INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject CT4 – Models
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

November 2014

General comments on Subject CT4

Subject CT4 comprises five main sections: (1) a study of the properties of models in general, and their uses for actuaries, including advantages and disadvantages (and a comparison of alternative models of the same processes); (2) stochastic processes, especially Markov chains and Markov jump processes; (3) models of a random variable measuring future lifetime; (4) the calculation of exposed to risk and the application of the principle of correspondence; (5) the reasons why mortality (or other decremental) rates are graduated, and a range of statistical tests used both to compare a set of rates with a previous experience and to test the adherence of a graduated set of rates to the original data. Throughout the subject the emphasis is on estimation and the practical application of models. Theory is kept to the minimum required in order usefully to apply the models to real problems.

Different numerical answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2014 paper

The general performance was disappointing compared with that in April 2014 and both 2013 sessions. Answers to questions on topics such as censoring in a survival model, and the memory less property of the exponential distribution, were often poor. Despite this, well-prepared candidates scored highly across the whole paper, with a substantial number of candidates scoring 70 per cent or more, and a highest mark of 96 per cent.

There was a tendency for candidates to fail to score marks by missing out the more 'wordy' sections of questions even when these were straightforward bookwork.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to include these areas in their revision.

1

	State Space	***Time Space***
Counting process	Discrete	Can be either
Simple random walk	Discrete	Discrete
Compound Poisson	Can be either	Continuous
Markov jump process	Discrete	Continuous

This question was well answered, with an average mark of more than 3 out of 4. [4]

2 (i) The objectives of the modelling exercise.

The validity of the model for the purpose to which it is to be put.

The validity of the data to be used.

The validity of assumptions used.

The possible errors associated with the model or parameters used not being a perfect representation of the real world situation being modelled.

The impact of correlations between the random variables that "drive" the model.

The extent of correlations between the various results produced from the model.

The current relevance of models written and used in the past.

The credibility of the data input.

The credibility of the results output.

The dangers of spurious accuracy.

Cost of buying or constructing, and of running the model.

Ease of use and availability of suitable staff to use it.

Risk of model being used incorrectly or with wrong inputs.

The ease with which the model and its results can be communicated.

Compliance with the relevant regulations.

Clear documentation. [4]

(ii) **Pension scheme for medium-sized client**

Validity of data/assumptions. Compliance with legislation.

It is a financially significant figure which you cannot afford to be way off the mark and is likely to make a big difference to the company making the contributions, so accurate data and calculations are important and compliance with legislation essential.

Business case for a bank loan

Ease of communication.

You must explain it to your friend who in turn must explain it to the bank manager.

Cake list

Dangers of spurious accuracy.

The sum of money concerned is so small anything which is time-consuming or expensive is a waste. [3]

[Total 7]

In part (i) not all the points listed here were required for full credit. In part (ii) the suggestions given here are just examples. Factors other than those listed here were given credit if sensible justifications were given. Most candidates scored reasonably well on part (i) of this question but answers to part (ii) were very disappointing, with many candidates appearing to treat part (ii) as completely unrelated to part (i). There is a tendency for candidates to learn by rote lists such as those required for part (i) of this question, without really thinking about the application of the lists to practical problems.

3 (i) In survival investigations, population counts will only be available at census dates.

Define $P_{x,t}$ to be the number of lives under observation, aged x last birthday, at any time t and suppose that we have the values of $P_{x,t}$ only if t is a census date.

We require the exposed to risk, E_x^c, over the interval between the first census and the last.

This is $E_x^c = \int_{t_1}^{t_2} P_{x,s} ds$, where t_1 and t_2 are the two census dates.

To evaluate this, we usually assume that Px,s is linear between census dates.

If the censuses are one year apart this leads to the trapezium approximation:

$$E_x^c = \frac{1}{2}(P_{x,t_1} + P_{x,t_2}) \ . \qquad [2]$$

(ii) **Company A**

Assume birthdays evenly distributed across calendar years.

Age 55 last = 0.5 * age 55 nearest + 0.5 * age 56 nearest.

The required exposed to risk is

$$\frac{1}{4}(3{,}390 + 3{,}100 + 3{,}020 + 2{,}950) = 3{,}115.$$

Company B

Data are based on age last birthday so no age adjustment needed.

Assuming population varies linearly between census dates, then

population on 1 January 2012 = $\frac{3}{4}1{,}300 + \frac{1}{4}1{,}190 = 1{,}272.5$

population on 1 January 2013 = $\frac{3}{4}1{,}440 + \frac{1}{4}1{,}300 = 1{,}405$

The required exposed to risk is then

$$\frac{3}{12}\frac{1}{2}(1{,}272.5 + 1{,}300) + \frac{9}{12}\frac{1}{2}\left(1{,}300 + 1{,}405\right) = 1{,}335.9375$$

Company C

Age 55 last birthday is equivalent to age 56 next birthday.

Assume that data for 31 December in year t apply to 1 January in year $t + 1$.

The required exposed to risk is then

$$\frac{1}{2}(5{,}950 + 5{,}980) = 5{,}965. \qquad [6]$$

[Total 8]

In part (i) equations were not required, explanations and diagrams are acceptable instead. Candidates' answers to this question varied considerably. Partial credit was given in part (ii) for a range of alternative approximations. A common error in part (ii) was to use the *wrong age when averaging (in particular supposing that age 55 last could be obtained by averaging age 54 nearest and age 55 nearest).*

4 (i) The force of mortality, μ_{x+t} at age $x + t$ is defined by the expression

$$\mu_{x+t} = \lim_{dt\to 0} \frac{1}{dt} \Pr[T \le x+t+dt \,|\, T > x+t] \qquad [1]$$

(ii) $_5p_0 = \mathrm{e}^{-5\mu}$ [1]

(iii) $_5p_5 = e^{-5\lambda}$

$_{15}p_5 = e^{-15\lambda}$

But $_5p_5 = 2\,{}_{15}p_5$

Hence

$$e^{-5\lambda} = 2e^{-15\lambda}$$

$$-5\lambda = \log_e 2 - 15\lambda = 0.693 - 15\lambda$$

so that $\lambda = 0.0693$. [3]

(iv) If $\lambda = \mu = 0.0693$, then $e_0 = \frac{1}{\mu} = \frac{1}{0.0693} = 14.43$ years. [1]

(v) If $\lambda \ne \mu$, then

$$e_0 = \int_0^5 e^{-\mu s}ds + e^{-5\mu}\int_0^\infty e^{-\lambda s}ds\,.$$

Evaluating the integrals gives

$$e_0 = \left[-\frac{1}{\mu}e^{-\mu s}\right]_0^5 + e^{-5\mu}\left[-\frac{1}{\lambda}e^{-\lambda s}\right]_0^\infty$$

$$= \left[-\frac{1}{\mu}e^{-5\mu} + \frac{1}{\mu}\right] + e^{-5\mu}\left[0 + \frac{1}{\lambda}\right]$$

$$= \frac{1}{\lambda}e^{-5\mu} + \frac{1}{\mu}(1 - e^{-5\mu}),$$

and putting $\lambda = 0.0693$ gives

$$e_0 = \frac{1}{0.0693} e^{-5\mu} + \frac{1}{\mu}(1 - e^{-5\mu})$$

$$= 14.43 e^{-5\mu} + \frac{1}{\mu}(1 - e^{-5\mu}). \quad \text{or} \quad = e^{-5\mu}\left(14.43 - \frac{1}{\mu}\right) + \frac{1}{\mu}.$$ [4]

OR

(v) If $\lambda \neq \mu$, then

$$e_0 = \int_0^\infty t\, {}_t p_x \mu dt.$$

$$e_0 = \int_0^5 t e^{-\mu t} \mu dt + e^{-5\mu} \int_5^\infty t e^{-\lambda(t-5)} \lambda dt$$

Integrating by parts

$$= \int_0^5 t\left(-\frac{\partial}{\partial t} e^{-\mu t}\right) dt + e^{-5(\mu-\lambda)} \int_5^\infty t\left(-\frac{\partial}{\partial t} e^{-\lambda t}\right) dt$$

$$= \left[-t e^{-\mu t}\right]_0^5 + \int_0^5 e^{-\mu t} dt + e^{-5(\mu-\lambda)}\left(\left[-t e^{-\lambda t}\right]_5^\infty + \int_5^\infty e^{-\lambda t} dt\right)$$

$$= -5e^{-5\mu} + \left[-\frac{1}{\mu} e^{-\mu t}\right]_0^5 + e^{-5(\mu-\lambda)}\left(\left[-t e^{-\lambda t}\right]_5^\infty + \left[-\frac{e^{-\lambda t}}{\lambda}\right]_5^\infty\right)$$

$$= -5e^{-5\mu} + \left(-\frac{e^{-5\mu}}{\mu} + \frac{1}{\mu}\right) + e^{-5(\mu-\lambda)}\left\{\left(0 - -5e^{-5\lambda}\right) + \left(0 + \frac{e^{-5\lambda}}{\lambda}\right)\right\}$$

$$= -5e^{-5\mu} + \left(-\frac{e^{-5\mu}}{\mu} + \frac{1}{\mu}\right) + e^{-5\mu}\left\{5 + \frac{1}{\lambda}\right\}$$

$$= \frac{1}{\lambda} e^{-5\mu} + \frac{1}{\mu}(1 - e^{-5\mu}),$$

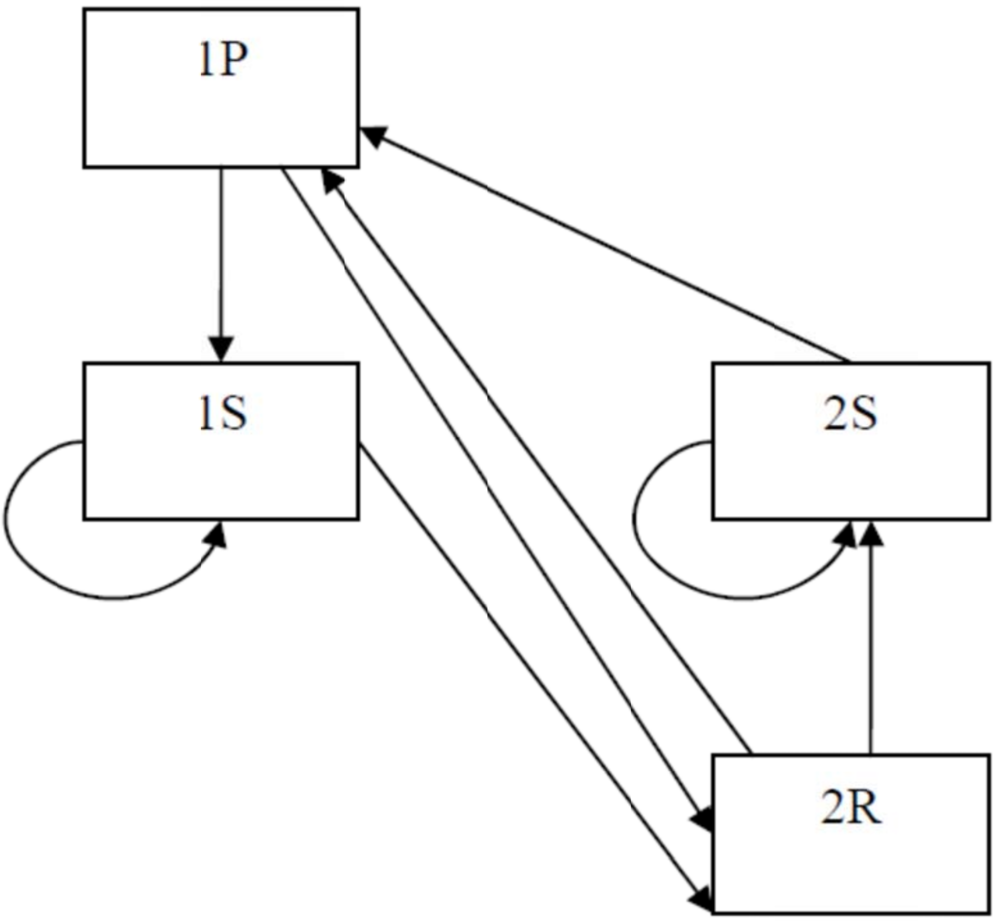
$$= 14.43 e^{-5\mu} + \frac{1}{\mu}(1 - e^{-5\mu}).$$ [4]

[Total 10]

Answers to part (i) were very disappointing, as this definition is in the Core Reading, Unit 5, page 3. A common error in part (iii) was to use ${}_{10}p_5$ instead of ${}_{15}p_5$ for the probability of surviving from age 5 years to age 20 years. Candidates who calculated an incorrect value of λ in part (iii) could score full credit for parts (iv) and (v) if they followed through correctly. A surprising number of candidates wasted time by deriving the answer in part (iv): the question said "calculate" and it was expected that candidates would know and could use the fact that the expectation of life in an exponential model is the reciprocal of the rate. Part (v) challenged most candidates. A common error was to omit $e^{-5\mu}$ from the second term in the initial expression.

5 (i) Four states. [1]

(ii) State space {1 just Promoted, 1 Same division, 2 Same division, 2 just Relegated}



[3]

(iii)

	1P	*1S*	*2S*	*2R*
1P	0	0.7	0	0.3
1S	0	0.85	0	0.15
2S	0.15	0	0.85	0
2R	0.25	0	0.75	0

[2]

(iv) (a) the chain is irreducible

because every state can eventually be reached from every other state.

(b) the chain is aperiodic

because it can loop in states 1S or 2S and, being irreducible, every state has the same period. [2]

(v) Probability of being relegated in first year is 0.3 and 0.15 in each subsequent year.

THEN EITHER

Require minimum integer x where $(0.7)0.85^{(x-1)} < (1-0.6)$

$$0.85^{(x-1)} < 0.57143$$
$$(x-1)\ln(0.85) < \ln(0.57143)$$
$$(x-1) > \frac{\ln(0.57143)}{\ln(0.85)}$$
$$(x-1) > 3.44$$

Since x must be an integer, $x\text{-}1 = 4$.

Hence, allowing for the first season when the probability of being relegated is 0.3, $x = 5$ years before probability of at least 60% of being relegated.

OR

Probability of at least 60% of being relegated equates to a probability of remaining in the division of less than 40%.

Year	*Probability of remaining in division*	
1	0.7	= 0.7
2	0.7×0.85	= 0.595
3	$0.7 \times 0.85 \times 0.85$	= 0.506
4	$0.7 \times 0.85 \times 0.85 \times 0.85$	= 0.430
5	$0.7 \times 0.85 \times 0.85 \times 0.85 \times 0.85$	= 0.365

So the first year at which the probability of being relegated exceeds 60% is Year 5. [3]

[Total 11]

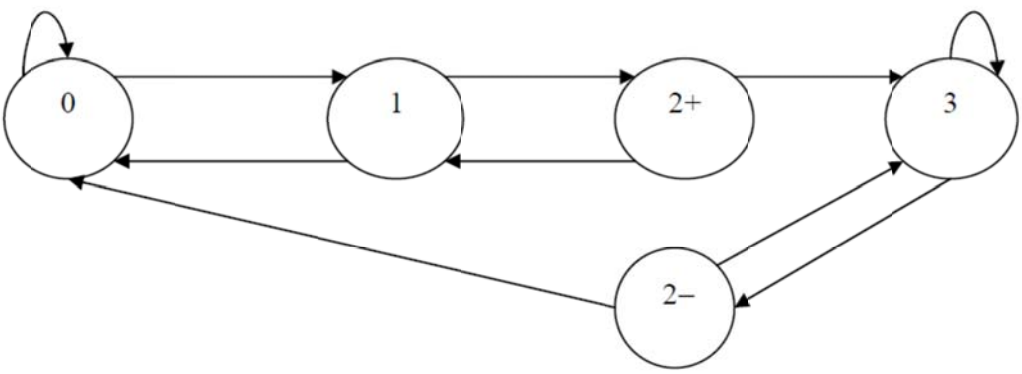
The quality of answers to this question varied greatly between Home and Overseas candidates, with Overseas candidates performing particularly poorly (many made only token attempts at the question). An alternative answer with 6 states (Top of Division 1, Bottom of Division 1, Elsewhere in Division 1, Top of Division 2, Bottom of Division 2, Elsewhere in Division 2) scores 0 for part (i) but could score full credit for other parts if followed through correctly (the diagram is much more complicated). Little credit was given for alternative state spaces which could not be used to model the situation as a Markov chain.

6 (i) (a) Five states are required

This is because if a person in discount level 2 makes a claim you need to know whether they made a claim in the previous year or not to determine whether they move down one or two levels.

Hence discount two must be split into
2^+ = no claim made previous year and
2^- = claim made previous year

(b)



[3]

(ii) Let p be the probability of making at least one claim.

$$P = \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 0 & 1-p \\ p & 0 & 0 & 0 & 1-p \\ 0 & 0 & 0 & p & 1-p \end{pmatrix}$$

where the levels are ordered 0, 1, 2^+, 2^-, 3.

Stationary distribution π satisfies $\pi = \pi P$

$$\pi_1 = p(\pi_1 + \pi_2 + \pi_4) \qquad (1)$$
$$\pi_2 = (1-p)\pi_1 + p\pi_3 \qquad (2)$$
$$\pi_3 = (1-p)\pi_2 \qquad (3)$$
$$\pi_4 = p\pi_5 \qquad (4)$$
$$\pi_5 = (1-p)(\pi_3 + \pi_4 + \pi_5) \qquad (5)$$

Also $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$ (6)

Working in terms of π_2

(3), (4) and (5) give $\pi_5 = (1-p)(\pi_3 + \pi_4 + \pi_5)$

$$p\pi_5 = (1-p)\pi_3 + (1-p)\pi_4$$
$$p\pi_5 = (1-p)^2\pi_2 + (1-p)p\pi_5$$
$$p^2\pi_5 = (1-p)^2\pi_2$$
$$\pi_5 = \frac{(1-p)^2}{p^2}\pi_2$$

But we are given that $\pi_5 = 9\pi_2$ so $\dfrac{(1-p)^2}{p^2} = 9$ which gives $p = 0.25$

(2) gives $\pi_1 = \dfrac{(1-p+p^2)}{(1-p)}\pi_2$

(4) gives $\pi_4 = \dfrac{(1-p)^2}{p}\pi_2$

and (6) gives $\pi_2\left\{\dfrac{1-p+p^2}{1-p} + 1 + (1-p) + \dfrac{(1-p)^2}{p} + \dfrac{(1-p)^2}{p^2}\right\} = 1$

Substituting $p = 0.25$ gives $\pi_2 = 0.071006$

$$\pi_3 + \pi_4 = \left\{ \frac{0.75^2}{0.25} + 0.75 \right\} \pi_2$$

$$= 3\pi_2$$

Thus the proportion of people at the 25% discount level is 0.213018. [6]

(iii) (a) Six states are now required

because the probability of a person in discount level 1 moving to discount level 2 depends upon whether a claim was made the previous year or not.

Hence discount level 1 must be split into
1^+ = no claim made previous year and
1^- = claim made previous year

(b) Let the probability of a claim in any year if there was a claim in the previous year be r(epeat) and the probability of a claim in any year if there was not a claim in the previous year be n(ew), then the new transition matrix is

$$\begin{pmatrix} r & 1-r & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 1-n & 0 & 0 \\ r & 0 & 0 & 1-r & 0 & 0 \\ 0 & 0 & n & 0 & 0 & 1-n \\ r & 0 & 0 & 0 & 0 & 1-r \\ 0 & 0 & 0 & 0 & n & 1-n \end{pmatrix}$$

where the levels are ordered 0, 1^+, 1^-, 2^+, 2^-, 3. [4]
[Total 13]

Most candidates worked out that five states were required, identified the correct state space and drew the correct diagram. Many also produced a correct matrix in part (ii) but few were able to solve the equations. Attempts at part (iii) were patchy, and only a minority of candidates attempted to write down the expanded matrix in part (iii)(b).

7 (i) A Poisson process is a counting process in continuous time $\{N_t, t \geq 0\}$, where N_t records the number of occurrences of a type of event within the time interval from 0 to t.

Events occur singly and may occur at any time;

the probability that an event occurs during the short time interval from time t to time $t + h$ is approximately equal to λh for small h, where the parameter λ is the rate of the Poisson process.

OR

A Poisson process is an integer valued process in continuous time $\{N_t, t \geq 0\}$, where

$$\Pr[N_{t+h} - N_t = 1 \mid F_t] = \lambda h + o(h)$$
$$\Pr[N_{t+h} - N_t = 0 \mid F_t] = 1 - \lambda h + o(h)$$
$$\Pr[N_{t+h} - N_t \neq 0,1 \mid F_t] = o(h)$$

and $o(h)$ is such that $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

OR

A Poisson process with rate λ is a continuous-time integer-valued process $N_t, t \geq 0$, with the following properties:

$N_0 = 0$

N_t has independent increments

N_t has Poisson distributed stationary increments

$$P\left[N_t - N_s = n\right] = \frac{\left[\lambda\left(t-s\right)\right]^n e^{-\lambda(t-s)}}{n!}, \qquad s < t,\ n = 0,1,\ldots$$ [2]

(ii) Consider the exponential distribution X with parameter x

$$P(X > t+s \mid X > s) = \frac{P(X > t+s, X > s)}{P(X > s)}$$

$$= \frac{P(X > t+s)}{P(X > s)} = \frac{\exp(-x(t+s))}{\exp(-xs)}$$

$$= \exp(-xt) = P(X > t)$$

Which is the memoryless property. [2]

(iii) (a) $P(\min(X,Y,Z) > t) = P(X > t, Y > t, Z > t)$

$$P(\min(X,Y,Z) > t) = \exp(-xt)\exp(-yt)\exp(-zt)$$

$$P(\min(X,Y,Z) > t) = \exp(-(x+y+z)t)$$

(b) Which is an exponential distribution with parameter $(x + y + z)$ [2]

(iv) Compound Poisson process [1]

(v) Motorcycles 60 * 2 * £1 = £120
Cars 60 * 5 * £2 = £600
Goods Vehicles 60 * 1.5 * £5 = £450

Total £1,170 [1]

(vi) Prob of n goods vehicles arriving

$$= \frac{\lambda^n . \exp(-\lambda)}{n!} \qquad \text{where } \lambda = 1.5$$

The probability of more than 2 arriving is

= 1 – Prob (zero) – Prob (one) – Prob (two)

$$= 1 - e^{-1.5}\left(\frac{1.5^0}{1} + \frac{1.5^1}{1} + \frac{1.5^2}{2}\right)$$

= 0.19115 [2]

(vii) The combinations which give rise to collect exactly £4 in tolls being collected are:

Motorcycles	*Cars*	*Goods Vehicles*
4	0	0
2	1	0
0	2	0

Probabilities of each event are:

Motorcycles	*Cars*	*Goods Vehicles*	*Combined probability*
$\frac{2^4 e^{-2}}{4!} = 0.09022$	$\frac{5^0 e^{-5}}{0!} = 0.006738$	$e^{-1.5} = 0.22313$	0.00014
$\frac{2^2 e^{-2}}{2!} = 0.27067$	$\frac{5^1 e^{-5}}{1!} = 0.03369$	$e^{-1.5} = 0.22313$	0.00203
$\frac{2^0 e^{-2}}{0!} = 0.13534$	$\frac{5^2 e^{-5}}{2!} = 0.08422$	$e^{-1.5} = 0.22313$	0.00254

Total probability 0.004714 [4]

[Total 14]

Overall, this was the least successfully answered question on the paper, with an average mark of between 4 and 5 out of 14. Very few candidates attempted part (ii) and attempts at part (iii) were very disappointing. In part (v) a common error was to use rates of 1/2, 1/5 and 1/1.5 per minute for motorcycles, cars and goods vehicles respectively, which gave an answer of £254. In part (vi) many candidates misread the question and incorrectly calculated the probability of <u>two or more</u> goods vehicles arriving. In part (vii) most candidates correctly identified the combinations of vehicles which could provide exactly £4 in tolls. A common error in the calculation was to omit the probabilities of observing zero vehicles of a particular type. This led to a final probability of 0.18357.

8 (i) **Right censoring**

Yes, of patients not experiencing the event of interest before 28 February either because they died, or because they had a second operation, or because they remained in the hospital until 28 February, each of which outcomes cut short observations in progress.

Type I censoring

Yes, of those patients remaining in hospital on 28 February, since this date was fixed in advance of the investigation.

Type II censoring
No, as the end of the investigation was determined by time, not by the number of patients who had left hospital.

Random censoring
Yes, of patients who died or who had a second operation, the times of which were not known in advance of the investigation and can be considered as random variables. [4]

(ii) Censoring is likely to be informative.

Those patients who died or who underwent a second operation were probably recovering less well than patients who left hospital.

Had they not died or undergone a second operation, they would probably have remained in hospital for longer than those patients who were not censored. [2]

(iii) We re-write the data as follows:

Patient	*Duration (days)*	*Experienced the event (1) or censored (0)*
1	28	0
2	2	0
3	14	1
4	31	1
5	14	1
6	15	1
7	1	0
8	36	0
9	7	0
10	24	1
11	14	1

The Kaplan-Meier estimate uses the table below:

t_j	N_j	d_j	c_j	d_j/N_j	$1-\frac{d_j}{N_j}$
0	11	0	3	0	0
14	8	3	0	0.375	0.625
15	5	1	0	0.2	0.8
24	4	1	1	0.25	0.75
31	2	1	1	0.5	0.5

The Kaplan-Meier estimate is then given by

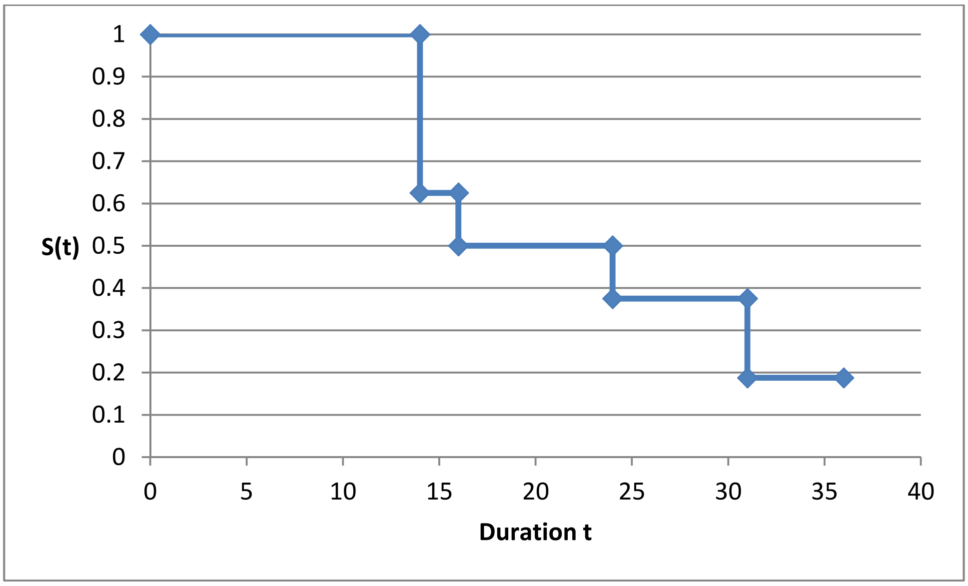
$$\hat{S}(t) = \Pi_{t_j < t}\left(1 - \frac{d_j}{N_j}\right).$$

This produces

t	$\hat{S}(t)$
$0 \le t < 14$	1.0000
$14 \le t < 15$	**0.6250**
$15 \le t < 24$	0.5000
$24 \le t < 31$	**0.3750**
$31 \le t < 36$	0.1875

[6]

(iv) See the sketch below.



[2]

(v) Deaths occur soon after the operation.

There is a high hazard of leaving the hospital after 14 days.

It may be that clinical protocols regard 14 days as the minimum period for which patients who have had this operation should remain in hospital, no matter how well they seem to be recovering.

The fact that censoring is informative is likely to bias the estimate.

The results may not be credible or may have a large variance because the sample size is very small.

The data only allow us to make estimates of "survival" up to a duration of 36 days. [2]

[Total 16]

Most candidates correctly identified and described the types of censoring present in this investigation, although most candidates also thought that the censoring was likely to be informative. Explanations of why this was the case were often vague and woolly. Part (iii) was reasonably well answered. Full credit was given in part (iv) for plots which were correct given the answer to part (iii). Answers to part (v) were disappointing, but good candidates noticed the tendency for patients to be discharged from hospital after 14 days, and for deaths to occur soon after surgery.

9 (i) To test for overall goodness of fit we use the χ^2 test.

The null hypothesis is that the graduated rates are the same as the true underlying rates applying to the new class of business.

The test statistic $\sum_x z_x^2 \approx \chi_m^2$ where m is the degrees of freedom.

The calculations are shown in the table below.

Age x	*Crude rates*	*Graduated rates*	*Exposed to risk*	*Observed deaths*	*Expected deaths*	z_x	z_x^2
70	0.0167	0.022661	1,200	20	27.193	–1.379	1.903
71	0.0209	0.024783	1,194	25	29.591	–0.844	0.712
72	0.0236	0.027204	973	23	26.469	–0.674	0.455
73	0.0324	0.029956	956	31	28.638	0.441	0.195
74	0.0362	0.033072	912	33	30.162	0.517	0.267
75	0.0402	0.036587	845	34	30.916	0.555	0.308
76	0.0561	0.040357	820	46	33.093	2.244	5.034
77	0.0623	0.044962	369	23	16.591	1.573	2.476
78	0.0552	0.049899	489	27	24.401	0.526	0.277
79	0.0640	0.055390	500	32	27.695	0.818	0.669

The observed test statistic is 12.295

The number of age groups is 10, but we lose some degrees of freedom because of the process of graduation

one for the parameter of the function linking the graduated rates to the standard table rates, and at least one more for the choice of standard table, so m = 7 or 8, say.

The critical value of the chi-squared distribution with 7 degrees of freedom at the 5% level is 14.07, and with 8 degrees of freedom is 15.51.

Since 12.295 < 14.07 (or 15.51)

we do not reject the null hypothesis at the 95% confidence level. [6]

(ii) There may be one or two large deviations at individual ages, the effect of which are insufficient to raise the chi-squared value above the critical level.

Small but consistent bias across the whole of the age range.

The graduation might be the wrong shape, in that the graduated rates might be higher than the crude rates in one part of the age range, and systematically lower in another part of the age range. This will lead to runs or clumps of deviations of the same sign.

The rates may not progress smoothly from age to age.

[Only three of these were required for full credit] [3]

(iii) **Large deviations**

For large deviations, use the Individual Standardised Deviations test.

The null hypothesis is the same as in part (i), that the graduated rates are the true underlying rates for the new class of business.

We would expect the individual deviations to be distributed Normal (0,1)

and therefore only 1 in 20 z_xs should have absolute magnitude greater than 1.96 (or none should be outside −3 to +3)
OR table showing split of deviations, actual versus expected as below

Range	*−∞,−2*	*−2,−1*	*−1,0*	*0,1*	*1,2*	*2,+∞*
Expected	0.2	1.4	3.4	3.4	1.4	0.2
Actual	0	1	2	5	1	1

Looking at the z_xs we see that the largest one is 2.24 and the next is 1.57.

This test is therefore inconclusive (1 deviation out of 10 ages is greater than 1.96).

Small but consistent bias

For small but consistent bias use the Signs test or the Cumulative Deviations test.

The null hypothesis is the same as in part (i), that the graduated rates are the true underlying rates for the new class of business.

EITHER SIGNS TEST

Under the null hypothesis, the number of positive signs amongst the z_x is distributed Binomial (10, ½)
We observe 7 positive signs.

The probability of observing 7 or more positive signs in 10 observations is 0.1719

OR

the probability of observing exactly 7 positive signs is 0.1172.

either of which implies that Pr[observing 6 or more] > 0.025 (a two-tailed test),

so we have no evidence to reject the null hypothesis

OR CUMULATIVE DEVIATIONS TEST

Under the null hypothesis

the test statistic $\dfrac{\sum_x (\text{Observed deaths} - \text{Expected deaths})}{\sqrt{\sum_x \text{Expected deaths}}} \sim \text{Normal}(0,1)$

So, using the results in the table in the solution to part (a) the value of the test statistic is

$$\frac{294 - 274.75}{\sqrt{274.75}} = 1.16$$

Since –1.96 < test statistic < +1.96

we have insufficient evidence to reject the null hypothesis.

Shape of graduation/runs or clumps

For the existence of runs or clumps of deviations of the same sign, we use the Grouping of Signs test or the Serial Correlations Test

The null hypothesis is the same as before, that the graduated rates are the true underlying rates for the new class of business.

EITHER GROUPING OF SIGNS TEST

G = Number of groups of positive deviations = 1

m = number of deviations = 10
n_1 = number of positive deviations = 7
n_2 = number of negative deviations = 3

THEN EITHER

We want k^* the largest k such that

$$\sum_{t=1}^{k} \frac{\binom{n_1-1}{t-1}\binom{n_2+1}{t}}{\binom{m}{n_1}} < 0.05$$

The test fails at the 5% level if $G \leq k^*$.

From the Gold Book $k^* = 1$, so we reject the null hypothesis.

OR

For $t = 1$

$$\binom{n_1-1}{t-1} = \binom{6}{0} = 1 \quad \text{and} \quad \binom{n_2+1}{t} = \binom{4}{1} = 4 \quad \text{and} \quad \binom{m}{n_1} = \binom{10}{7} = 120$$

So $\Pr[t = 1]$ if the null hypothesis is true is
4/120 = 0.0333, which is less than 5% so we reject the null hypothesis.

OR SERIAL CORRELATIONS TEST

The calculations are shown in the table below.

Age	z_x	z_{x+1}	$A = z_x - z'_x$	$B = z_{x+1} - z'_{x+1}$	AB	A^2	B^2
70	–1.379	–0.844	–1.708	–1.417	2.420	2.917	2.008
71	–0.844	–0.674	–1.173	–1.247	1.462	1.375	1.555
72	–0.674	0.441	–1.003	–0.132	0.132	1.006	0.017
73	0.441	0.517	0.112	–0.056	–0.006	0.013	0.003
74	0.517	0.555	0.188	–0.018	–0.003	0.035	0.000
75	0.555	2.244	0.226	1.671	0.378	0.051	2.793
76	2.244	1.573	1.915	1.000	1.915	3.668	1.000
77	1.573	0.526	1.244	–0.047	–0.058	1.548	0.002
78	0.526	0.818	0.194	0.245	0.048	0.039	0.060
z'	0.329	0.573		Sums	6.288	10.652	7.438

$6.288/\sqrt{(10.652*7.438)} = 0.706$

Test 0.706 ($\sqrt{10}$) = 2.232 against Normal (0,1), and, since

2.232 > +1.645, we have sufficient evidence to reject the null hypothesis (NB one-sided test)

Smoothness of the graduated rates

To check the smoothness of the graduated rates we do the Third Differences test.

Graduated rates	*First difference*	*Second difference*	*Third Difference*
0.022661	0.002122	0.000299	0.000032
0.024783	0.002421	0.000331	0.000033
0.027204	0.002752	0.000364	0.000034
0.029956	0.003116	0.000399	–0.000144
0.033072	0.003515	0.000255	0.000580
0.036587	0.003770	0.000835	–0.000503
0.040357	0.004605	0.000332	0.000222
0.044962	0.004937	0.000554	
0.049899	0.005491		
0.055390			

These should be small in magnitude compared with the rates themselves and progress regularly which does not seem to be the case here.

So we conclude that the graduated rates are not sufficiently smooth. [6]

(iv) Although the overall fit of the graduated rates to the crude rates is acceptable,

EITHER

The result of the grouping of signs/serial correlation test suggests that the graduated rates are the wrong shape: too high at younger ages and too low at older ages.

OR

The results of the individual standardised deviation test indicate that there might be an outlier at age 76

The graduation should be carried out again with either a different standard table or a different link functions

It seems a multiplicative function such as $\mathring{\mu}_x = a\mu_x^s$ might be better than just adding a constant to the μ_x^s s.

IF SMOOTHNESS TEST NOT DONE

The graduated rates should be sufficiently smooth to use for financial calculations.

OR, IF SMOOTHNESS TEST DONE

The graduated rates do not seem to be especially smooth.

Because the shape of the graduated rates seems incorrect, the company would be unwise to use these rates for financial calculations. [2]

[Total 17]

In part (i) the null hypothesis was often vaguely expressed. Statements that the graduated rates are "a good fit", "consistent with", "similar to" or "representative of" the underlying rates were not given full credit. In part (iii) it was acceptable if candidates did not use whole numbers of actual deaths but obtained the actual deaths as the result of multiplying the (rounded) crude rates by the exposed to risk. In part (iii) the null hypothesis should be stated for each test, but candidates could obtain credit by a statement that the null hypotheses for each test are the same as than in part (i). To obtain credit, the comments in part (iv) had to reflect the tests actually carried out by the candidate. Many candidates scored highly on parts (i) and (ii). Performance on part (iii) was less convincing. In part (iv) few candidates made comments beyond noting the immediate implications of the test results (i.e. that although the overall fit is satisfactory the graduation failed the Grouping of Signs test or that there was "clumping of signs").

END OF EXAMINERS' REPORT