INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

26 September 2014 (am)

Subject CT4 – Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all nine questions, beginning your answer to each question on a new page.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 For each of the following processes:

- counting process
- simple random walk
- compound Poisson process
- Markov jump process

(i) State whether the state space is discrete, continuous or can be either. [2]

(ii) State whether the time set is discrete, continuous, or can be either. [2]
[Total 4]

2 (i) List eight factors which should be considered when assessing whether a model is suitable for a particular application. [4]

(ii) State, giving reasons, a factor which would be particularly important in each of the following applications:

- Calculating the pension contribution for a medium sized pension scheme.

- Helping a friend construct a business case to secure a loan from a bank for his new ice-cream van venture.

- Working out how much it will cost to buy each member of your team their favourite cake on your birthday in six months' time.

[3]
[Total 7]

3 (i) Explain the census approximation for calculating the exposed to risk between any two census dates. [2]

A mortality investigation bureau has collected the following information on number of policies in-force each year from different companies.

Age and year	*Year*	*Company A*	*Company B*	*Company C*
Age 54	2011	3,400	1,250	5,780
	2012	3,350	1,450	5,500
	2013	3,000	1,500	6,010
Age 55	2011	3,250	1,190	6,000
	2012	3,390	1,300	5,960
	2013	3,100	1,440	6,030
Age 56	2011	3,270	1,150	5,950
	2012	3,020	1,300	5,980
	2013	2,950	1,500	5,990

- Company A has provided in-force policy data as at the beginning of each calendar year using age nearest birthday.

- Company B has provided in-force policy data as at the financial year closing date (which was 31 March in each year) using age last birthday.

- Company C has provided in-force policy data as at the end of each calendar year using age next birthday.

(ii) Calculate the contribution to central exposed to risk for lives aged 55 last birthday for the calendar year 2012 for each of the companies. [6]

[Total 8]

4 (i) Define the force of mortality, μ_{x+t} of a random variable T denoting length of life. [1]

The mortality of a certain species of animal has been studied. It is known that at ages under five years the force of mortality, μ, is constant.

(ii) Write down an expression, in terms of μ, for the probability that an animal will survive from birth to exact age five years. [1]

Mortality of these animals at ages over five years exact is incompletely understood.

However it is known that the probability that an animal aged exactly five years will survive until exact age 10 years is twice the probability that an animal aged exactly five years will survive until exact age 20 years.

Assume that the force of mortality, λ, is constant at ages over five years exact.

(iii) Calculate λ. [3]

(iv) Calculate the expectation of life at birth for these animals if $\lambda = \mu$. [1]

(v) Derive an expression, in terms only of μ, for the expectation of life at birth for these animals if $\lambda \neq \mu$, [4]

[Total 10]

5 A sports league has two divisions {1,2} with Division 1 being the higher. Each season the bottom team in Division 1 is relegated to Division 2, and the top team in Division 2 is promoted to Division 1.

Analysis of the movements of teams between divisions indicates that the probabilities of finishing top or bottom of a division differs if a team has just been promoted or relegated, compared with the probabilities in subsequent seasons.

The probabilities are as follows:

Finishing position	*If promoted previous season*	*If relegated previous season*	*If neither promoted nor relegated previous season*
Top	0.1	0.25	0.15
Bottom	0.3	0.25	0.15
Other	0.6	0.5	0.7

(i) Write down the minimum number of states required to model this as a Markov chain. [1]

(ii) Draw a transition graph for the Markov chain. [3]

(iii) Write down the transition matrix for the Markov chain. [2]

(iv) Explain whether the Markov chain is:

(a) irreducible.
(b) aperiodic. [2]

Team A has just been promoted to Division 1.

(v) Calculate the minimum number of seasons before there is at least a 60% probability of Team A having been relegated to Division 2. [3]
[Total 11]

6 A motor insurance company offers annually renewable policies. To encourage policyholders to renew each year it offers a No Claims Discount system which reduces the premiums for those people who claim less often. There are four levels of premium:

0: no discount
1: 15% discount
2: 25% discount
3: 40% discount

A policyholder who does not make a claim in the year, moves up one level of discount the following year (or stays at the maximum level).

A policyholder who makes one or more claims in a year moves down one level of discount if they did not claim in the previous year (or remains at the lowest level) but if they made at least one claim in the previous year they move down two levels of discount (subject to not going below the lowest level).

(i) (a) Explain how many states are required to model this as a Markov chain.

(b) Draw the transition graph of the process. [3]

The probability, p, of making at least one claim in any year is constant and independent of whether a claim was made in the previous year.

(ii) Calculate the proportion of policyholders who are at the 25% discount level in the long run given that the proportion at the 40% level is nine times that at the 15% level. [6]

(iii) (a) Explain how the state space of the process would change if the probability of making a claim in any one year depended upon whether a claim was made in the previous year.

(b) Write down the transition matrix for this new process. [4]

[Total 13]

7 (i) Define a Poisson process. [2]

(ii) Prove the memoryless property of the exponential distribution. [2]

Suppose there are three independent exponential distributions:

X with parameter x
Y with parameter y
Z with parameter z

(iii) (a) Demonstrate that min(X,Y,Z) is also an exponential distribution.

(b) Give the parameter of this exponential distribution. [2]

The arrivals of different types of vehicles at a toll bridge are assumed to follow Poisson processes whereby:

Type of Vehicle	*Rate*
Motorcycle	2 per minute
Car	5 per minute
Goods vehicle	1.5 per minute

The toll for a motorcycle is £1, for a car £2 and for a goods vehicle £5.

(iv) State the name of the stochastic process that describes the total value of tolls collected. [1]

(v) Calculate the expected value of tolls collected per hour. [1]

On the advice of a structural engineer, no more than two goods vehicles are allowed across the bridge in any given minute. If more than two goods vehicles arrive then some goods vehicles have to wait to go across.

(vi) Calculate the probability that more than two goods vehicles arrive in any given minute. [2]

(vii) Calculate the probability that exactly £4 in tolls is collected in a given minute. [4]

[Total 14]

8 An investigation was undertaken into the length of post-operative stay in hospital after a particular type of surgery. All patients undergoing this surgery between 1 January and 31 January 2013 were observed until either they left the hospital, died, or underwent a second operation. The event of interest was leaving the hospital. Patients who died or underwent a second operation during the period of investigation were treated as censored at the date of death or second operation respectively. The investigation ended on 28 February 2013, and patients who were still in the hospital at that time were treated as censored.

(i) State, with reasons, whether the following types of censoring are present in this investigation:

- right
- Type I
- Type II
- random

[4]

(ii) Comment on whether censoring in this investigation is likely to be informative. [2]

The following data relate to 11 patients included in the investigation.

Date of operation	*Date observation ended*	*Reason that observation ended*
2 January	30 January	Second operation
5 January	7 January	Died
10 January	24 January	Left hospital
12 January	12 February	Left hospital
15 January	29 January	Left hospital
20 January	4 February	Left hospital
20 January	21 January	Died
23 January	28 February	End of investigation
24 January	31 January	Second operation
27 January	20 February	Left hospital
31 January	14 February	Left hospital

(iii) Calculate the Kaplan-Meier estimate of the survivor function for remaining in the hospital. [6]

(iv) Sketch the Kaplan-Meier estimate of the survivor function, labelling the axes. [2]

(v) Comment on the results of the investigation. [2]

[Total 16]

9 A life insurance company is developing a new class of annuity business. It has conducted a study of mortality among lives it believes represent this new business. It wishes to graduate the data so that they are suitable for use in financial calculations. It decides to use a standard table as a basis for graduation and the function:

$$\overset{\circ}{\mu}_x = \mu_x^s + 0.01$$

where $\overset{\circ}{\mu}_x$ are the graduated rates and μ_x^s are the rates from the standard table.

The table below gives some results from the graduation.

Age x	*Crude rates* $\hat{\mu}_x$	*Graduated rates* $\overset{\circ}{\mu}_x$	*Exposed to risk*
70	0.0167	0.022661	1,200
71	0.0209	0.024783	1,194
72	0.0236	0.027204	973
73	0.0324	0.029956	956
74	0.0362	0.033072	912
75	0.0402	0.036587	845
76	0.0561	0.040357	820
77	0.0623	0.044962	369
78	0.0552	0.049899	489
79	0.0640	0.055390	500

(i) Carry out an overall test of the goodness-of-fit of this graduation to the crude rates. [6]

(ii) List three defects of a graduation which the test you conducted in (i) may not detect. [3]

(iii) Perform, for each of two of the defects listed in (ii), an additional test which can detect the defect. [6]

(iv) Comment on the results of the tests carried out in parts (i) and (iii). [2]
[Total 17]

END OF PAPER