INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2013 examinations

Subject CT4 – Models
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

December 2013

General comments on Subject CT4

Subject CT4 comprises five main sections: (1) a study of the properties of models in general, and their uses for actuaries, including advantages and disadvantages (and a comparison of alternative models of the same processes); (2) stochastic processes, especially Markov chains and Markov jump processes; (3) models of a random variable measuring future lifetime; (4) the calculation of exposed to risk and the application of the principle of correspondence; (5) the reasons why mortality (or other decremental) rates are graduated, and a range of statistical tests used both to compare a set of rates with a previous experience and to test the adherence of a graduated set of rates to the original data. Throughout the subject the emphasis is on estimation and the practical application of models. Theory is kept to the minimum required in order usefully to apply the models to real problems.

Different numerical answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2013 paper

The general performance was very similar to that in April 2013, and better than that in September 2012. Well-prepared candidates scored highly across the whole paper, with an above average proportion of candidates scoring 70 per cent or more. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to include these areas in their revision.

1

(i) All our models and analyses are based on the assumption that we can observe groups of identical lives (or at least, lives whose mortality characteristics are the same).

In practice, this is never possible.

However, we can at least subdivide our data according to characteristics known, from experience, to have a significant effect on mortality.

This ought to reduce the heterogeneity of each class so formed.

(ii) The number of lives in each subdivision may become small. This will lead to estimates of mortality that are unreliable, with large standard errors.

OR

Information about the factors which affect mortality may be unavailable because it was not asked on the insurance proposal form, or population census

OR

Information about the factors which affect mortality may be unreliable because respondents gave inaccurate or false answers to questions.

(iii) Sex
Age
Type of policy (which often reflects the reason for insuring)
Smoker/non-smoker status
Level of underwriting
Duration in force
Sales channel
Policy size
Occupation of policyholder
Known impairments
Postcode/geographical location
Marital status

Answers to part (i) of this question were disappointing, with few candidates relating the need for homogeneity to the models we use. Parts (ii) and (iii) were generally well answered. In part (ii) the instruction was to describe a single limitation, so no credit was given for second or subsequent limitations. In part (iii) credit was given in some cases for wording different from that indicated, such as "state of health", or for certain other factors which are known to affect mortality, and about which information is asked, for example, in population censuses. However, genetic factors were not given credit.

2

(i) Need to rearrange data as tally chart of next states:

Previous state	*Number where next state is:* U	C	D
U	1	11	111
C	11	111	11
D	11	111	1

So the transition probabilities are estimated as:

From/To	*U*	*C*	*D*
U	1/6	1/3	1/2
C	2/7	3/7	2/7
D	1/3	1/2	1/6

(ii) The possible sequences with at least 2 wins for United are:

UUU, UUC, UUD, DUU, CUU, UDU, UCU

The probabilities if the last match was won by City are:

UUU = 2/7*1/6*1/6 = 1/126
UUC = 2/7*1/6*1/3 = 1/63
UUD = 2/7*1/6*1/2 = 1/42
DUU = 2/7*1/3*1/6 = 1/63
CUU = 3/7*2/7*1/6 = 1/49
UDU = 2/7*1/2*1/3 = 1/21
UCU = 2/7*1/3*2/7 = 4/147

OR (quicker)

UUX = 2/7*1/6 = 1/21
DUU = 2/7*1/3*1/6 = 1/63
CUU = 3/7*2/7*1/6 = 1/49
UDU = 2/7*1/2*1/3 = 1/21
UCU = 2/7*1/3*2/7= 4/147

where X refers to any result

Total = 140/882 = 10/63 = 0.15873

Answers to this question were generally disappointing. In both parts (i) and (ii) the question said "estimate" so some explanation of where the answer is coming from was required for full credit (e.g. in part (i) a statement that n_{ij}/n_i is needed, or a suitable diagram were acceptable). A common error was to use 8 as the denominator for the C row. A more serious error was to use 19 as the denominator for all the transition probabilities. Many candidates *did not take account of the fact that City had won the last match in the string given and thus only used pairs, rather than triplets, of probabilities.*

3

(i) A Poisson process is a counting process in continuous time $\{N_t, t \geq 0\}$, where N_t records the number of occurrences of a type of event within the time interval from 0 to t.

Events occur singly and may occur at any time;

the probability that an event occurs during the short time interval from time t to time $t+h$ is approximately equal to λh for small h, where the parameter λ is the rate of the Poisson process.

OR

A Poisson process is an integer valued process in continuous time $\{N_t, t \geq 0\}$, where

$$\Pr[N_{t+h} - N_t = 1 \mid F_t] = \lambda h + o(h)$$
$$\Pr[N_{t+h} - N_t = 0 \mid F_t] = 1 - \lambda h + o(h)$$
$$\Pr[N_{t+h} - N_t \neq 0,1 \mid F_t] = o(h)$$

and $o(h)$ is such that $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

OR

A Poisson process with rate λ is a continuous-time integer-valued process $N_t, t \geq 0$, with the following properties:

$N_0 = 0$
N_t has independent increments
N_t has Poisson distributed stationary increments

$$P\left[N_t - N_s = n\right] = \frac{\left[\lambda\left(t-s\right)\right]^n e^{-\lambda(t-s)}}{n!}, \qquad s < t,\ n = 0,1,\ldots$$

(ii) The probability that no bus arrives in the first 60 minutes is $e^{-60/15} = 0.0183$.

By the memoryless property / by independence of increments / because the holding times are exponential.

(iii) The expected time between buses is 15 minutes.

By independence of increments / memoryless property this is the time Mr Bean can expect to wait for the second bus.

(iv) The rate at which buses arrive per 10 minute period is 10/15.

Therefore the probability of no buses arriving between 1.10 and 1.20 p.m. is $e^{-10/15} = 0.5134$.

The probability of one bus arriving is $e^{-10/15}(10/15) = 0.3423$.

The probability of two or more buses arriving is therefore

$1 - 0.5134 - 0.3423 = 0.1443$.

Answers to this question were disappointing. Most candidates managed to score reasonably well on part (i). In parts (ii) and (iii) some explanations of the answers were required. In part (iv) several candidates calculated the probability of exactly two buses arriving. Candidates who used an incorrect rate in part (ii) could score full credit for part (iv) if they correctly calculated the probability of two or more buses arriving given the incorrect rate.

4

(i) The objectives of the modelling exercise.

The validity of the model for the purpose to which it is to be put.

The validity of the data to be used.

The validity of assumptions used.

The possible errors associated with the model or parameters used not being a perfect representation of the real world situation being modelled.

The impact of correlations between the random variables that "drive" the model.

The extent of correlations between the various results produced from the model.

The current relevance of models written and used in the past.

The credibility of the data input.

The credibility of the results output.

The dangers of spurious accuracy.

The cost of buying or constructing, and of running the model.

The ease of use and availability of suitable staff to use it.

Risk of model being used incorrectly or with wrong inputs.

The ease with which the model and its results can be communicated.

Compliance with the relevant regulations.

(ii) The objectives are to determine the number of rooms the council needs.

But we have no information about "down time" between occupants, or requirement for seasonal variation or any built-in surplus, and so on.

So with these data alone the requirements cannot be fulfilled.

Local mortality may be different from national experience.

Care home residents may experience significantly different mortality to the national population (the council may have data on deaths to care home residents which could be used to adjust the national mortality table).

The data are readily available and should be reliable.

However they may not be suitable for projecting more than a couple of years into the future because for example:

- the distribution of the local population may be skewed, e.g. there may be a huge number of 55 to 59 year olds compared to 60 to 65 year olds;
- age of entry into care homes is likely to change as medical advances help people stay healthier longer;
- the proportion of people going into council homes versus private homes may change as economic conditions change;
- the national mortality table may have no mortality improvements built in.
- social habits may change e.g. families may start living more as a unit so adult children/grandchildren may be available to care for the elderly at home for longer, especially if the ethnic mix of the city changes;
- the size/age mix of the city may change.

The average age at entry into a care home needs to be converted to a distribution by age. This may be subjective.

There might be different types of room for different levels of care needed. In this case facilities may be inadequate to meet needs even if there are sufficient rooms in total.

The data used for the model are taken from different sources so should not be unduly correlated.

We are not told of any models written in the past. If these existed, it would be useful to compare its past projections with what has happened in reality.

The resultant "number of rooms" occupied at any one time will need to be adjusted for example more elderly may decide to enter homes at the start of winter when it becomes harder/more expensive to stay warm at home, or when epidemics of influenza happen.

The model is relatively simple to explain.

The local mix of public and private care homes available locally may affect the proportion of elderly who go into council homes.

Most candidates made a good attempt at part (i). Answers to part (ii) were more variable, and tended to focus more on the problems with the mortality data rather than issues connected with the supply and provision of care homes. In both parts of this question, not all the points listed were required for full marks.

5

(i) $$P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0.2 & 0 & 0 & 0.8 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

where the levels are ordered 0%, 25%, 50%, 60%.

(ii) (a) The chain is irreducible as it is clear that any state can eventually be reached from any other state.

(b) The process is aperiodic because, for example, the process can loop round in the 0% or 60% states giving no set return period to any state.

(iii) Stationary distribution π satisfies $\pi = \pi P$

$$0.2\pi_0 + 0.2\pi_{25} + 0.2\pi_{50} = \pi_0 \qquad (1)$$
$$0.8\pi_0 + 0.2\pi_{60} = \pi_{25} \qquad (2)$$
$$0.8\pi_{25} = \pi_{50} \qquad (3)$$
$$0.8\pi_{50} + 0.8\pi_{60} = \pi_{60} \qquad (4)$$

Also $\pi_0 + \pi_{25} + \pi_{50} + \pi_{60} = 1 \qquad (5)$

Working in terms of π_{60}

$$\pi_{50} = 0.25\pi_{60}$$

$$\pi_{25} = \frac{5}{16}\pi_{60}$$

$$\pi_0 = \frac{9}{64}\pi_{60}$$

Hence $\dfrac{(64+16+20+9)}{64}\pi_{60} = 1$

So the e stationary distribution is $\dfrac{1}{109}\begin{bmatrix} 9 \\ 20 \\ 16 \\ 64 \end{bmatrix}$

and the proportion of drivers at each level is

0%	9/109 = 0.08257
25%	20/109 = 0.18349
50%	16/109 = 0.14679
60%	64/109 = 0.58716.

(iv) The 60% discount level becomes an absorbing state and so it is no longer irreducible.

However it is still aperiodic because you cannot get out of the absorbing state 60% and the other states still have no period.

The process would now be stationary when all drivers are in the absorbing 60% discount level.
OR
The new stationary distribution is [0,0,0,1] because the 60% state is now absorbing.

This question was well answered, with many candidates scoring close to full marks. In part (iii) the correct numerical probabilities scored full marks, provided that it was clear to which level each probability applied. In part (iv) some candidates made vague statements that the probability of being in the 60% state would increase. While this is true, it was not given full credit, as the key point is that the stationary distribution has <u>everyone</u> in the 60% state.

6

(i) Censoring is the mechanism which prevents us from knowing when an individual entered the investigation or the exact date of death.

(ii) Right Censoring. The trial is cut short after four weeks when some patients had still not recovered.
OR
The trial is cut short when some patients left the trial before their symptoms disappeared.

Type I Censoring. Censoring times are known in advance for all those patients still not recovered at the end of the trial.
Random Censoring. The time at which patients left the trial before their symptoms disappeared is a random variable.

Non-Informative Censoring. There is no reason to believe that those who left the trial had more or less chance of being cured by the cream than those who remained.

(iii) Rearranging the data:

Day	0	2	6	7	10	10	13	14
People in trial	100	100	97	95	94	93	92	89
No of exits	0	3	2	1	1	1	3	2
Reason for exit		Left	cured	cured	cured	left	left	cured

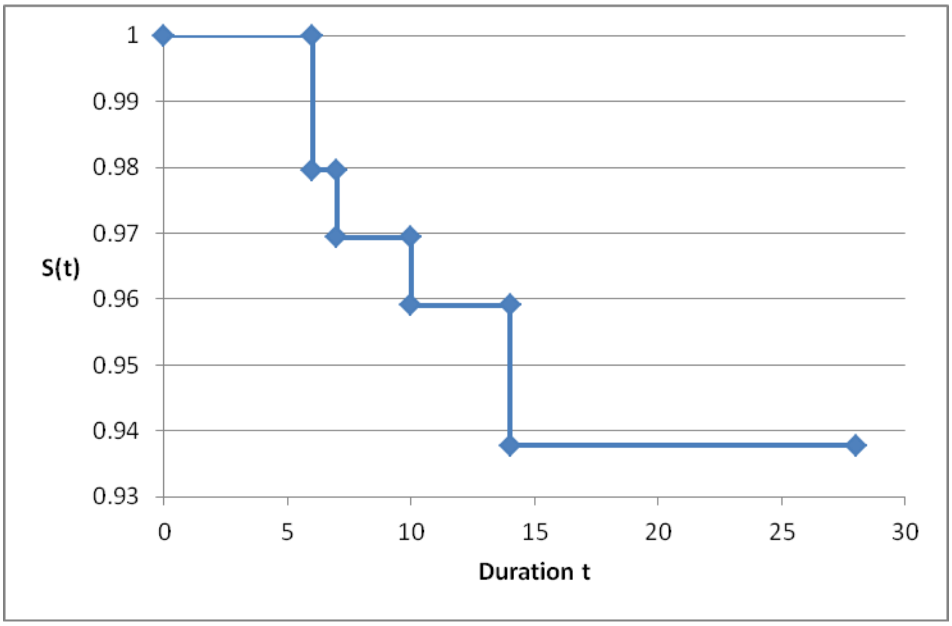
The Nelson-Aalen estimate is $\Lambda_t = \sum_{x_j \le x} \frac{d_j}{n_j}$

t_j	n_j	d_j	c_j	d_j/n_j	Λ_t
0	100	0	0		
2	100	0	3		
6	97	2	0	2/97	.020619
7	95	1	0	1/95	.031145
10	94	1	1	1/94	.041783
13	92	0	3		
14	89	2	0	2/89	.064255

Since $S(t) = \exp\left(-\Lambda_t\right)$ we have

t	$S(t)$
$0 \le t < 6$	1
$6 \le t < 7$	0.97959
$7 \le t < 10$	0.96934
$10 \le t < 14$	0.95908
$14 \le t < 28$	0.93777

(iv)



(v) The survival probability at t = 14 is 0.93777, so there is approximately a 94% chance of still having symptoms after two weeks.

Many candidates answered this question well. In part (ii) Informative Censoring was acceptable if a sensible argument was made for it, for example those who left may be allergic to the cream and therefore less likely to be cured by it than those who remain. In part (ii) some candidates did not answer both parts of the question (that is, both describing the censoring and stating to whom it applied). In part (iii) and part (iv) it was expected that candidates would recognise that no information about what happened after 28 days could be gained from the data. In part (v) the answer given should be consistent with the S(t) estimated in part (iii).

7

(i) The Gompertz model is simple to understand and to apply, having only two parameters.

It also fits human mortality at older ages well (e.g. 30–85 years).

(ii) $\log_e \mu_x = \alpha + \beta_0 x + \beta_1 U + \beta_2 I$

So $\mu_x = \exp(\alpha + \beta_0 x + \beta_1 U + \beta_2 I) = \exp(\beta_0 x)\exp(\alpha + \beta_1 U + \beta_2 I)$

This is equal to Bc^x where $B = \exp(\alpha + \beta_1 U + \beta_2 I)$ and $c = \exp \beta_0$, hence Gompertz.

$\mu_x = \exp(\alpha + \beta_0 x + \beta_1 U + \beta_2 I) = \exp(\alpha + \beta_0 x)\exp(\beta_1 U + \beta_2 I)$

EITHER

Hence the force of mortality factorises into a term $\exp(\alpha+\beta_0 x)$ depending on age x but not the covariates, and a term $\exp(\beta_1 U+\beta_2 I)$ depending on the covariates but not x, SO proportional hazards.

OR

Consider any two individuals, i and j, with values of the covariates U_i and I_i, and U_j and I_j respectively. Then the hazards for individuals i and j at age x are

$$\mu_{x,i} = \exp(\alpha+\beta_0 x)\exp(\beta_1 U_i + \beta_2 I_i)$$

and

$$\mu_{x,j} = \exp(\alpha+\beta_0 x)\exp(\beta_1 U_j + \beta_2 I_j)$$

The ratio between the hazards is thus

$$\frac{\mu_{x,i}}{\mu_{x,j}} = \frac{\exp(\alpha+\beta_0 x)\exp(\beta_1 U_i + \beta_2 I_i)}{\exp(\alpha+\beta_0 x)\exp(\beta_1 U_j + \beta_2 I_j)} = \frac{\exp(\beta_1 U_i + \beta_2 I_i)}{\exp(\beta_1 U_j + \beta_2 I_j)},$$

which does not depend on x, hence proportional hazards.

(iii) $\log_e \mu_{40} = -9 + 0.09(40) + 0.3 - 0.0001(20{,}000) = -7.1$

so $\mu_{40} = 0.000825$.

(iv) $\mu_x = \exp(\alpha + \beta_0 x + \beta_1 U + \beta_2 I)$

Let the income of the urban resident be I_U and that of the rural resident be I_R.

$$\begin{aligned}
&\exp(\alpha)\exp(\beta_0 x)\exp(\beta_1 + \beta_2 I_U) = \exp(\alpha)\exp(\beta_0 x)\exp(\beta_2 I_R)\\
&\exp(\beta_1 + \beta_2 I_U) = \exp(\beta_2 I_R)\\
&\exp(0.3 - 0.0001 I_U) = \exp(-0.0001 I_R)\\
&0.3 - 0.0001 I_U = -0.0001 I_R\\
&3{,}000 = I_U - I_R
\end{aligned}$$

So the difference is \$3,000.

Survival probability is

$$\exp\left\{-\left[\frac{e^{0.09s}}{0.09}\right]_{40}^{50} e^{-9}e^{0.3}e^{-0.0001(20,000)}\right\} = \exp\left[-\frac{0.00002254(90.017-36.598)}{0.09}\right]$$

$$= \exp(-0.01338) = 0.9867.$$

(vi) Since $S_x(t) = \exp\left[-\int_x^{x+t} \mu_s ds\right]$,

then if the rural resident is *a* years older than the urban resident we have

$$\exp\left[-\int_x^{x+t} e^{0.09s}e^{\alpha}e^{\beta_1}e^{\beta_2 I} ds\right] = \exp\left[-\int_{x+a}^{x+a+t} e^{0.09s}e^{\alpha}e^{\beta_2 I} ds\right]$$

Therefore

$$e^{\alpha}e^{\beta_2 I}\int_{x+a}^{x+a+t} e^{0.09s}ds = e^{\alpha}e^{\beta_2 I}\int_x^{x+t} e^{\beta_1}e^{0.09s}ds$$

$$\int_{x+a}^{x+a+t} e^{0.09s}ds = \int_x^{x+t} e^{\beta_1}e^{0.09s}ds$$

$$\left[\frac{e^{0.09s}}{0.09}\right]_{x+a}^{x+a+t} = \left[\frac{e^{\beta_1}e^{0.09s}}{0.09}\right]_x^{x+t}$$

$$e^{0.09(x+a+t)} - e^{0.09(x+a)} = e^{\beta_1}(e^{0.09(x+t)} - e^{0.09x})$$

$$e^{0.09x}e^{0.09a}(e^{0.09t}-1) = e^{\beta_1}e^{0.09x}(e^{0.09t}-1)$$

$$e^{0.09a} = e^{\beta_1}$$

$$a = \beta_1 / 0.09 = 0.3/0.09 = 3.33$$

So the rural dweller is aged 40 + 3.33 = 43.33 years.

In part (i) very few candidates made the point that the Gompertz model is simple and convenient to use. Part (ii) was very poorly answered. When demonstrating that the model was a proportional hazards (PH) model, many candidates simply factorised the expression as $\mu_{x,i} = exp(\alpha)exp(\beta_0 x + \beta_1 U_i + \beta_2 I_i)$ *and said that therefore exp(α) was the baseline hazard. This is incorrect because the second term includes both duration and the covariates. It was* *acceptable in part (ii) only to break up the equation once as long as the argument was developed further for both the Gompertz and the PH cases. A common error in part (v) was to assume that the hazard was constant at its value at age 40 years. This produced a survival probability of 0.99178. In part (vi) the derivation shown in above was not required for full credit. Candidates who spotted that, if 40+a is the age of the rural dweller in years, then* $e^{0.09a} = e^{\beta_1}$*, scored full credit.*

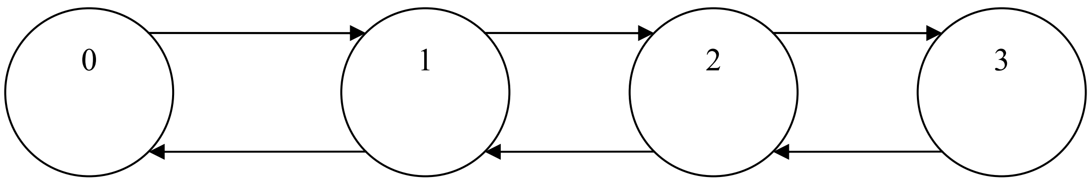
Since the question was missing a comma after "were $\alpha = -9.0$*"a small number of candidates interpreted the parameters differently i.e.* $\alpha = 0.09$*,* $\beta_0 = -0.01$*,* $\beta_1 = 0.3$ *and* $\beta_3 = -0.0001$*. This interpretation was given full credit, if followed through correctly.*

In part (vi) the approach using ${}_t p_x = \left[\exp\left(\frac{-B}{\log c}\right)\right]^{c^x(c^t-1)}$ *was acceptable.*

8

(i) The state space is {0,1,2,3} where the number indicates the number of available spaces.

(ii)



(iii) $\begin{pmatrix} -3B & 3B & 0 & 0 \\ A & -A-2B & 2B & 0 \\ 0 & A & -A-B & B \\ 0 & 0 & A & -A \end{pmatrix}$

where the order of the rows/columns is {0, 1, 2, 3}.

(iv) $\frac{d}{dt}P_{00}(t) = -3BP_{00}(t)$ (as probability of returning to state 0 not of interest)

OR

$$P_{00}(t) = \exp\left[-\int_0^t 3B\,dt\right]$$

$$P_{00}(t) = \exp(-3Bt)$$

$$P_{00}(2) = \exp(-6B)$$

(v) If a Markov jump process X_t is examined only at the times of transition, the resulting process is called the jump chain associated with X_t.
OR
A jump chain is each distinct state visited in the order visited where the time set is the times when states are moved between.

(vi)
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ A/_{A+2B} & 0 & 2B/_{A+2B} & 0 \\ 0 & A/_{A+B} & 0 & B/_{A+B} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where the order of the rows/columns is $\{0, 1, 2, 3\}$.

(vii) This is $\dfrac{A}{A+B}.\dfrac{A}{A+2B}$

(viii) Consider the paths by which the car park can become full before it becomes empty

Required probability = $P_{21}P_{10} + P_{21}P_{12}P_{21}P_{10} + P_{21}P_{12}P_{21}P_{12}P_{21}P_{10} + ...$

$$= \frac{A}{A+B}.\frac{A}{A+2B}\left[1+\frac{A}{A+B}.\frac{2B}{A+2B}+\frac{A}{A+B}.\frac{2B}{A+2B}\frac{A}{A+B}.\frac{2B}{A+2B}+....\right]$$

$$=\frac{A}{A+B}.\frac{A}{A+2B}/\left[1-\frac{A}{A+B}.\frac{2B}{A+2B}\right]$$

OR

This can be done by defining a function of the probability full before empty from the current state, say D_x

Then D_0 = 1and D_3 =0

and $D_1 = D_0.\dfrac{A}{A+2B} + D_2.\dfrac{2B}{A+2B}$

$$D_2 = D_1.\frac{A}{A+B} + D_3.\frac{B}{A+B}$$

Solving these gives

$$D_2 = \frac{A}{A+B}.\frac{A}{A+2B}/\left[1-\frac{A}{A+B}.\frac{2B}{A+2B}\right]$$

(ix) A time inhomogeneous model may be more appropriate.

Residents may come and go at particular times, for example if they drive to work.

They are unlikely to be moving their car as regularly in the middle of the night

Independent arrivals questionable because a family might have two cars arriving/leaving at the same time OR people might arrive and wait until a space becomes available thus leading to a queue

The Markov assumption may not be valid because neighbours may know from at experience when cars are moved and time their arrival accordingly.

The model assumes those parking cars are competent drivers, and do not park so as to take up 2 spaces.

The problem can be worked in terms of the number of occupied spaces. This was not given full credit for part (i) as the question said "model the number of spaces free, but could gain full credit for the other parts. In part (ii) it was not necessary to mark the probabilities on the diagram. A common error was to omit the 2s and 3s in the matrix in part (iii). Part (v) was not as well answered as might have been expected, with many candidates writing vague descriptions which did not make it clear that they understood what a jump chain is.

Overall, this question was poorly answered by many candidates. A large proportion of candidates did not attempt parts (iii)–(viii).

9

(i) Graduation by parametric formula.

Advantage: If a small number of parameters is used the resultant rates are automatically smooth;
OR sometimes when comparing several investigations it is useful to fit the same parametric formula to all of them;
OR the approach is well suited to the production of standard tables from large amounts of data.

Disadvantage: It can be difficult to find a suitable curve which fits the experience at all ages;
OR care is needed when extrapolating from ages where there is most data.

Graduation by reference to a standard table.

Advantage: Provided a simple function is chosen the resultant rates are automatically smooth;
OR it can be useful to fit relatively small data sets when a suitable standard table exists;

OR the standard table can be very good at deciding the shape of the graduation at extreme ages where data are sparse.

Disadvantage: It can be difficult to find a suitable standard table for the data;
OR it is not suitable for the preparation of standard tables.

Graphical graduation.

Advantage: It can be used for small data where no suitable standard table exists;
OR can allow for known features of the experience for example the accident hump.

Disadvantage: It is hard to achieve accuracy;
OR it takes a skilled practitioner;
OR it is very difficult to achieve adequate smoothness.

(ii) To test for overall goodness of fit we use the χ^2 test.

The null hypothesis is that the graduated rates are the underlying rates of the experience.

The test statistic $\sum_x z_x^2 \approx \chi_m^2$ where m is the degrees of freedom.

Age	*Exposed to risk*	*Observed deaths*	*Table Rates*	*Expected deaths*	z_x	z_x^2
30	36,254	26	0.000590	21.38986	0.9968	0.9936
31	37,259	20	0.000602	22.42992	–0.5131	0.2632
32	28,057	23	0.000617	17.31117	1.3673	1.8695
33	31,944	23	0.000636	20.31638	0.5954	0.3545
34	30,005	26	0.000660	19.80330	1.3925	1.9390
35	28,389	12	0.000689	19.56002	–1.7094	2.9220
36	36,124	31	0.000724	26.15378	0.9476	0.8980
37	28,152	22	0.000765	21.53628	0.0999	0.0100
38	24,001	25	0.000813	19.51281	1.2422	1.5430
39	30,448	31	0.000870	26.48976	0.8763	0.7679
				Total	5.2956	11.5608

The observed test statistic is 11.56

The number of age groups is 10, but we lose some degrees of freedom for the choice of the standard table and one degree of freedom for each parameter in the link function. So $m < 10$.

The critical value of the chi-squared distribution with 9 degrees of freedom at the 5% level is 16.92 (or with 8 d.f. is 15.51, or with 7 is 14.07).

Since 11.56 < 16.92 (or 15.51, or 14.07), we do not reject the null hypothesis at 95% level of significance.

(iii) Any two from:

Individual Standardised Deviations Test

Under the null hypothesis that the graduated rates are the true rates underlying the observed data

we should expect individual deviations to be distributed Normal (0,1).

Only 1 in 20 of the z_xs should lie above 1.96 in absolute value;
OR
none should lie above 3 in absolute value;
OR
table showing split of deviations, actual versus expected as below.

Range	$-\infty, -2$	$-2, -1$	$-1, 0$	$0, 1$	$1, 2$	$2, +\infty$
Expected	0	1.4	3.4	3.4	1.4	0
Actual	0	1	1	5	3	0

The largest deviation we have here is −1.71,

which is within the range −1.96 to 1.96,

therefore we have no reason to reject the null hypothesis at the 95% level of significance.

Signs Test

Under the null hypothesis that the graduated rates are the true rates underlying the observed data

the number of positive signs amongst the z_x is distributed Binomial (10, ½).

We observe 8 positive signs.

The probability of observing 8 or more positive signs in 10 observations is 0.0547 OR the probability of observing exactly 8 positive signs is 0.044.

This implies that Pr[observing 8 or more] > 0.025 (a two-tailed test),

so we have insufficient evidence to reject the null hypothesis at the 95% level.

Cumulative Deviations Test

Under the null hypothesis that the graduated rates are the true rates underlying the observed data, the test statistic

$$\frac{\sum_x (\text{Observed deaths} - \text{Expected deaths})}{\sqrt{\sum_x \text{Expected deaths}}} \sim \text{Normal}(0,1).$$

So, calculating as follows:

Age x	*Observed deaths*	*Expected deaths*	*Observed minus expected deaths*
30	26	21.38986	4.6101
31	20	22.42992	–2.4299
32	23	17.31117	5.6888
33	23	20.31638	2.6836
34	26	19.80330	6.1967
35	12	19.56002	–7.5600
36	31	26.15378	4.8462
37	22	21.53628	0.4637
38	25	19.51281	5.4872
39	31	26.48976	4.5102
Totals		214.5033	24.4967

The value of the test statistic is $\frac{24.50}{\sqrt{214.50}} = 1.6726$.

Since $-1.96 <$ test statistic $< +1.96$,

we have insufficient evidence to reject the null hypothesis at the 95% level.

Grouping of Signs Test

Under the null hypothesis that the graduated rates are the true rates underlying the observed data

G = Number of groups of positive deviations = 3
m = number of deviations = 10
n_1 = number of positive deviations = 8
n_2 = number of negative deviations = 2

THEN EITHER

We want k^* the largest k such that $\sum_{t=1}^{k} \frac{\binom{n_1-1}{t-1}\binom{n_2+1}{t}}{\binom{m}{n_1}} < 0.05$

The test fails at the 5% level if $G \le k^*$.

From the Gold Book there is no entry for k^*.

So we have insufficient evidence to reject the null hypothesis at the 95% level.

OR

For $t = 3$

$$\binom{n_1-1}{t-1} = \binom{7}{2} = 21 \quad \text{and} \quad \binom{n_2+1}{t} = \binom{3}{3} = 1 \quad \text{and} \quad \binom{m}{n_1} = \binom{10}{8} = 45.$$

So Pr[$t = 3$] if the null hypothesis is true is 21/45 = 0.467, which is greater than 5% .

We have insufficient evidence reject the null hypothesis at the 95% level.

(iv) The chi-squared test suggests that the graduated rates adhere satisfactorily overall to the crude rates which gave rise to the observed deaths.

The Signs Test suggests that small but consistent bias is not a problem.

The shape of the graduated rates is not significantly different from the crude rates, as evidenced by the result of the Grouping of Signs Test.

The shape of the graduated rates is not significantly different from the crude rates, as evidenced by the result of the Cumulative Deviations Test.

There are no individual ages with suspiciously large deviations between the crude rates and the graduated rates.

Therefore it would seem reasonable for the company to use the graduated rates to price life insurance policies for this particular block of businesses.

In part (iii) there were 3 marks available for each test. ANY two of the tests described above were allowed, even if they are testing for effectively the same thing, but the test for smoothness was not given credit as the graduation had been carried out with reference to a standard table. Credit was also given for the serial correlations test, and one or two candidates attempted this (none especially successfully). Candidates who carried out more *than two tests, were credited with the marks for their two highest-scoring. In part (iv) many candidates made vague statements that the graduation "passed all the tests". This was given only limited credit. For full credit, details of the aspects of the graduation that each test focuses on were required. Also in part (iv) some candidates (despite finding no small bias in the data) decided that the presence of eight out of ten positive deviations merited further investigation, or that since observed deaths were generally higher than expected deaths, life products should be priced cautiously. Credit was given for these sensible comments.*

END OF EXAMINERS' REPORT