INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT4 — Models
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

July 2011

Question 1

We can calculate the maximum likelihood estimate (MLE) of the transition intensities directly using the two-state model, whereas the Binomial model requires additional assumptions.

The variance of the Binomial estimate is greater than that of the estimate from the two-state model (though the difference is tiny unless the transition intensities are large).

The MLE in the two-state model is consistent and unbiased, whereas the Binomial estimate is only consistent and unbiased if lives are observed for exactly one year, which is rarely the case.

The two-state model is easily extended to encompass increments and additional decrements, whereas the Binomial model is not.

The two-state model uses the exact times of the transitions, whereas the Binomial model only uses the number of transitions.

This question was poorly answered by many candidates, despite being straightforward bookwork. Many candidates commented that the two-state model and the Binomial model make different assumptions about the shape of the force of mortality within the year of age. This was only be given credit if candidates also explained why the multiple state model's assumption is BETTER than the Binomial model's assumption (which it might be, for example, at younger ages).

Full marks could be obtained for giving three reasons. It was not necessary to give all the points listed above in order to obtain full marks.

Question 2

(a) A Markov chain with a finite state space has at least one stationary probability distribution.

(b) An irreducible Markov chain with a finite state space has a unique stationary probability distribution.

(c) A Markov chain with a finite state space which is irreducible, and which is also aperiodic converges to a unique stationary probability distribution.

Many candidates scored full marks on this question. The question asked candidates to "distinguish". Therefore for full credit it is important that candidates did, indeed, understand and make the relevant distinction.

Question 3

Individual variables may behave differently, for example a model over 50 years may be more sensitive to differences in the input values of certain variables than one over the short term.

A variable which has an ignorable effect in the short term may have a non-ignorable effect over 50 years.

Over the short term, it may be reasonable to assume the values of some variables to be constant or to vary linearly, whereas this would not be reasonable over 50 years. For example, growth which is exponential may appear linear if studied over a short time frame.

The interaction between variables in the short-term may be different from that over the long-term.

Higher order relationships between variables may be ignored for simplicity if modelling over a short time frame.

The time units used in the model might be shorter for a model projecting over a short time frame, so that the total number of time units used in each model is roughly the same.

Over 50 years, regulatory changes and other "shock" events are more likely to occur, and the model design may need to consider the circumstances in which the results or conclusions may be materially impacted (e.g. in the short term the tax basis may be known, but in the long run it is likely to change).

The marks on this question were the lowest on any question. The question was a "higher skills" question and so required candidates to think about the context. Little credit was given to candidates who uncritically reproduced sections of the Core Reading. In particular, the question is about model DESIGN, so the points made should relate to the design of the model.

Question 4

(i)

$$\begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0.1 & 0.2 & 0.7 \end{pmatrix}$$

(ii) If the probability distribution in the first week is Π, and the transition matrix is M, then the probability distribution at the end of the third week is

$$\Pi M^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0.1 & 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0.1 & 0.2 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.56 & 0.24 & 0.15 & 0.05 \\ 0.35 & 0.27 & 0.21 & 0.17 \\ 0.17 & 0.21 & 0.27 & 0.35 \\ 0.05 & 0.15 & 0.24 & 0.56 \end{pmatrix}$$

so that there is a probability of

35% that a child will be graded Poor',
27% that a child will be graded Satisfactory,
21% that a child will be graded Good and
17% that a child will be graded Excellent..

There were two common errors on this question. The first was to assume that if a child could not move up or down two levels, he or she would not move at all. The phrase in the question "[s]ubject to a maximum level of Excellent and a minimum level of Poor" was intended to indicate that children could not move beyond these limits in either direction, but would move as far as they could. Thus a child at level "Good", who had a 20% chance of moving up one level and a 10% chance of moving up two levels, would have a 30% chance of moving to level Excellent, as the 10% who would have moved up two levels will only be able to move up one level. The second error was to use ΠM^3 in part (ii). Candidates who made the first error were penalised in part (i) but could gain full credit for part (ii) if they followed through correctly.

Question 5

(i) We believe that mortality varies smoothly with age (and evidence from large experiences supports this belief).

Therefore the crude estimate of mortality at any age carries information about mortality at adjacent ages.

By smoothing the experience, <u>we can make use of data at adjacent ages</u> to improve the estimates at each age.

This reduces sampling (or random) errors.

The mortality experience may be used in financial calculations.

Irregularities, jumps and anomalies in financial quantities (such as premiums for life insurance contracts) are hard to justify to customers.

(ii) (a) Female members of a medium-sized pension scheme.

With reference to a standard table, because there are many extant tables dealing with female pensioners.

(b) Male population of a large industrial country.

By parametric formula, because the experience is large.
OR
because the graduated rates may form a new standard table for the country.

(c) Population of a particular species of reptile in the zoological collections of the southern hemisphere.

Graphical, because no suitable standard table is likely to exist and the experience is small.

This question was well answered. In part (i)(c) BOTH elements of the reason were needed for credit (i.e. that no suitable table is likely to exist AND the experience is small).

Question 6

(i) The probability that an insect will survive for 10 days, ${}_{10}p_0$, is given by the formula

$$ {}_{10}p_0 = \exp\left(-\int_0^{10} \mu_x dx \right). $$

Since the force of mortality is constant up to age 30 days at a value of 0.05,

$$ {}_{10}p_0 = \exp\left(-\int_0^{10} 0.05dx \right) = \exp\left(-\left[0.05x\right]_0^{10}\right) = \exp\left(-0.5\right) = 0.6065\,. $$

(ii) The probability that an insect 10 days old will survive for a further 30 days (that is to exact age 40 days) is given by

$$ {}_{30}p_{10} = \exp\left(-\int_{10}^{40} \mu_x dx \right). $$

Since ${}_{30}p_{10} = {}_{20}\,p_{10} \cdot {}_{10}p_{30}$, this is equal to

$$ \exp\left(-\int_{10}^{30} 0.05dx \right) \exp\left(-\int_0^{10} 0.05\exp(0.01x)dx \right) $$

$$ = \exp\left(-\left[0.05x\right]_{10}^{30}\right) \exp\left(-\left[\frac{0.05}{0.01}\exp(0.01x) \right]_0^{10} \right) $$

$$ = e^{-(1.5-0.5)} e^{-(5\exp(0.1)-5\exp(0))} $$

$$ = e^{-1} e^{-0.5258} = 0.3679 \times 0.5911 = 0.2174. $$

(iii) If the required age is $30+a$, then we have

$$ {}_{30+a}p_0 = {}_{30}p_0 \cdot {}_a p_{30} = 0.1\,. $$

Now

$$ {}_{30}p_0 = \exp\left[-\int_0^{30} 0.05dx \right] = \exp(-1.5) = 0.2231\,. $$

So $\quad {}_a p_{30} = \dfrac{0.1}{0.2231} = 0.4483\,.$

Using the result from part (ii), we have

$${}_a p_{30} = \exp\left(-\left[\frac{0.05}{0.01}e^{0.01x}\right]_0^a\right) = \exp\left(-\left[\frac{0.05}{0.01}e^{0.01a} - \frac{0.05}{0.01}\right]\right) = \exp(5 - 5e^{0.01a})$$

Therefore

$$e^{5(1-\exp(0.01a))} = 0.4483,$$

whence

$$\log_e 0.4483 = 5(1 - e^{0.01a}),$$

so that

$$1 - e^{0.01a} = -0.1605$$
$$e^{0.01a} = 1.1605$$
$$0.01a = 0.1488$$
$$a = 14.88$$

Therefore the required age is 14.88+30 = 44.88 days.

Most candidates answered part (i) of this question correctly. Part (ii) was less well answered, and only a minority of candidates managed to obtain the correct answer to part (iii). A common error was to use the limits 40 and 30 when integrating 0.05exp(0.01x).

Question 7

(i) It is a stochastic process in discrete or continuous time.

The state space is all the natural numbers {0, 1, 2, ... }

The value of the process $X(t)$ is a non-decreasing (OR an increasing) function of time t
OR
the value of the process goes up one at a time.

(ii) (a)

Process	*State space*	*Time set*
Simple random walk	Discrete	Discrete
Compound Poisson process	Either	Continuous
Markov Chain	Discrete	Discrete

(b)

Process	*Application*
Simple random walk	The number of customers in the shop each time the door is opened
Compound Poisson process	The weight of almonds remaining in stock at any time in the day. OR value of goods sold at any time during the day
Markov chain	The number of customers owning loyalty cards at the end of each week.

In part (i), it was not sufficient just to say "discrete state space". The fact the state space is all natural numbers should be indicated for credit. In part (ii)(B), some candidates only gave one example IN TOTAL, whereas the question asked for an example FOR EACH PROCESS. In part (ii)(b) other examples were given credit. The criterion used to award credit were whether the example COULD be modelled using the relevant process and how USEFUL to the shopkeeper such a model might be!

Question 8

(i) EITHER

The central exposed to risk at age x, E_x^c, is the waiting time in a multiple-state or Poisson model.

The initial exposed to risk is equal to the central exposed to risk plus the time elapsing between the date of death and the end of the rate interval for those who are observed to die during the rate interval.

OR

If the age at entry of life i is $x + a_i$, and the age at exit is $x+b_i$ for lives which do not die, and $x+t_i$ for lives who die, then the central exposed to risk is equal to

$\sum_i [(x_i + b_i) - (x_i - a_i)] = \sum_i (b_i - a_i)$ for lives who do not die, and

$\sum_i [(x_i + t_i) - (x_i - a_i)] = \sum_i (t_i - a_i)$ for lives who die.

The initial exposed to risk is given by the central exposed to risk plus a quantity equal to $\sum_i (1 - t_i)$ for the lives who die.

If the rate interval is the year of age between exact ages x and x+1, and if deaths are approximately uniformly distributed across the year of age, the initial exposed to risk is approximately equal to $E_x^c + 0.5d_x$, where d_x is the number of deaths between exact ages x and x+1.

The central exposed to risk estimates μ_x whereas the initial exposed-to risk estimates q_x.

(ii) The maximum likelihood estimate of the force of mortality in the two-state model is deaths divided by the central exposed to risk.

The central exposed to risk is calculated as shown in the table below.

Life	*Entry into observation*	*Exit from observation*	*Months exposed to risk*
1	1 August 2008	1 August 2009	12
2	1 September 2008	1 September 2009	12
3	1 December 2008	1 February 2009	2
4	1 January 2009	1 January 2010	12
5	1 February 2009	1 February 2010	12
6	1 March 2009	1 December 2009	9
7	1 June 2009	1 June 2010	12
8	1 July 2009	1 July 2010	12
9	1 September 2009	1 September 2010	12
10	1 November 2009	1 December 2009	1

The total number of months exposed to risk is therefore

12 + 12 + 2 + 12 + 12 + 9 + 12 + 12 + 12 + 1 = 96

which is 8 years

There were 3 deaths.

Therefore the maximum likelihood estimate of the force of mortality is $\frac{3}{8} = 0.375$.

(iii) If the force of mortality is μ_0, then

$$q_0 = 1 - \exp(-\mu_0) = 1 - \exp(-0.375) = 0.3127.$$

EITHER ALTERNATIVE 1

(iv) The initial exposed to risk, E_0 is approximately equal to $E_0^c + 0.5d_0$, where E_0^c is the central exposed to risk and d_0 is the number of deaths.

Therefore we have

$$q_0{}^* = \frac{d_0}{E_0^c + 0.5d_0} = \frac{3}{8+0.5(3)} = \frac{3}{9.5} = 0.3158\,.$$

(v) $q_0{}^*$ is calculated assuming a uniform distribution of deaths over the year of age between birth and exact age 1 year, whereas q_0 assumes a constant force of mortality between exact ages 0 and 1.

These assumptions are different, implying a different distribution of deaths over the first year of life.

OR ALTERNATIVE 2

(iv) As the only way of leaving observation is through death, the initial exposed to risk is 10 and $q_0{}^* = \frac{3}{10} = 0.3$.

(v) $q_0{}^*$ is calculated using the exact initial exposed to risk, making no assumptions about the shape of the force of mortality during the interval,

OR

In the calculation of $q_0{}^*$ lives could die at any time during the year of age, so they are treated as being exposed to risk for the entire year, whereas q_0 assumes a constant force of mortality between exact ages 0 and 1, which implies an assumption about the distribution of deaths over this interval.

In part (i) full credit could be obtained for rather less than is written in the solution above. Credit can be given for any clear algebraic expressions in terms of the entry age $x+a_i$, the age at death, $x+t_i$ and the age at exit if the life did not die, $x+b_i$, which made clear the difference between the central and initial exposeds to risk.

In part (v) the wording did not have to be precise. The Examiners were looking for some understanding of the idea that different assumptions are made about the shape of the force of mortality over the rate interval.

Question 9

(i) A Markov jump process is a <u>continuous time</u>, <u>discrete state</u> process

THEN EITHER

in which, given the present state of the process, additional knowledge of the past is irrelevant for the calculation of the probability distribution of future values of the process.

OR

$$P[X_t \in A \mid X_{s_1} = x_1, X_{s_2} = x_2, ..., X_{s_n} = x_n] = P[X_t \in A \mid X_s = x]$$

for all times $s_1 < s_2 < ... < s_n < s < t$, all states $x_1, x_2, ..., x_n, x$ in S and all subsets A of S.

(ii) Using the Markov property, and conditioning on the state occupied at age $x + t$, we have

$$_{t+dt}p_x^{13} = {}_t p_x^{11} \, {}_{dt}p_{x+t}^{13} + {}_t p_x^{12} \, {}_{dt}p_{x+t}^{23} + {}_t p_x^{13} \, {}_{dt}p_{x+t}^{33}$$

Assume that ${}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij} dt + o(dt), \;\; i \neq j$

where $\lim\limits_{dt \to 0^+} \dfrac{o(dt)}{dt} = 0$.

Substituting for the ${}_{dt}p_{x+t}^{ij}$ in the equation above, and noting that ${}_{dt}p_{x+t}^{33} = 1$ since return from the state "Dead" is impossible, produces

$$_{t+dt}p_x^{13} = {}_t p_x^{11} \mu_{x+t}^{13} dt + {}_t p_x^{12} \mu_{x+t}^{23} dt + {}_t p_x^{13} + o(dt)$$

so that

$$_{t+dt}p_x^{13} - {}_t p_x^{13} = {}_t p_x^{11} \mu_{x+t}^{13} dt + {}_t p_x^{12} \mu_{x+t}^{23} dt + o(dt)$$

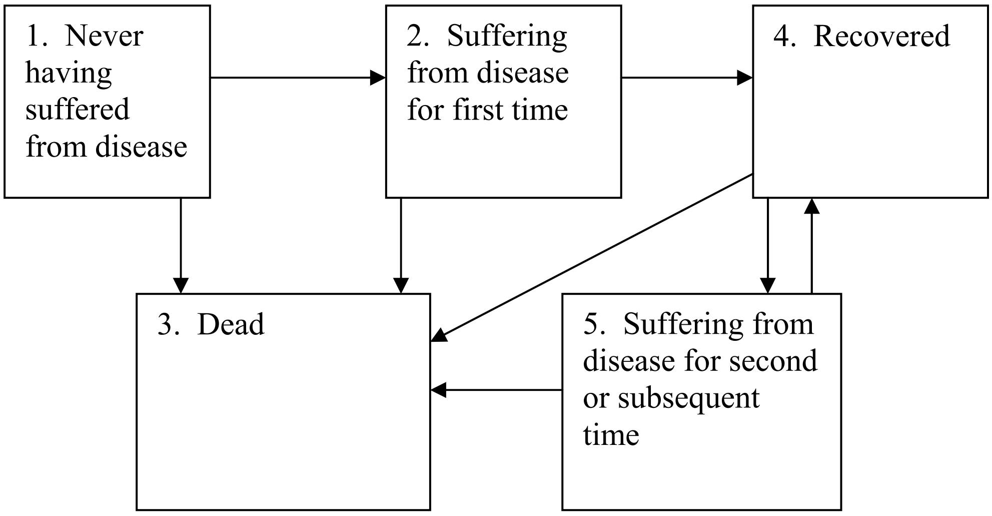
and, taking limits, we have

$$\lim_{dt \to 0^+} \frac{_{t+dt}p_x^{13} - {}_t p_x^{13}}{dt} = {}_t p_x^{11} \mu_{x+t}^{13} + {}_t p_x^{12} \mu_{x+t}^{23}$$

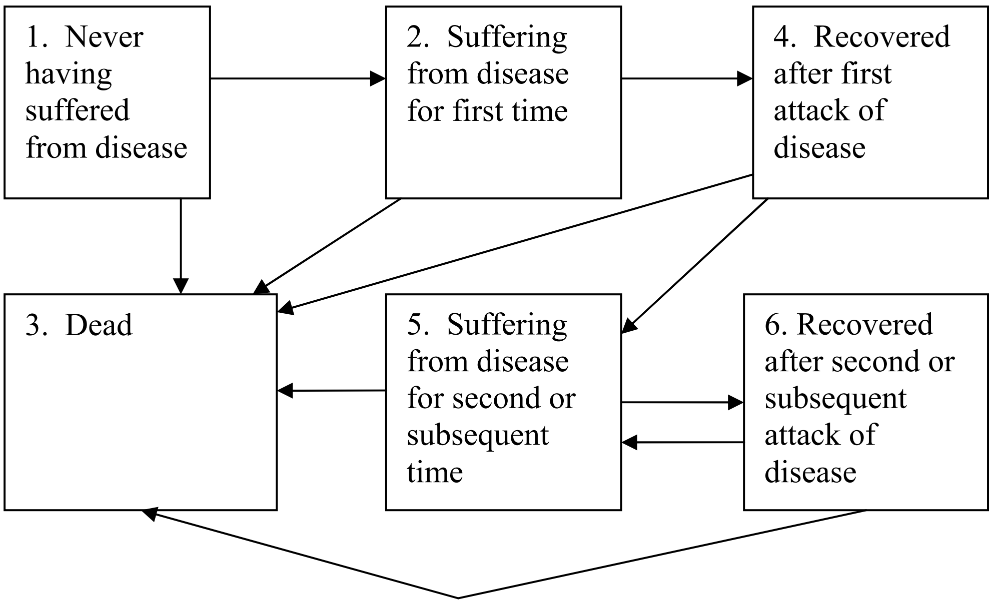
So

$$\frac{d}{dt} {}_t p_x^{13} = {}_t p_x^{11} \mu_{x+t}^{13} + {}_t p_x^{12} \mu_{x+t}^{23}.$$

(iii) EITHER



OR



This question was fairly well answered, though many candidates omitted the initial point in part (ii), that we need the Markov property to be able to condition on the state occupied at x+t. A common error in part (iii) was a four-state solution with the states "1 – Never having suffered from the disease", "2 – Suffering from the disease", "3 – Dead", and "4 – Recovered". This is not correct, as the probability of moving from the state "Suffering from the disease" to the state "Dead" depends on whether the person is suffering from the disease for the first time or the second or subsequent time.

In part (iii) both alternatives were accepted. The second allows for the possibility that the effect of contracting the disease for the second time in raising the risk of death persists even after the patient recovers from the second or subsequent attack.

Question 10

(i) Right censoring is present
for those still alive and in hospital at the end of August
OR
for those who left hospital while still alive

Left censoring is not present

The censoring is likely to be informative, since those leaving hospital are likely to be in much better health than those who remain. (The idea of going home to die when you have had a lung transplant is a little tenuous.)

(ii) The durations and outcomes are shown in the table below.

Patient	*Died/Censored*	*Duration*
A	Died	2
G	Died	5
J	Died	5
B	Censored	29
E	Died	32
M	Censored	38
H	Censored	56
K	Died	56
N	Censored	62
L	Censored	66
I	Censored	70
F	Censored	80
D	Censored	84
C	Censored	87

EITHER ALTERNATIVE 1

Assuming that at duration 56 the death occurred before the life was censored, the Kaplan-Meier estimate is as follows:

t_j	n_j	d_j	c_j	$\lambda_j = \frac{d_j}{n_j}$
0	14	0	0	0
2	14	1	0	1/14
5	13	2	1	2/13
32	10	1	1	1/10
56	8	1	7	1/8
+½	+½	+½	+½	

The Kaplan-Meier estimate at duration t is given by the product of $1-\frac{d_j}{n_j}$ over durations up to and including t. Thus the Kaplan-Meier estimate of the survival function is

t	$\hat{S}(t)$	
$0 \le t < 2$	1.0000	
$2 \le t < 5$	0.9286	OR 13/14
$5 \le t < 32$	0.7857	OR 11/14
$32 \le t < 56$	0.7071	OR 99/140
$56 \le t < 92$	0.6188	OR 99/160

OR ALTERNATIVE 2

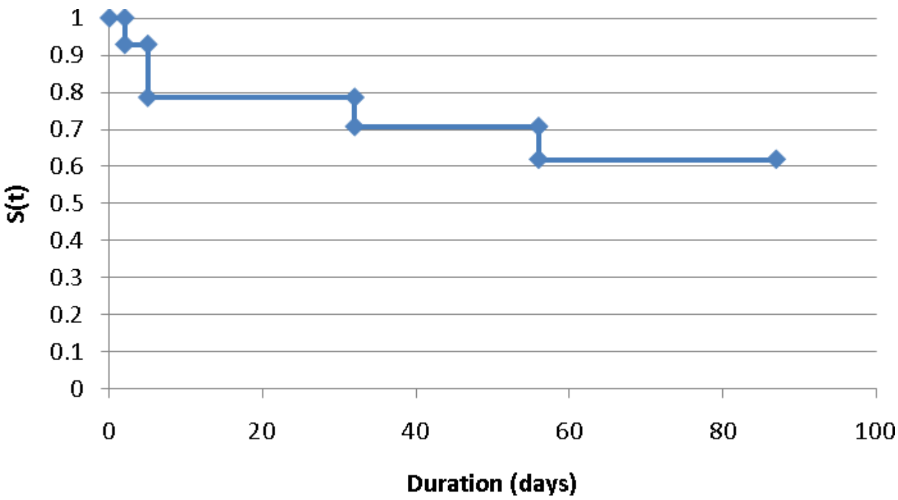
Assuming that at duration 56 the death occurred after the life was censored, the Kaplan-Meier estimate is as follows:

t_j	n_j	d_j	c_j	$\lambda_j = \frac{d_j}{n_j}$
0	14	0	0	0
2	14	1	0	1/14
5	13	2	1	2/13
32	10	1	2	1/10
56	7	1	6	1/7

The Kaplan-Meier estimate at duration t is given by the product of $1-\dfrac{d_j}{n_j}$ over durations up to and including t. Thus the Kaplan-Meier estimate of the survival function is

t	$\hat{S}(t)$		
$0 \le t < 2$	1.0000		
$2 \le t < 5$	0.9286	OR	13/14
$5 \le t < 32$	0.7857	OR	11/14
$32 \le t < 56$	0.7071	OR	99/140
$56 \le t < 92$	0.6061	OR	297/490

(iii)



(iv) The probability of death within 4 weeks is $1 - S(28) = 0.2143$.

In part (i) candidates could receive credit for saying that left censoring was present IF they gave a valid reason (which typically involved the imprecise measurement of the times of surgery or of events – the left censoring arising as a special case of interval censoring).
In part (ii) each error was only penalised once. Correct calculations which carried forward earlier errors were given full credit. However, candidates who did not list the durations they were using, but then presented incorrect estimates of the survival function, were more heavily penalised, as it was not clear how many errors they had made.

In part (ii) candidates who assume that the death at duration 56 takes place after the censoring at the same duration (ALTERNATIVE 2) were required to state this assumption for full credit. For ALTERNATIVE 1, the assumption that the death at duration 56 takes place before the censoring does not need to be stated for full credit, as it is the convention when calculating Kaplan-Meier estimates.

In part (iii) the plotted function should be consistent with the answer to part (ii). If the answer to part (ii) was incorrect but the incorrect answer to part (ii) was correctly plotted in part (iii), full credit could be awarded to part (iii).

Question 11

(i) The chi-squared test is a suitable overall test.

Let μ_x be the force of mortality in age-group x in the sample.
Let μ_x^s be the force of mortality in age group x in the national population.
Let E_x^c be the central exposed to risk in the sample.

Then if $z_x = \dfrac{E_x^c \mu_x - E_x^c \mu_x^s}{\sqrt{E_x^c \mu_x^s}}$

the test statistic is $\sum_x {z_x}^2 \sim {\chi^2}_m$,

THEN EITHER

where m is the number of age groups, which in this case is 8.

The calculations are shown below.

Age-group	*Expected deaths*	z_x	z_x^2
5–14	18.7935	–1.3364	1.7860
15–24	50.5460	–0.4988	0.2488
25–34	59.8842	–1.0188	1.0380
35–44	53.3092	–0.4532	0.2054
45–54	39.6396	–1.0546	1.1121
55–64	23.4140	–0.0856	0.0073
65–74	13.3926	–0.1073	0.0115
75–84	1.8200	0.8747	0.7651

Therefore the value of the test statistic is 5.1742.

The critical value of the chi-squared distribution at the 5% level of significance with 8 degrees of freedom is 15.51.

Since 5.1742 < 15.51 we do not reject the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population.

OR

where m is the number of age groups, which in this case is 7, because we should combine age groups 65–74 and 75–84 as the expected number of deaths in age group 75–84 years is less than 5

The calculations are shown below.

Age-group	*Expected deaths*	z_x	z_x^2
5–14	18.7935	–1.3364	1.7860
15–24	50.5460	–0.4988	0.2488
25–34	59.8842	–1.0188	1.0380
35–44	53.3092	–0.4532	0.2054
45–54	39.6396	–1.0546	1.1121
55–64	23.4140	–0.0856	0.0073
65–84	15.2126	0.2019	0.0408

Therefore the value of the test statistic is 4.438.

The critical value of the chi-squared distribution at the 5% level of significance with 7 degrees of freedom is 14.07.

Since 4.438 < 14.07 we do not reject the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population

(ii) (a) Small bias which is not great enough for the chi-squared test to detect.

EITHER

(b) **Signs test**

Under the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population,

the number of positive signs is distributed Binomial (m, 0.5), where m is the number of ages.

We have 1 positive sign.

The probability of 1 or fewer positive signs is given by

$$\binom{8}{0}0.5^8 + \binom{8}{1}0.5^8 = 0.0352.$$

OR (if only 7 age groups are being used)

$$\binom{7}{0}0.5^7 + \binom{7}{1}0.5^7 = 0.0625\,.$$

We use a two-tailed test (since too few or too many positive signs would be a problem)

so we reject the null hypothesis if the probability of 1 or fewer positive signs is less than 0.025.

Since 0.0352 (or 0.0625) > 0.025

we do not reject the null hypothesis.

OR

(b) **Cumulative deviations test**

Under the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population

the test statistic

$$\frac{\sum_x \left(E_x^c \mu_x - E_x^c \mu_x^s\right)}{\sqrt{\sum_x E_x^c \mu_x^s}} \sim \text{Normal}(0,1)$$

The calculations are shown in the table below

Age-group	$E_x^c \mu_x - E_x^c \mu_x^s$	$E_x^c \mu_x^s$
5–14	–5.7935	18.7935
15–24	–3.5460	50.5460
25–34	–7.8842	59.8842
35–44	–3.3092	53.3092
45–54	–6.6396	39.6396
55–64	–0.4140	23.4140
65–74	–0.3926	13.3926
75–84	1.1800	1.8200
Σ	–26.7991	260.7991

So the value of the test statistic is $\frac{-26.7991}{\sqrt{260.7991}} = 1.6595\,.$

Using a 5% level of significance, we see that $-1.96 < 1.6596 < 1.96$.

We do not reject the null hypothesis.

(a) Individual ages at which there are unusually large differences between the sample and the national experience.

(b) **Individual standardised deviations**

Under the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population

we would expect the individual deviations to be distributed Normal (0,1)

and therefore only 1 in 20 z_x s should have absolute magnitudes greater than 1.96
OR
none should lie outside the range (–3, +3)
OR
diagram showing split of deviations actual versus expected.

Since the largest deviation is less in absolute magnitude than 1.96 we do not reject the null hypothesis.

(a) Sections of the data where there is appreciable bias, revealed by runs or clumps of signs of the same type.

EITHER

(b) **Grouping of signs test**

Under the null hypothesis that the mortality rate from tuberculosis in the sample is the same as that in the national population

G = Number of groups of positive zs = 1

m = number of deviations = 8 (or 7 if last two age groups combined)
n_1 = number of positive deviations = 1
n_2 = number of negative deviations = 7 (or 6 if last two age groups combined)

THEN EITHER

We want k^* the largest k such that

$$\sum_{t=1}^{k} \frac{\binom{n_1-1}{t-1}\binom{n_2+1}{t}}{\binom{m}{n_1}} < 0.05$$

The test fails at the 5% level if $G \leq k^*$.

In the table in the Gold Book a value for k^* is not given,
OR
The table in the Gold Book shows that $k^* = 0$,

so we are not able to reject the null hypothesis
OR
so there is no evidence of clumping.

OR

For $t = 1$

$$\binom{n_1 - 1}{t - 1} = \binom{0}{0} \quad \text{which is 1}$$

So this test is automatically passed
OR
There is no evidence of clumping
OR
We cannot reject the null hypothesis.

OR

(b) **Serial correlations (lag 1)**

The calculations are shown in the tables below.

EITHER USING SEPARATE MEANS FOR THE z_x AND z_{x+1}

Age group	z_x	z_x	$A = z_x - \overline{z}$	$B = z_{x+1} - \overline{z}$	AB	A^2	B^2
5–14	–1.336	–0.499	–0.686	–0.164	0.342	0.470	0.027
15–24	–0.499	–1.019	0.152	–0.684	–0.155	0.023	0.468
25–34	–1.019	–0.453	–0.368	–0.118	0.167	0.136	0.014
35–44	–0.453	–1.055	0.197	–0.720	–0.208	0.039	0.518
45–54	–1.055	–0.086	–0.404	0.249	0.035	0.163	0.062
55–64	–0.086	–0.107	0.565	0.228	–0.061	0.319	0.052
65–74	–0.107	0.875	0.543	1.210	0.475	0.295	1.463
75–84							
$\overline{z}$	–0.651	–0.335		Sum	0.595	1.446	2.604

0.595/√(1.446*2.604) = 0.307

Test 0.307 (√8) = 0.868 against Normal (0,1), and, since 0.868 < 1.645, we do not reject the null hypothesis.

that the mortality rate from tuberculosis in the sample is the same as that in the national population

OR USING THE FORMULA IN THE GOLD BOOK

Age group	z_x	z_x	$A = z_x - \overline{z}$	$B = z_{x+1} - \overline{z}$	AB	A^2
5–14	–1.336	–0.499	–0.876	–0.039	0.034	0.767
15–24	–0.499	–1.019	–0.039	–0.559	0.022	0.002
25–34	–1.019	–0.453	–0.559	0.007	–0.004	0.312
35–44	–0.453	–1.055	0.007	–0.595	–0.004	0.000
45–54	–1.055	–0.086	–0.595	0.374	–0.223	0.354
55–64	–0.086	-0.107	0.374	0.353	0.132	0.140
65–74	–0.107	0.875	0.353	1.335	0.471	0.125
75–84	0.875		1.335			1.782
$\overline{z}$	–0.460			Sum	0.428	3.481

$$\frac{\frac{1}{7}(0.428)}{\frac{1}{8}(3.481)} = 0.141$$

Test 0.141 (√8) = 0.397 against Normal (0,1), and, since 0.397 < 1.645, we do not reject the null hypothesis.

that the mortality rate from tuberculosis in the sample is the same as that in the national population

(iii) In none of the tests we have performed do we reject the null hypothesis.

Therefore it seems that the mortality from tuberculosis in the town is the same as the national force of mortality.

In part (ii) the null hypothesis should be stated somewhere for each test. It could be stated at the beginning, or in the conclusion. As long as it is correctly stated somewhere, full credit was given. In part (iii), the comment should be consistent with the results of the tests performed in parts (i) and (ii) to gain credit.

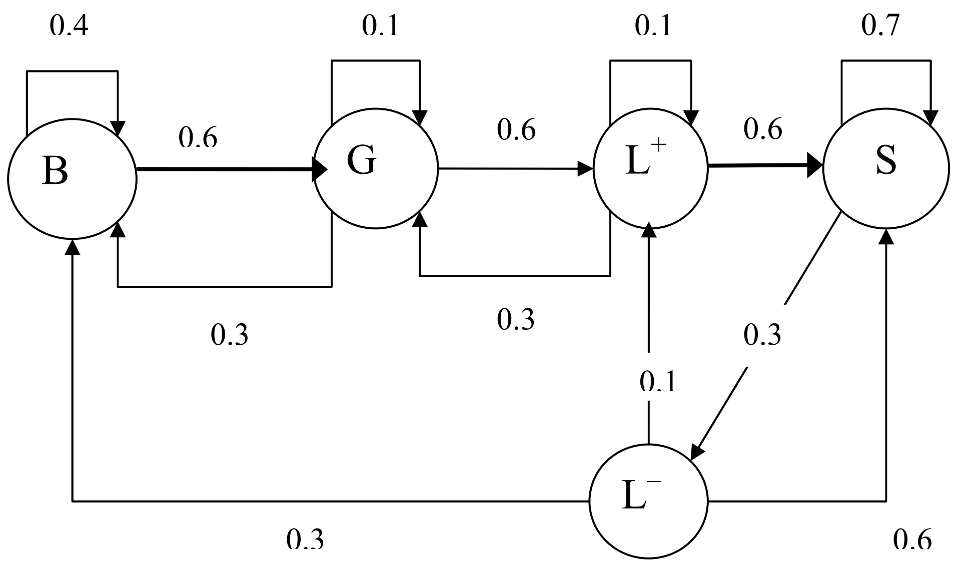
Most candidates made a good attempt at part (i). Attempts at part (ii) were more varied. In particular, most candidates did not point out that the chi-squared test only fails to detect SMALL (but consistent) bias. If the bias is large and consistent, the chi-squared test will detect it.

Question 12

(i) Past history is needed to decide where to go in the chain.

If a customer is at L and reduces his or her order, you need to know what level of discount he was at the previous year to determine whether he or she drops one or two levels of discount.

(ii) The L level needs to be split into two.
L^+ is Loyalty Price with no reduction in demand last year
L^- is Loyalty Price with reduction in demand last year



The probabilities were not required for full credit for this diagram.

(iii) (a)

$$\begin{array}{ccccc} B & G & L^+ & L^- & S \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{array}$$

$$\begin{pmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0.6 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0 & 0.6 \\ 0.3 & 0 & 0.1 & 0 & 0.6 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

(b) $\pi = \pi P$

$\pi_1 = 0.4\,\pi_1 + 0.3\,\pi_2 + 0.3\,\pi_4$ (1)
$\pi_2 = 0.6\,\pi_1 + 0.1\,\pi_2 + 0.3\,\pi_3$ (2)
$\pi_3 = 0.6\,\pi_2 + 0.1\,\pi_3 + 0.1\,\pi_4$ (3)
$\pi_4 = 0.3\,\pi_5$ (4)
$\pi_5 = 0.6\,\pi_3 + 0.6\,\pi_4 + 0.7\,\pi_5$ (5)
$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

(4) gives $\pi_4 = 0.3\,\pi_5$

(5) gives
$$\begin{aligned} 0.6\pi_3 &= 0.3\,\pi_5 - 0.6(0.3\pi_5) \\ &= 0.12\,\pi_5 \\ \pi_3 &= 0.2\,\pi_5 \end{aligned}$$

(3) gives
$$\begin{aligned} 0.6\,\pi_2 &= 0.9\,\pi_3 - 0.1\,\pi_4 \\ &= 0.18\,\pi_5 - 0.03\,\pi_5 \\ &= 0.15\,\pi_5 \\ \pi_2 &= 0.25\,\pi_5 \end{aligned}$$

(2) gives
$$\begin{aligned} 0.6\,\pi_1 &= 0.9\,\pi_2 - 0.3\,\pi_3 \\ &= 0.9(0.25)\,\pi_5 - 0.3(0.2)\,\pi_5 \\ &= 0.225\,\pi_5 - 0.06\,\pi_5 \\ &= 0.165\,\pi_5 \\ \pi_1 &= 0.275\,\pi_5 \end{aligned}$$

$\pi_5\,(0.275 + 0.25 + 0.2 + 0.3 + 1) = 1$
$\pi_5 = 1 / 2.025$
$= 0.49382716$

$\pi_1 = 0.13580$			OR 11/81
$\pi_2 = 0.12346$			OR 10/81
$\pi_3 = 0.09877$	)		OR 8/81
$\pi_4 = 0.14815$	)	0.24692	OR 12/81
$\pi_5 = 0.49383$			OR 40/81

(c) Average price for a bale of hay is

£8 × (1 × 0.1358 + 0.9 × 0.12346 + 0.8 × (0.09877 + .14815) + 0.75 ×.49383)

= £6.5181

(iv)
$$\begin{pmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0.6 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0 & 0.6 \\ 0.3 & 0 & 0.1 & 0 & 0.6 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0.6 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0 & 0.6 \\ 0.3 & 0 & 0.1 & 0 & 0.6 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} .16+.18 & .24+.06 & .36 & - & - \\ .12+.03 & .18+.01+.18 & .06+.06 & - & .36 \\ .09 & .03+.03 & .18+.01 & .18 & .18 \\ .12 & .18+.03 & .01 & .18 & .06+.42 \\ .09 & - & .03 & .21 & .18+.42 \end{pmatrix} = \begin{pmatrix} .34 & .3 & .36 & - & - \\ .15 & .37 & .12 & - & .36 \\ .09 & .06 & .19 & .18 & .48 \\ .12 & .21 & .01 & .18 & .48 \\ .09 & - & .03 & .21 & .67 \end{pmatrix}$$

THEN ALTERNATIVE 1

Using the long-run probabilities of being in L^+ and L^-, therefore

the chance of being at L in two years' time is

(0.19 + 0.18)*0.4 + (0.18 + 0.01)*0.6 = 0.262.

OR ALTERNATIVE 2

Assuming there is an equal probability of being L^+ and L^-,

the chance of being at L in two years' time is

(0.19 + 0.18)*0.5 + (0.18 + 0.01)*0.5 = 0.28.

OR ALTERNATIVE 3

We do not know the relative proportions in L^+ and L^-,

but for those in L^+ the chance of being in L in two years' time is 0.19 + 0.18 = 0.37, and for those in L^+ the chance of being in L in two years' time is 0.18 + 0.01 = 0.19.

OR ALTERNATIVE 4

We do not know the relative proportions in L^+ and L^-,

and so it is not possible to evaluate the overall probability that a customer in L will be in L in two years' time.

(v) A constant figure takes no account of the amount of hay which Farmer Giles has to sell: for example a drought year could produce very little which one large customer may buy in its entirety.

The amount of hay in the local market is important.

Another supplier may try a heavy discounted year to get into the market.

Customers' behaviour may depend on the discount level they are at.

There may be national trends in the demand for hay e.g. a sudden trend towards vegetarianism.

A 60% chance of increasing may be implausible, as field space is likely to be limited, so a constant increase in numbers unlikely.

Customers' behaviour may depend on the amount of hay they typically purchase.

A common error in part (ii) was to split state G into two states as well as splitting state L. This is not required to model the system with the Markov property and so was penalised. However, candidates who split state G and then followed through with a correct matrix in part (iii)(a) and correct solutions in part (iii)(b) were not penalised again. Note that splitting state G should produce the same answer to part (iii)(b), though more work will be needed!

In part (iv) candidates who adopted ALTERNATIVE 4, in which they declined to give an overall answer on the grounds that they do not know the proprotions in states L^{+} *and* L^{-} *, were only given credit if they presented a reasoned argument with evidence.*

In part (v) credit was given for other sensible suggestions.

END OF EXAMINERS' REPORT