·5727050	3 738566	9·158	19	·7772425	5·987458	13·280
15	·6136125	4·107830	9·954	20	·8181500	6·578850	14·143

I conclude with an example of the actual process of determining from the table the value of an annuity on two joint lives, and also of the equivalent common age in a case of three lives. The corresponding annuity in the latter case would, of course, be found from the table of three lives in precisely the same way as the annuity on the two lives.

Example 1.—Required the value of an annuity on two joint lives aged respectively 30 and 45.

Here, $k=15$ and (Table II.) $d_{15}=9{\cdot}954$; wherefore the equivalent common age$=39{\cdot}954$. By Table I. we find the value of an annuity for the age $39{\cdot}9=13{\cdot}2904$, and the corresponding difference for one-tenth of a year$=-233$. Therefore,

$$\begin{array}{lr}
 & 13{\cdot}2904 \\
233\times{\cdot}5\;\;=117 & \\
233\times{\cdot}04=\;\;\;9 & \\
\overline{} & {\cdot}0126 \\
\hline
 & 13{\cdot}2778 = \text{value required.}
\end{array}$$

Example 2.—Required the common age equivalent to the three ages, 25, 40, and 45,

Here, $k=15$, $l=20$.

$$\begin{array}{r}
1{\cdot} \\
q^{15}=4{\cdot}10783 \\
q^{20}=6{\cdot}57885 \\
\hline
3\,)\,11{\cdot}68668 \\
\hline
3{\cdot}89556 \quad \log.={\cdot}59057 \\
\log.\ q={\cdot}04091
\end{array}$$

$$\therefore\ d_{15,\,20}=\frac{{\cdot}59057}{{\cdot}04091}=14{\cdot}436$$

and common age required$=39{\cdot}436$.