EXAMINATION

1 October 2010 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 12 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Following a review of the results of a stochastic model run, an actuary requests that a parameter is changed. The change is not expected to alter the results significantly, but results on the final basis are required in order to complete a report. Unfortunately the actuarial student who produced the original model run is away on study leave, and so the revised run is assigned to a different student.

When the revised results are produced, they are significantly different from the original results.

Discuss possible reasons why the results are different. [3]

2 Compare the characteristics of deterministic and stochastic models, by considering the relationship between inputs and outputs. [4]

3 The government of a small island state intends to set up a model to analyse the mortality of the island's population over the past 50 years.

Describe the process that would be followed to carry out the analysis. [6]

4 A large pension scheme conducts an investigation into the mortality of its younger male pensioners. The crude mortality rates are graduated using a standard table by subtracting a constant from the rates given in the table.

A trainee has been asked to test the goodness-of-fit of the proposed graduation using a chi-squared test. The trainee's workings are reproduced below:

"Test H_0: good fit against H_1: bad fit.

Age	*Actual Deaths*	*Expected Deaths*	*(Actual Deaths – Expected Deaths)^2 /Actual Deaths*
60	8	8.23	0.00661
61	8	10.01	0.50501
62	10	10.52	0.02704
63	12	14.80	0.65333
64	14	14.21	0.00315
65	13	17.37	1.46899
Test Statistic			2.66413

Age range is 65–60 = 5 years so 5 degrees of freedom.

Two-tailed test so take 2 * 2.66413 = 5.32826 and compare against tabulated value of chi-square distribution with 5 degrees of freedom at 2.5% level, which is 12.833.

So we accept the null hypothesis."

Identify the errors in the trainee's workings, without performing any detailed calculations. [6]

5 (i) Write down a formula for ${}_tq_x$ ($0 \le t \le 1$) under each of the following assumptions:

(a) uniform distribution of deaths
(b) constant force of mortality
(c) the Balducci assumption

[2]

(ii) Calculate ${}_{0.5}p_{60}$ to six decimal places under each assumption given $q_{60} = 0.05$. [2]

(iii) Comment on the relative magnitude of your answers to part (ii). [2]

[Total 6]

6 (i) Outline the circumstances under which graphical graduation of crude mortality rates might be useful. [1]

(ii) List the steps involved in graphical graduation. [5]

[Total 6]

7 Two neighbouring small countries have for many years taken annual censuses of their populations on 1 January in which each inhabitant must give his or her age. Country *A* uses an "age last birthday" definition of age, whereas Country *B* uses an "age nearest birthday" definition. Each country has also operated a system in which deaths are recorded on an "age nearest birthday at date of death" basis.

On 30 June 2009 Country *A* invaded Country *B* and the two countries became one state. The new government wishes to estimate a single set of age-specific death rates, μ_x, for the new unified state using the census data taken in the years before the invasion.

Derive a formula which the new government may use to estimate μ_x in terms of the recorded number of deaths in each country, and the population of each country recorded as being aged x in the censuses. State any assumptions you make. [8]

8 Rocky Bay is a small seaside town in the north of Europe. In a leaflet advertising the town, the tourist office has claimed that "in August, Rocky Bay has a Mediterranean climate". An actuarial student spent August 2009 on holiday in Rocky Bay with his family, and became sceptical of this claim. When he returned home, he thought it might be interesting to examine the claim by applying some of the methods he had learned while studying for the Core Technical subjects. For each of the 31 days in August 2009 he collected data recorded by various meteorological offices on the maximum temperature in Rocky Bay and the mean of the maximum temperatures reported on the same day at a range of places in the Mediterranean region.

The data are shown below, where, for each of the days in August, "+" means that Rocky Bay had the higher maximum temperature and "–" means that the Mediterranean average was higher.

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
- - - - - - - - - -  -  -  +  +  +  +  -  -  -  -

21 22 23 24 25 26 27 28 29 30 31
-  -  -  -  -  -  -  -  +  +  +
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(i) Carry out a statistical test to examine the tourist office's claim. [5]

(ii) Suggest reasons why the test might not be an appropriate way to examine the tourist office's claim. [2]

[Total 7]

9 A researcher is reviewing a study published in a medical journal into survival after a certain major operation. The journal only gives the following summary information:

- the study followed 16 patients from the point of surgery
- the patients were studied until the earliest of five years after the operation, the end of the study or the withdrawal of the patient from the study
- the Nelson-Aalen estimate, $S(t)$, of the survival function was as follows:

Duration since operation t (years)	*S(t)*
$0 \le t < 1$	1
$1 \le t < 3$	0.9355
$3 \le t < 4$	0.7122
$4 \le t < 5$	0.6285

(i) Describe the types of censoring which are present in the study. [2]

(ii) Calculate the number of deaths which occurred, classified by duration since the operation. [6]

(iii) Calculate the number of patients who were censored. [1]

[Total 9]

10 A study is undertaken of marriage patterns for women in a country where bigamy is not permitted. A sample of women is interviewed and asked about the start and end dates of all their marriages and where the marriages had ended, whether this was due to death or divorce (all other reasons can be ignored). The investigators are interested in estimating the rate of first marriage for all women and the rate of re-marriage among widows.

(i) Draw a diagram illustrating a multiple-state model which the investigators could use to make their estimates, using the four states: "Never married", "Married", "Widowed" and "Divorced". [1]

(ii) Derive from first principles the Kolmogorov differential equation for first marriages. [5]

(iii) Write down the likelihood of the data in terms of the waiting times in each state, the numbers of transitions of each type, and the transition intensities, assuming the transition intensities are constant. [3]

(iv) Derive the maximum likelihood estimator of the rate of first marriage. [2]

[Total 11]

11 At a certain airport, taxis for the city centre depart from a single terminus. The taxis are all of the same make and model, and each can seat four passengers (not including the driver). The terminus is arranged so that empty taxis queue in a single line, and passengers must join the front taxi in the line. As soon as it is full, each taxi departs. A strict environmental law forbids any taxi from departing unless it is full. Taxis are so numerous that there is always at least one taxi waiting in line.

Customers arrive at the terminus according to a Poisson process with a rate β per minute.

(i) Explain how that the number of passengers waiting in the front taxi can be modelled as a Markov jump process. [2]

(ii) Write down, for this process:

(a) the generator matrix
(b) Kolmogorov's forward equations in component form

[4]

(iii) Calculate the expected time a passenger arriving at the terminus will have to wait until his or her taxi departs. [4]

The four-passenger taxis were highly polluting, and the government instituted a "scrappage" scheme whereby taxi drivers were given a subsidy to replace their old four-passenger taxis with new "greener" models. Two such models were on the market, one of which had a capacity of three passengers and the other of which had a capacity of five passengers (again, not including the driver in each case). Half the taxis were replaced with three-passenger models, and half with five-passenger models.

Assume that, after the replacement, three-passenger and five-passenger models arrive randomly at the terminus.

(iv) Write down the transition matrix of the Markov jump chain describing the number of passengers in the front taxi after the vehicle replacement. [2]

(v) Calculate the expected waiting time for a passenger arriving at the terminus after the vehicle scrappage scheme and compare this with your answer to part (iii). [3]

[Total 15]

12 A pet shop has four glass tanks in which snakes for sale are held. The shop can stock at most four snakes at any one time because:

- if more than one snake were held in the same tank, the snakes would attempt to eat each other and
- having snakes loose in the shop would not be popular with the neighbours

The number of snakes sold by the shop each day is a random variable with the following distribution:

Number of Snakes Potentially Sold in Day (if stock is sufficient)	*Probability*
None	0.4
One	0.4
Two	0.2

If the shop has no snakes in stock at the end of a day, the owner contacts his snake supplier to order four more snakes. The snakes are delivered the following morning before the shop opens. The snake supplier makes a charge of C for the delivery.

(i) Write down the transition matrix for the number of snakes in stock when the shop opens in a morning, given the number in stock when the shop opened the previous day. [2]

(ii) Calculate the stationary distribution for the number of snakes in stock when the shop opens, using your transition matrix in part (i). [4]

(iii) Calculate the expected long term average number of restocking orders placed by the shop owner per trading day. [2]

If a customer arrives intending to purchase a snake, and there is none in stock, the sale is lost to a rival pet shop.

(iv) Calculate the expected long term number of sales lost per trading day. [2]

The owner is unhappy about losing these sales as there is a profit on each sale of P. He therefore considers changing his restocking approach to place an order before he has run out of snakes. The charge for the delivery remains at C irrespective of how many snakes are delivered.

(v) Evaluate the expected number of restocking orders, and number of lost sales per trading day, if the owner decides to restock if there are fewer than two snakes remaining in stock at the end of the day. [5]

(vi) Explain why restocking when two or more snakes remain in stock cannot optimise the shop's profits. [2]

The pet shop owner wishes to maximise the profit he makes on snakes.

(vii) Derive a condition in terms of C and P under which the owner should change from only restocking where there are no snakes in stock, to restocking when there are fewer than two snakes in stock. [2]

[Total 19]

END OF PAPER