INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

15 April 2011 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 12 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Give three advantages of the two-state model over the Binomial model for estimating transition intensities where exact dates of entry into and exit from observation are known. [3]

2 Distinguish between the conditions under which a Markov chain:

(a) has at least one stationary distribution.
(b) has a unique stationary distribution.
(c) converges to a unique stationary distribution. [3]

3 Describe the ways in which the design of a model used to project over only a short time frame may differ from one used to project over fifty years. [4]

4 Children at a school are given weekly grade sheets, in which their effort is graded in four levels: 1 "Poor", 2 "Satisfactory", 3 "Good" and 4 "Excellent". Subject to a maximum level of Excellent and a minimum level of Poor, between each week and the next, a child has:

- a 20 per cent chance of moving up one level.
- a 20 per cent chance of moving down one level.
- a 10 per cent chance of moving up two levels.
- a 10 per cent chance of moving down two levels.

Moving up or down three levels in a single week is not possible.

(i) Write down the transition matrix of this process. [2]

Children are graded on Friday afternoon in each week. On Friday of the first week of the school year, as there is little evidence on which to base an assessment, all children are graded "Satisfactory".

(ii) Calculate the probability distribution of the process after the grading on Friday of the third week of the school year. [3]
[Total 5]

5 (i) Explain why a mortality experience would need to be graduated. [3]

An actuary has conducted investigations into the mortality of the following classes of lives:

(a) the female members of a medium-sized pension scheme

(b) the male population of a large industrial country

(c) the population of a particular species of reptile in the zoological collections of the southern hemisphere

The actuary wishes to graduate the crude rates.

(ii) State an appropriate method of graduation for each of the three classes of lives and, for each class, briefly explain your choice. [3]
[Total 6]

6 A study of the mortality of a certain species of insect reveals that for the first 30 days of life, the insects are subject to a constant force of mortality of 0.05. After 30 days, the force of mortality increases according to the formula:

$$\mu_{30+x} = 0.05\exp(0.01x),$$

where x is the number of days after day 30.

(i) Calculate the probability that a newly born insect will survive for at least 10 days. [1]

(ii) Calculate the probability that an insect aged 10 days will survive for at least a further 30 days. [3]

(iii) Calculate the age in days by which 90 per cent of insects are expected to have died. [4]
[Total 8]

7 (i) Define a counting process. [2]

For each of the following processes:

- simple random walk
- compound Poisson
- Markov chain

(ii) (a) State whether each of the state space and the time set is discrete, continuous or can be either.

(b) Give an example of an application which may be useful to a shopkeeper selling dried fruit and nuts loose.

[6]
[Total 8]

8 (i) Explain the difference between the central and the initial exposed to risk, in the context of mortality investigations. [2]

An investigation studied the mortality of infants aged under 1 year. The following table gives details of 10 lives involved in the investigation. Infants with no date of death given were still alive on their first birthday.

Life	*Date of birth*	*Date of death*
1	1 August 2008	-
2	1 September 2008	-
3	1 December 2008	1 February 2009
4	1 January 2009	-
5	1 February 2009	-
6	1 March 2009	1 December 2009
7	1 June 2009	-
8	1 July 2009	-
9	1 September 2009	-
10	1 November 2009	1 December 2009

(ii) Calculate the maximum likelihood estimate of the force of mortality, using a two-state model and assuming that the force is constant. [3]

(iii) Hence estimate the infant mortality rate, q_0. [1]

(iv) Estimate the infant mortality rate, q_0, using the initial exposed to risk. [1]

(v) Explain the difference between the two estimates. [2]

[Total 9]

9 (i) Define a Markov jump process. [2]

A study of a tropical disease used a three-state Markov process model with states:

1. Not suffering from the disease
2. Suffering from the disease
3. Dead

The disease can be fatal, but most sufferers recover. Let ${}_t p_x^{ij}$ be the probability that a person in state i at age x is in state j at age $x+t$. Let μ_{x+t}^{ij} be the transition intensity from state i to state j at age $x+t$.

(ii) Show from first principles that:

$$\frac{d}{dt}\,{}_t p_x^{13} = {}_t p_x^{11}\,\mu_{x+t}^{13} + {}_t p_x^{12}\mu_{x+t}^{23}.$$ [4]

The study revealed that sufferers who contract the disease a second or subsequent time are more likely to die, and less likely to recover, than first-time sufferers.

(iii) Draw a diagram showing the states and possible transitions of a model which allows for this effect yet retains the Markov property. [3]

[Total 9]

10 At Miracle Cure hospital a pioneering new surgery was tested to replace human lungs with synthetic implants. Operations were carried out throughout June 2010. Patients who underwent the surgery were monitored daily until the end of August 2010, or until they died or left hospital if sooner. The results are shown below. Where no date is given, the patient was alive and still in hospital at the end of August.

Patient	*Date of surgery*	*Date of leaving observation*	*Reason for leaving observation*
A	June 1	June 3	Died
B	June 3	July 2	Left Hospital
C	June 5		
D	June 8		
E	June 9	July 11	Died
F	June 12		
G	June 16	June 21	Died
H	June 17	Aug 12	Left Hospital
I	June 22		
J	June 24	June 29	Died
K	June 25	Aug 20	Died
L	June 26		
M	June 29	Aug 6	Left Hospital
N	June 30		

(i) Explain whether each of the following types of censoring is present and for those present explain where they occur:

- right censoring
- left censoring
- informative censoring

[3]

(ii) Calculate the Kaplan-Meier estimate of the survival function for these patients, stating all assumptions that you make. [6]

(iii) Sketch, on a suitably labelled graph, the Kaplan-Meier estimate of the survival function. [2]

(iv) Estimate the probability that a patient will die within four weeks of surgery. [1]

[Total 12]

11 An historian has investigated the force of mortality from tuberculosis in a particular town in a developed country in the 1860s using a sample of records from a cemetery. He wishes to test whether the underlying mortality from tuberculosis in the town is the same as the national force of mortality from this cause of death, as reported in death registration data. The data are shown in the table below.

Age-group	*Deaths in sample*	*Central exposed to risk in sample*	*National force of mortality*
5–14	13	3,685	0.0051
15–24	47	2,540	0.0199
25–34	52	1,938	0.0309
35–44	50	1,687	0.0316
45–54	33	1,386	0.0286
55–64	23	1,018	0.0230
65–74	13	663	0.0202
75–84	3	260	0.0070

(i) Carry out an overall test of the null hypothesis that the underlying mortality from tuberculosis in the town is the same as the national force of mortality, and state your conclusion. [6]

(ii) (a) Identify two differences between the experience of the sample and the national experience which the test you performed in (i) might not detect.

(b) Carry out a test for each of the differences in (ii)(a). [7]

(iii) Comment on the results from all the tests carried out in (i) and (ii). [1]
[Total 14]

12 Farmer Giles makes hay each year and he makes far more than he could possibly store and use himself, but he does not always sell it all. He has decided to offer incentives for people to buy large quantities so it does not sit in his field deteriorating. He has devised the following "discount" scheme.

He has a Base price, *B* of £8 per bale. Then he has three levels of discount: Good price, *G*, is a 10% discount, Loyalty price, *L* is a 20% discount and Super price, *S*, is a 25% discount on the Base price.

- Customers who increase their order compared with last year move to one higher discount level, or remain at level *S*.
- Customers who maintain their order from last year stay at the same discount level.
- Customers who reduce their order from last year drop one level of discount or remain at level *B* provided that they maintained or increased their order the previous year.

- Customers who reduce their order from last year drop two levels of discount if they also reduced their order last year, subject to remaining at the lowest level B.

(i) Explain why a process with the state space of $\{B, G, L, S\}$ does not display the Markov property. [2]

(ii) (a) Define any additional state(s) required to model the system with the Markov property.

(b) Construct a transition graph of this Markov process clearly labelling all the states. [3]

Farmer Giles thinks that each year customers have a 60% likelihood of increasing their order and a 30% likelihood of reducing it, irrespective of the discount level they are currently in.

(iii) (a) Write down the transition matrix for the Markov process.

(b) Calculate the stationary distribution.

(c) Hence calculate the long run average price he will get for each bale of hay.

[8]

(iv) Calculate the probability that a customer who is currently paying the Loyalty price, L, will be paying L in two years' time. [3]

(v) Suggest reasons why the assumptions Farmer Giles has made about his customers' behaviour may not be valid. [3]

[Total 19]

END OF PAPER