# INSTITUTE AND FACULTY OF ACTUARIES

# EXAMINATION

7 October 2011 (am)

## Subject CT4 — Models
## Core Technical

*Time allowed: Three hours*

***INSTRUCTIONS TO THE CANDIDATE***

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

***Graph paper is NOT required for this paper.***

*AT THE END OF THE EXAMINATION*

*Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.*

*In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.*

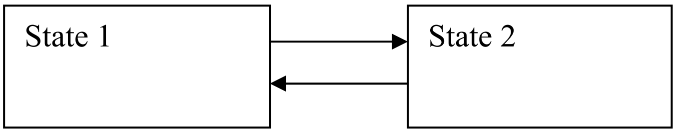
 

**1** The diagrams below show three Markov chains, where arrows indicate a non-zero transition probability.

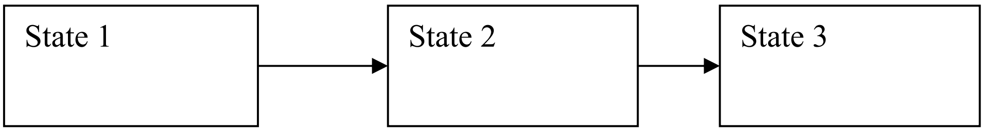
State whether each of the chains is:

(a) irreducible.
(b) periodic, giving the period where relevant. [3]

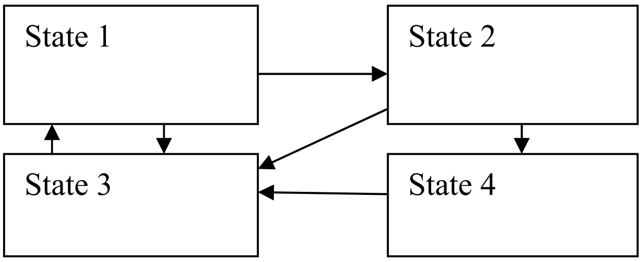
A.




B.




C.




**2** (i) Describe what is represented by each of the central rate of mortality, $m_x$ , and the initial rate of mortality, $q_x$. [2]

(ii) State the circumstance in which $m_x = \mu_x$. [1]

[Total 3]

**3** Describe how a strictly stationary stochastic process differs from a weakly stationary stochastic process. [3]

**4** A new weedkiller was tested which was designed to kill weeds growing in grass. The weedkiller was administered via a single application to 20 test areas of grass. Within hours of applying the weedkiller, the leaves of all the weeds went black and died, but after a time some of the weeds re-grew as the weedkiller did not always kill the roots.

The test lasted for 12 months, but after six months five of the test areas were accidentally ploughed up and so the trial on these areas had to be discontinued. None of these five areas had shown any weed re-growth at the time they were ploughed up.

- Ten of the remaining 15 areas experienced a re-growth of weeds at the following durations (in months): 1, 2, 2, 2, 5, 5, 8, 8, 8, 8.
- Five areas still had no weed re-growth when the trial ended after 12 months.

(i) Describe, giving reasons, the types of censoring present in the data. [2]

(ii) Estimate the probability that there is no re-growth of weeds nine months after application of the weedkiller using either the Kaplan-Meier or the Nelson-Aalen estimator. [4]
[Total 6]

**5** (i) List the factors which should be considered in assessing the suitability of a model for a particular exercise. [3]

(ii) Assess the suitability of a multiple state model with three states: Healthy, Sick and Dead, for estimating the transition intensities in an analysis of claims for sickness benefit, in the light of your answer to (i). [4]
[Total 7]

**6** A recording instrument is set up to observe a continuous time process, and stores the results for the most recent 250 transitions. The data collected are as follows:

| *State i* | *Total time spent in state i (hours)* | *Number of transitions to* State A | State B | State C |
|---|---|---|---|---|
| *A* | 35 | Not applicable | 60 | 45 |
| *B* | 150 | 50 | Not applicable | 25 |
| *C* | 210 | 55 | 15 | Not applicable |

It is proposed to fit a Markov jump model using the data.

(i) (a) State all the parameters of the model.
(b) Outline the assumptions underlying the model. [4]

(ii) (a) Estimate the parameters of the model.
(b) Write down the estimated generator matrix of the model. [4]

(iii) Specify the distribution of the number of transitions from state $i$ to state $j$, given the number of transitions out of state $i$. [1]
[Total 9]

**7** A study is made of the impact of regular exercise and gender on the risk of developing heart disease among 50–70 year olds. A sample of people is followed from exact age 50 years until either they develop heart disease or they attain the age of 70 years. The study uses a Cox regression model.

(i) List reasons why the Cox regression model is a suitable model for analyses of this kind. [3]

The investigator defined two covariates as follows:

- $Z_1$ = 1 if male, 0 if female.
- $Z_2$ = 1 if takes regular exercise, 0 otherwise.

The investigator then fitted three models, one with just gender as a covariate, a second with gender and exercise as covariates, and a third with gender, exercise and the interaction between them as covariates. The maximised log-likelihoods of the three models and the maximum likelihood estimates of the parameters in the third model were as follows:

| | |
|---|---|
| null model | –1,269 |
| gender | –1,256 |
| gender + exercise | –1,250 |
| gender + exercise + interaction | –1,246 |

| *Covariate* | *Parameter* |
|---|---|
| Gender | 0.2 |
| Exercise | –0.3 |
| Interaction | –0.35 |

(ii) Show that the interaction term is required in the model by performing a suitable statistical test. [5]

(iii) Interpret the results of the model. [3]

[Total 11]

**8** A continuous-time Markov process with states {Able to work ($A$), Temporarily unable to work ($T$), Permanently unable to work ($P$), Dead ($D$)} is used to model the cost of providing an incapacity benefit when a person is permanently unable to work. The generator matrix, with rates expressed per annum, for the process is estimated as:

$$\begin{array}{ccccc} & A & T & P & D \\ A & -0.15 & 0.1 & 0.02 & 0.03 \\ T & 0.45 & -0.6 & 0.1 & 0.05 \\ P & 0 & 0 & -0.2 & 0.2 \\ D & 0 & 0 & 0 & 0 \end{array}$$

(i) Draw the transition graph for the process. [2]

(ii) Calculate the probability of a person remaining in state $A$ for at least 5 years continuously. [2]

Define $F(i)$ to be the probability that a person, currently in state $i$, will never be in state $P$.

(iii) Derive an expression for:

(a) $F(A)$ by conditioning on the first move out of state $A$.
(b) $F(T)$ by conditioning on the first move out of state $T$. [3]

(iv) Calculate $F(A)$ and $F(T)$. [2]

(v) Calculate the expected future duration spent in state $P$, for a person currently in state $A$. [2]
[Total 11]

**9** (i) State the principle of correspondence as it applies to the estimation of mortality rates. [1]

(ii) Explain why it might be difficult to ensure the principle of correspondence is adhered to, and give a specific example of an investigation where this may be the case. [2]

An actuary was asked to investigate the mortality of lives in a particular geographical area. Data are available of the population of this area, classified by age last birthday, on 1 January in each year. Data on the number of deaths in this area in each calendar year, classified by age nearest birthday at death, are also available.

(iii) Derive a formula which would allow the actuary to estimate the force of mortality at age $x + f$, $\mu_{x+f}$, in a particular calendar year, in terms of the available data, and derive a value for $f$. [6]

(iv) List four factors other than geographical location which a government statistical office might use to subdivide data for national mortality analysis. [2]
[Total 11]

**10** (i) Describe three shortcomings of the $\chi^2$ test for comparing crude estimates of mortality with a standard table and why they may occur. [3]

The following table gives an extract of data from a mortality investigation conducted in the rural highlands of a developed country. The raw data have been graduated by reference to a standard mortality table of assured lives.

| *Age* $x$ | *Expected deaths* | *Observed deaths* | $z_x$ | $z_x^2$ |
|---|---|---|---|---|
| 60 | 36.15 | 35 | –0.191 | 0.037 |
| 61 | 28.92 | 24 | –0.915 | 0.837 |
| 62 | 31.34 | 27 | –0.775 | 0.601 |
| 63 | 38.01 | 35 | –0.488 | 0.238 |
| 64 | 26.88 | 32 | 0.988 | 0.975 |
| 65 | 37.59 | 36 | –0.259 | 0.067 |
| 66 | 33.85 | 34 | 0.026 | 0.001 |
| 67 | 26.66 | 32 | 1.034 | 1.070 |
| 68 | 22.37 | 26 | 0.767 | 0.589 |
| 69 | 18.69 | 33 | 3.310 | 10.956 |
| 70 | 18.24 | 22 | 0.880 | 0.775 |

(ii) For each of the three shortcomings you described in (i):

(a) name a test that would detect that shortcoming.
(b) carry out the test on the data above. [12]

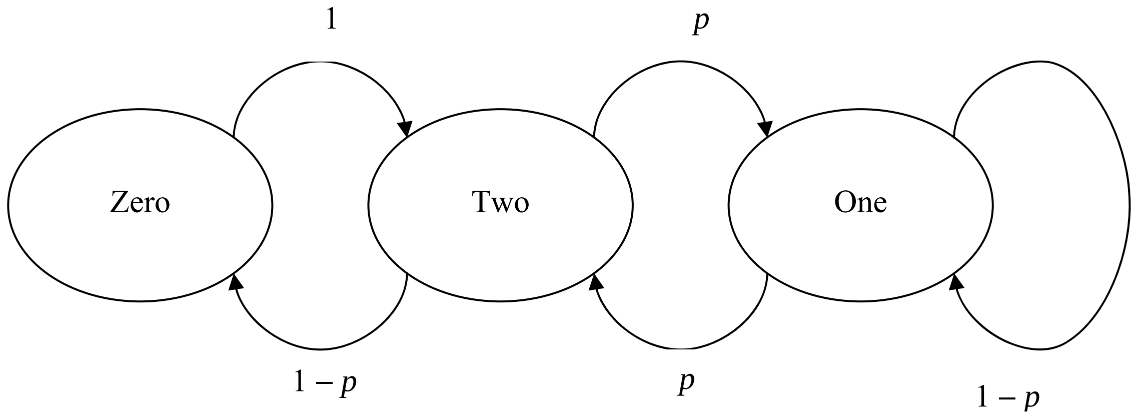
(iii) Comment on your results from (ii). [3]
[Total 18]

**11** An actuary walks from his house to the office each morning, and walks back again each evening. He owns two umbrellas. If it is raining at the time he sets off, and one or both of his umbrellas is available, he takes an umbrella with him. However if it is not raining at the time he sets off he always forgets to take an umbrella.

Assume that the probability of it raining when he sets off on any particular journey is a constant $p$, independent of other journeys.

This situation is examined as a Markov Chain with state space {0,1,2} representing the number of his umbrellas at the actuary's current location (office or home) and each time step representing one journey.

(i) Explain why the transition graph for this process is given by:




[3]

(ii) Derive the transition matrix for the number of umbrellas at the actuary's house before he leaves each morning, based on the number before he leaves the previous morning. [3]

(iii) Calculate the stationary distribution for the Markov Chain. [3]

(iv) Calculate the long run proportion of journeys (to or from the office) on which the actuary sets out in the rain without an umbrella. [2]

The actuary considers that the weather at the start of a journey, rather than being independent of past history, depends upon the weather at the start of the previous journey. He believes that if it was raining at the start of a journey the probability of it raining at the start of the next journey is $r$ ($0 < r < 1$), and if it was not raining at the start of a journey the probability of it raining at the start of the next journey is $s$ ($0 < s < 1$, $r \neq s$).

(v) Write down the transition matrix for the Markov Chain for the weather. [1]

(vi) Explain why the process with three states {0,1,2}, being the number of his umbrellas at the actuary's current location, would no longer satisfy the Markov property. [2]

(vii) Describe the additional state(s) needed for the Markov property to be satisfied, and draw a transition diagram for the expanded system. [4]

[Total 18]

**END OF PAPER**