INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2013 examinations

Subject CT4 – Models
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

July 2013

General comments on Subject CT4

Subject CT4 comprises five main sections: (1) a study of the properties of models in general, and their uses for actuaries, including advantages and disadvantages (and a comparison of alternative models of the same processes); (2) stochastic processes, especially Markov chains and Markov jump processes; (3) models of a random variable measuring future lifetime; (4) the calculation of exposed to risk and the application of the principle of correspondence; (5) the reasons why mortality (or other decremental) rates are graduated, and a range of statistical tests used both to compare a set of rates with a previous experience and to test the adherence of a graduated set of rates to the original data. Throughout the subject the emphasis is on estimation and the practical application of models. Theory is kept to the minimum required in order usefully to apply the models to real problems.

Different numerical answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2013 paper

The general performance was slightly inferior to that in April 2011 or April 2012, but better than that in September 2012. Despite this, well-prepared candidates scored highly across the whole paper, with an above average proportion of candidates scoring 70 per cent or more. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to include revision of these areas in their preparation.

1

A stochastic model is one that recognises the random nature of the input components.

A model that does not contain any random component is deterministic in nature.

In a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined.

By contrast, in a stochastic model the output is random in nature. The output is only a snapshot or an estimate of the characteristics of the model for a given set of inputs.

A deterministic model is really just a special (simplified) case of a stochastic model.

A deterministic model will give one set of results of the relevant calculations for a single scenario; a stochastic model will be run many times with the same input and gives distributions of the relevant results for a distribution of scenarios

The results for a deterministic model can often be obtained by direct calculation.

The results of stochastic models often require Monte Carlo simulation, although some stochastic models can have an analytical solution.

Correlations can be important in stochastic models as they indicate when the behaviour of one variable is associated with that of another.

Stochastic models are more complex and more difficult to interpret than deterministic models and so require more expertise, expense and computer power.

Not all the points listed above were required for full marks. Credit was also given for sensible points not included in the above list.

2

(i) **Right censoring.** The duration to the event is not known exactly, but is known to exceed some value.
OR
the censoring mechanism cuts short observations in progress.

Type I censoring. The durations at which observations will be censored are specified in advance.

Type II censoring. Observation continues until a pre-determined number/proportion of individuals have experienced the event of interest.

(ii) An investigation of mortality based on life office data in which individuals are censored who discontinue paying their premiums.

Those whose premiums lapse tend, on average, to be in better health than do those who carry on paying their premiums.

In part (ii) any suitable example was given credit. However, for full credit it was necessary to describe a comparison between the risk of the event happening in the censored and uncensored observations (e.g. "in better health than" or "less likely to die than"). Most candidates made a good attempt at this question.

3

(a)

<table>
<tr><td colspan="2"></td><td colspan="2">Time Space</td></tr>
<tr><td colspan="2"></td><td>Discrete</td><td>Continuous</td></tr>
<tr><td rowspan="2">State Space</td><td>Discrete</td><td>Counting process</td><td>Poisson process</td></tr>
<tr><td>Continuous</td><td>General random walk</td><td>Compound Poisson process</td></tr>
</table>

(b)

<table>
<tr><td colspan="2"></td><td colspan="2">Time Space</td></tr>
<tr><td colspan="2"></td><td>Discrete</td><td>Continuous</td></tr>
<tr><td rowspan="2">State Space</td><td>Discrete</td><td>Simple random walk</td><td>Counting process</td></tr>
<tr><td>Continuous</td><td>White noise</td><td>Compound Poisson process</td></tr>
</table>

This question was answered well, with many candidates scoring full marks. Some candidates lost marks by failing to follow the instructions in the question precisely. To obtain full credit, candidates were required to place the processes in grids like those shown above with ONE process in each of the four cells. What is shown above is the only solution which fulfils this criterion for groups (a) and (b). In some cases, processes could correctly be placed in cells other than those shown in the grids above, and credit was given for each process thus classified correctly.

4

(i) If the force of mortality, μ, is constant, then the expected waiting time is $\frac{1}{\mu}$.

Hence expected age at death is $5 + \frac{1}{\mu} = \frac{5\mu+1}{\mu}$.

[1]

(ii) EITHER

We need ${}_{10}p_0 - {}_{15}p_0$.

Since ${}_{x}p_0 = {}_{x-5}p_5 \cdot {}_{5}p_0$

and for $x > 5$, ${}_{x}p_5 = e^{-\mu x}$,

then

$${}_{10}p_0 - {}_{15}p_0 = {}_{5}p_5 \cdot {}_{5}p_0 - {}_{10}p_5 \cdot {}_{5}p_0 = {}_{5}p_0 e^{-5\mu} - {}_{5}p_0 e^{-10\mu} = {}_{5}p_0 (e^{-5\mu} - e^{-10\mu}).$$

OR

We need ${}_{10}p_0 \cdot {}_{5}q_{10}$

$= {}_{10}p_0 (1 - {}_{5}p_{10})$

Since for $x > 5$, ${}_{x}p_5 = e^{-\mu x}$,

$${}_{10}p_0 (1 - {}_{5}p_{10}) = {}_{5}p_0 ({}_{5}p_5 - {}_{10}p_5) = {}_{5}p_0 (e^{-5\mu} - e^{-10\mu})$$

[3]

(iii) EITHER

${}_{5}p_0 e^{-5\mu} = 0.3$ and ${}_{5}p_0 e^{-10\mu} = 0.2$.

So $\quad \frac{{}_{5}p_0 e^{-5\mu}}{{}_{5}p_0 e^{-10\mu}} = \frac{0.3}{0.2}$

and $e^{-5\mu} = 1.5 e^{-10\mu}$

so that $-5\mu = \log_e 1.5 - 10\mu$

$5\mu = 0.4055$
$\mu = 0.0811.$

Therefore ${}_5p_0 e^{-5(0.0811)} = 0.3$

and $\quad {}_5p_0 = \dfrac{0.3}{e^{-5(0.0811)}} = 0.4500.$

OR

${}_{10}p_0 = {}_5p_0 \cdot {}_5p_5 = 0.3$

With a constant force after age 5 years, ${}_5p_5 = {}_5p_{10}$,

so ${}_{15}p_0 = {}_5p_0 \cdot {}_{10}p_5 = {}_5p_0 \cdot {}_5p_5 \cdot {}_5p_{10} = {}_5p_0 ({}_5p_5)^2 = 0.2$.

Hence ${}_5p_5 = \dfrac{0.2}{0.3}$

and ${}_5p_0 = \dfrac{0.3}{{}_5p_5} = \dfrac{(0.3)^2}{0.2} = 0.45.$

Then $\mu = -\dfrac{\log_e {}_5p_5}{5} = \dfrac{0.4055}{5} = 0.0811.$

Answers to this question were extremely disappointing. Few candidates could even attempt part (i) correctly, and there were similarly few correct attempts at parts (ii) and (iii). In part (iii) the question asked "calculate" so candidates giving both correct numerical answers scored full credit. If one of either μ or ${}_5p_0$ was correct, a minimum of +2 was scored. Where candidates made the same theoretical error in parts (ii) and (iii), the error was only penalised once.

5

(i) We adjust the exposed to risk so that the age definition corresponds with that of the deaths data.

Let the population at age 65 nearest birthday be P_{65} and let the central exposed to risk at age 65 nearest birthday be E_{65}^c.

In 2006 P_{65} = 0.5(300,000 + 290,000) = 295,000
In 2009 P_{65} = 0.5(320,000 + 310,000) = 315,000

In 2010 $P_{65} = 0.5(350{,}000 + 330{,}000) = 340{,}000$,

assuming that birthdays are uniformly distributed across calendar time.

Using the census approximation (trapezium method) for the period 2006-2009 then

assuming that the population varies linearly between census dates,

$E_{65}^{c} = 1.5(295{,}000 + 315{,}000) = 915{,}000$

and for the period 2009–2010

$E_{65}^{c} = 0.5(315{,}000 + 340{,}000) = 327{,}500$.

Assuming that the force of mortality is constant within each year of age

$$\mu_{65} = \frac{3{,}000}{915{,}000} = 0.003279 \text{ for the period 2006–2009, and}$$

$$\mu_{65} = \frac{1{,}000}{327{,}500} = 0.003053 \text{ for the period 2009–2010.}$$

We also assuming that the President doesn't change (so the birthday is on the same day each year), or if the President does change the new President's birthday is the same as the birthday of the old President.

(ii) The rate interval is the life year, starting at age x – 0.5.

The age in the middle of the rate interval is thus x, so the estimate relates to exact age 65 years.

A common error in part (i) was to use equal time periods, whereas the period 2006-2009 is three years and 2009-2010 only one year. For full credit, the assumptions had to appear in the script close to the relevant bit of calculation. Candidates who listed many assumptions, both necessary and unnecessary, in a block at the end of the answer were penalised. In part (i), some candidates calculated q_x rather than μ_x. Full credit was given for this provided that the initial exposed-to-risk was used as the denominator. In part (ii) the age to which q_x applies is 64.5 years (i.e. the age at the start of the rate interval), and for full credit the answers to parts (i) and (ii) had to be consistent.

6

(i) $\lambda(t{:}Z_i) = \lambda_0(t)\exp(\beta Z_i^T)$

Where:

$\lambda(t{:}Z_i)$ is the hazard at time t

$\lambda_0(t)$ is the baseline hazard

Z_i is a vector of covariates

β is a vector of regression parameters

(ii) It ensures the hazard is always positive.

The log-hazard is linear.

You can ignore the shape of the baseline hazard and calculate the effect of covariates directly from the data.

It is widely available in standard computer packages OR is a popular, well-established model.

(iii) Ben, self-employed, first attempt, no study has hazard $\lambda_0(t)\exp(0.4)$

Bill, employee, re-sit, study leave has hazard $\lambda_0(t)\exp(0.95)$

So Ben is only $\exp(-0.55) = 57.7\%$ as likely to pass as Bill OR 42.3% less likely to pass than Bill.

OR

Bill is 73% more likely to pass than Ben

(iv) The model could be adjusted by including a covariate measuring the interaction between the number of attempts and employment status.

The covariate would be equal to Z_1Z_2 and would take the value 1 for a self-employed person on his or her second or subsequent attempt, and 0 otherwise.

The effect of the number of attempts for an employee would be equal to $\exp(\beta_2)$, where β_2 is the parameter related to Z_2, For a self-employed person, the effect of the number of attempts would be equal to $\exp(\beta_2 + \beta_3)$, where β_3 is the parameter related to the interaction term.

This question was well answered by many candidates. In part (iii) the question asked candidates to "calculate" so the correct numerical answer scored full credit. However a common error was to use ambiguous or incorrect wording in the final comparison (e.g. Bill *is 57.7 per cent less likely to pass than Ben). In part (iv) no credit was given for the addition of covariates with no bearing on the interaction term. However redundant parameters were not penalised provided the modification to the model allowed the interaction to be quantified.*

7

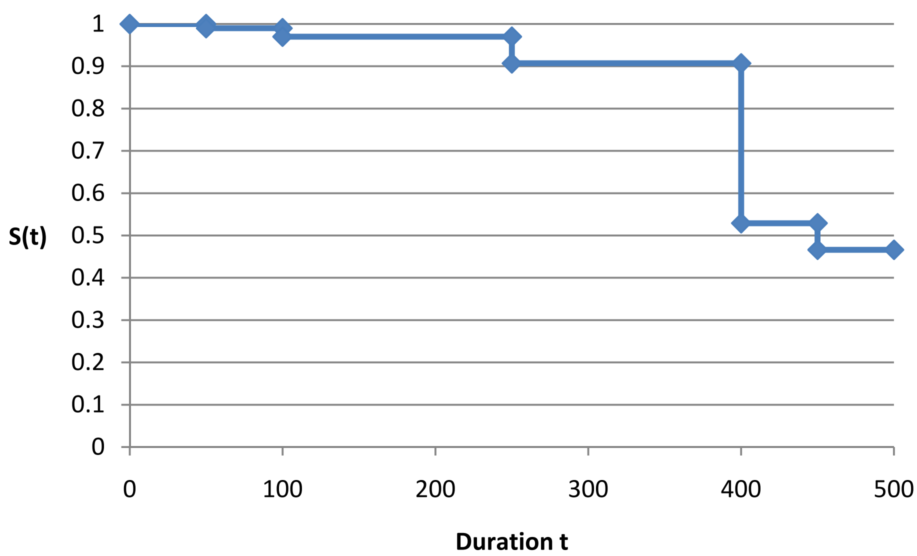
(i)

t_j	N_j	d_j	c_j	d_j / N_j	$1 - d_j / N_j$	
0	1,000					
50	1,000	10	0	0.0100	0.9900	or 99/100
100	990	20	0	0.0202	0.9798	or 97/99
200	970	0	200			
250	770	50	0	0.0649	0.9351	or 72/77
400	720	300	0	0.4167	0.5833	or 7/12
450	420	50	370	0.1190	0.8810	or 37/42

The Kaplan-Meier estimate is $\hat{S}(t) = \prod_{t_j \le t} (1 - \frac{d_j}{n_j})$

t	*Kaplan-Meier estimate of S(t)*	
$0 \le t < 50$	1.0000	or 1
$50 \le t < 100$	0.9900	or 99/100
$100 \le t < 250$	0.9700	or 97/100
$250 \le t < 400$	0.9070	or 1,746/1,925
$400 \le t < 450$	0.5291	or 291/550
$450 \le t < 500$	0.4661	or 3,589/7,700

(ii)



(iii) $S(300) = 0.9070$.

$S(400) = 0.5291$.

$S(600)$ cannot be estimated without additional assumptions

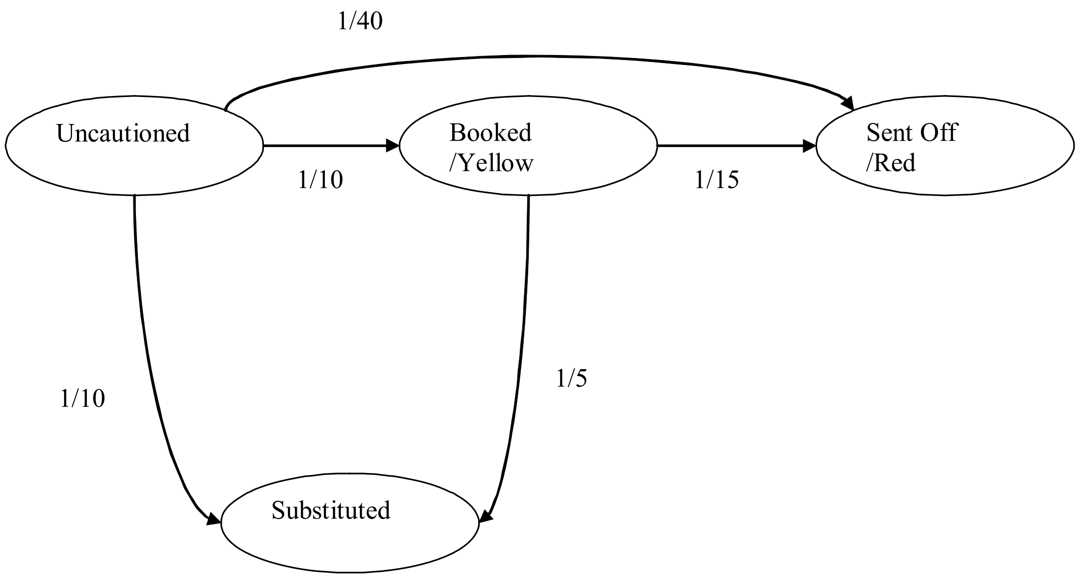
as it lies outside the range of our data.

[3]
[Total 11]

This question was very well answered, with many candidates scoring 10 or more marks out of a possible 11. Some of the sketches in part (ii) were very scrappy and were penalised: though great accuracy was not required, the sketch did need to be sufficiently clear to demonstrate that the candidate understood the nature of the function being plotted. In part (iii) some candidates suggested an assumption which would enable them to give an answer for S(600). Such candidates were given full credit provided they explained why the assumption was needed, and provided that the stated assumption was consistent with the numerical answer offered.

8

(i)



(ii) $\frac{d}{dt}P(t) = P(t)A$

where generator matrix

$$A = \begin{pmatrix} -9/40 & 1/10 & 1/40 & 1/10 \\ 0 & -4/15 & 1/15 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In order of states $\{U, Y, R, S\}$

(iii) $\frac{d}{dt}P_{UU}(t) = -\frac{9}{40}P_{UU}(t)$

$P_{UU}(t) = \exp\left(-\frac{9}{40}t\right) + const = \exp\left(-\frac{9}{40}t\right)$ as looking for probability on pitch throughout match.

At end $t = 3/2$ so require exp(−27/80) = 71.36%.

(iv) Prob[sent off without being booked] = $\int_{s=0}^{3/2} P_{UU}(s).\frac{1}{40}ds$

$$= \int_{s=0}^{3/2} \exp(-\frac{9}{40}s).\frac{1}{40}ds$$

$$= \left[-\frac{1}{9}\exp\left(-\frac{9}{40}s\right)\right]_0^{3/2} = \frac{1}{9}\left(1 - \exp\left(-\frac{27}{80}\right)\right) = 0.03183$$

(v) (a) EITHER

The upper limit of the integral tends to infinity

so result becomes 1/9.

OR

We need

Pr[sent off directly]/Pr[leaves state U]

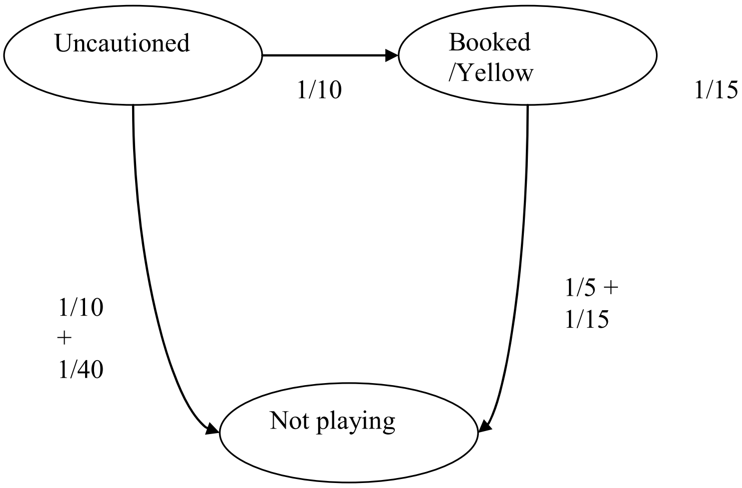
$$= \frac{1/40}{9/40} = \frac{1}{9}$$

(b) This is the ratio of the transition rate to "straight to sent off" to the total transition rate out of state *U*.

This was one of the more demanding questions on the paper, and a high proportion of candidates struggled to get past part (ii). In part (i) the transition rates were not required. In part (iii) the question asked candidates to solve an equation, so for full credit the equation had to be written down, and the method of solution described. In part (v) candidates who used the rationale in (b) to do the calculation in (a) scored full credit.

Candidates who interpreted the question in a manner not intended, but instead combined the categories "sent off" and "substituted" were not penalised. This interpretation leads to a three state solution for parts (i) and (ii) as follows.

(i)



(ii) $\frac{d}{dt}P(t) = P(t)A$

where generator matrix

$$A = \begin{pmatrix} \frac{-9}{40} & \frac{1}{10} & \frac{1}{8} \\ 0 & \frac{-4}{15} & \frac{4}{15} \\ 0 & 0 & 0 \end{pmatrix}$$

In order of states {U, Y, N} [3]

This was given full credit in parts (i), (ii) and (iii). The answer to part (iii) is the same as for the four-state solution. Credit was given in parts (iv) and (v) for following this alternative through correctly.

9

(i) **Signs Test**

Under the null hypothesis that the underlying mortality of the life office policyholders is the same as the CMI mortality,

the number of positive deviations is distributed Binomial(m,0.5)

THEN EITHER ALTERNATIVE 1 (NORMAL APPROXIMATION)

Here we have $m = 31$, so as $m > 20$ we can use the Normal approximation, that the number of positive deviations is distributed Normal ($m/2$, $m/4$).

the number of positive deviations is Normal $\left(\frac{31}{2}, \frac{31}{4}\right)$.

In this case we have 22 positive deviations.

The z-score corresponding to 22 is $\frac{22-15.5}{\sqrt{7.75}} = \frac{6.5}{2.78} = 2.33$

OR Using a continuity correction

The z-score corresponding to 22 is $\frac{21.5-15.5}{\sqrt{7.75}} = \frac{6}{2.78} = 2.16$

Using a 2-tailed test, we reject the null hypothesis at the 5% level of significance if $|z|>1.96$.

Since 2.33 (or 2,16) > 1.96 we reject the null hypothesis.

OR ALTERNATIVE 2 (EXACT CALCULATION)

In this case we have 22 positive deviations.
The probability of observing exactly 22 positive deviations is 0.009388
OR
The probability of observing ≥ 22 positive deviations is 0.014725

Using a 2-tailed test, we reject the null hypothesis at the 5% level of significance if the probability is <0.025

Since 0.014725 < 0.025 we reject the null hypothesis.

Grouping of Signs Test

Define the test statistic:

G = Number of groups of positive deviations.

THEN EITHER ALTERNATIVE 1 (NORMAL APPROXIMATION)

Since $m = 31$ (which is ≥ 20), we can use a Normal approximation as follows:

$$G \sim \text{Normal}\left(\frac{n_1(n_2+1)}{n_1+n_2}, \frac{(n_1 n_2)^2}{(n_1+n_2)^3}\right).$$

In this case $m = 31$, $n_1 = 22$ and $n_2 = 9$.

Thus $G \sim \text{Normal}(7.10, 1.32)$.

We have 4 groups of positive signs.

The z-score corresponding to 4 is $\dfrac{4-7.10}{\sqrt{1.32}} = \dfrac{-3.10}{1.15} = -2.70$

Using a 1-tailed test, we reject the null hypothesis at the 5% level of significance if $z < -1.645$.

Since $-2.70 < -1.645$ we reject the null hypothesis.

OR ALTERNATIVE 2 USING TABLE IN GOLD BOOK

m = total number of deviations = 31
n_1 = number of positive deviations = 22
n_2 = number of negative deviations = 9

We want k^* the largest k such that

$$k = \sum_{t=1}^{x} \binom{n_1-1}{t-1}\binom{n_2+1}{t} \Big/ \binom{n_1+n_2}{n_1} < 0.05$$

We have 4 groups of positive signs.

The test fails at the 5% level if $G \leq k^*$

From the table in the Gold Book $k^* = 4$

Since G is not greater than this, we reject the null hypothesis

(ii) The life office's rates are, overall, different from the CMI rates (actually they are higher).

Additional tests are needed to examine the magnitude of the difference between the two sets of rates.

The shape of the life office's mortality rates is also rather different from the CMI schedule, and this might require further investigation,
OR
The Grouping of Signs test suggests clumping of the deviations.

It is possible that the difference between the shape of the two sets of rates is so small in magnitude as to be negligible.

(iii) We can no longer be sure that we are observing a collection of independent claims.

It is quite possible that two distinct death claims are the result of the death of the same life.

The effect of this is to increase the variance of the number of claims,

by a factor which may depend on age.

This may affect tests based on standardised deviations.

Answers to this question were very disappointing, especially part (i) . The two tests were often performed in a rather cursory way, with important steps being missed out. In part (i) the null hypothesis only needed stating once. Common errors were to work with only 30 ages or, more seriously, to use eight ages (i.e. to treat each run of consecutive ages of the same sign as a single age).

10

(i) The future development of the process depends only on the state currently occupied and not on any previous history.

OR

$$P[X_t \in A \mid X_{s_1} = x_1,\ X_{s_2} = x_2, ...,\ X_{s_n} = x_n, X_s = x] = P[X_t \in A \mid X_s = x]$$

for all times $s_1 < s_2 < ... < s_n < s < t$, all states $x_1, x_2, ..., x_n, x$ in S and all subsets A of S.

(ii) Condition on the state occupied at x+t, using the Markov assumption:

$$_{t+dt}p_x^{12} = {}_tp_x^{11}\,{}_{dt}p_{x+t}^{12} + {}_tp_x^{12}\,{}_{dt}p_{x+t}^{22}$$

But by Law of Total Probability $_{dt}p_{x+t}^{22} = 1 - {}_{dt}\,p_{x+t}^{21}$, so

$$_{t+dt}p_x^{12} = {}_tp_x^{11}\,{}_{dt}p_{x+t}^{12} + {}_tp_x^{12}(1 - {}_{dt}\,p_{x+t}^{21}).$$

Now, since by assumption, for small *dt*, ${}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}dt + o(dt)$,

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$,

we can substitute to give

$$\begin{aligned} {}_{t+dt}p_x^{12} &= {}_tp_x^{11}\mu_{x+t}^{12}dt + {}_tp_x^{12}(1-\mu_{x+t}^{21}dt) + o(dt) \\ &= {}_tp_x^{11}\mu_{x+t}^{12}dt + {}_tp_x^{12} - {}_tp_x^{12}\mu_{x+t}^{21}dt + o(dt) \end{aligned}$$

so that

$${}_{t+dt}p_x^{12} - {}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12}dt - {}_tp_x^{12}\mu_{x+t}^{21}dt + o(dt).$$

Dividing by *dt* and taking limits gives

$$\lim_{dt \to 0}\left[\frac{{}_{t+dt}p_x^{12} - {}_tp_x^{12}}{dt}\right] = \frac{d}{dt}{}_tp_x^{12} = \lim_{dt \to 0}\left[\frac{{}_tp_x^{11}\mu_{x+t}^{12}dt}{dt} - \frac{{}_tp_x^{12}\mu_{x+t}^{21}dt}{dt} + \frac{o(dt)}{dt}\right]$$

$$= {}_tp_x^{11}\mu_{x+t}^{12} - {}_tp_x^{12}\mu_{x+t}^{21}.$$

(iii) State 1 to state 2: $\mu^{12} = 4{,}330/21{,}650 = 0.2$.

State 2 to state 1: $\mu^{21} = 4{,}160/5{,}200 = 0.8$.

(iv) $$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu^{12} - {}_tp_x^{12}\mu^{21} = \mu^{12}(1-{}_tp_x^{12}) - {}_tp_x^{12}\mu^{21}$$

and substituting the values from the answer to part (iii) gives

$$\frac{d}{dt}{}_tp_x^{12} = 0.2(1-{}_tp_x^{12}) - 0.8\,{}_tp_x^{12} = 0.2 - {}_tp_x^{12}.$$

$$\frac{d}{dt}{}_tp_x^{12} + {}_tp_x^{12} = 0.2$$

$$\left[\frac{d}{dt}{}_tp_x^{12}\right]e^t + {}_tp_x^{12}e^t = \frac{d}{dt}({}_tp_x^{12}e^t) = 0.2e^t$$

$${}_tp_x^{12}e^t = 0.2e^t + C$$

Since ${}_0p_x^{12} = 0$

$C = -0.2$

and ${}_t p_x^{12} = 0.2 - 0.2e^{-t}$.

For t = 3 days, ${}_t p_x^{12} = 0.2 - 0.2e^{-3} = 0.1900$.

In part (ii) minor variations on the exact derivation given above were permitted, but all the steps were required for full credit. In part (iii) some candidates attempted a solution using integral equations. A relatively common example argued that the required probability could be obtained from:

$${}_3 p_0^{12} = \int_0^3 e^{-0.2w}.0.2.e^{-0.8(3-w)}dw = 0.2\int_0^3 e^{-2.4}e^{0.6w}dw = 0.2e^{-2.4}\int_0^3 e^{0.6w}dw.$$

Evaluating this integral produces ${}_3 p_0^{12} = \dfrac{0.2}{0.6}e^{-2.4}\left[e^{0.6w}\right]_0^3 = \dfrac{0.2}{0.6}(e^{-0.6} - e^{-2.4}) = 0.1527$.

This is incorrect as it ignores the possibility that lives might oscillate between states 1and 2 between t = 0 and t = 3. It only considers those lives who move between states 1 and 2 with exactly one transition and do not return to state 1. However, this alternative shows considerable understanding of the process, and was given some credit.

11

(i) A Markov chain is a discrete time, discrete space Markov process

For a time-inhomogeneous Markov chain, the transition probabilities depend on the absolute values of time, rather than just the time difference.

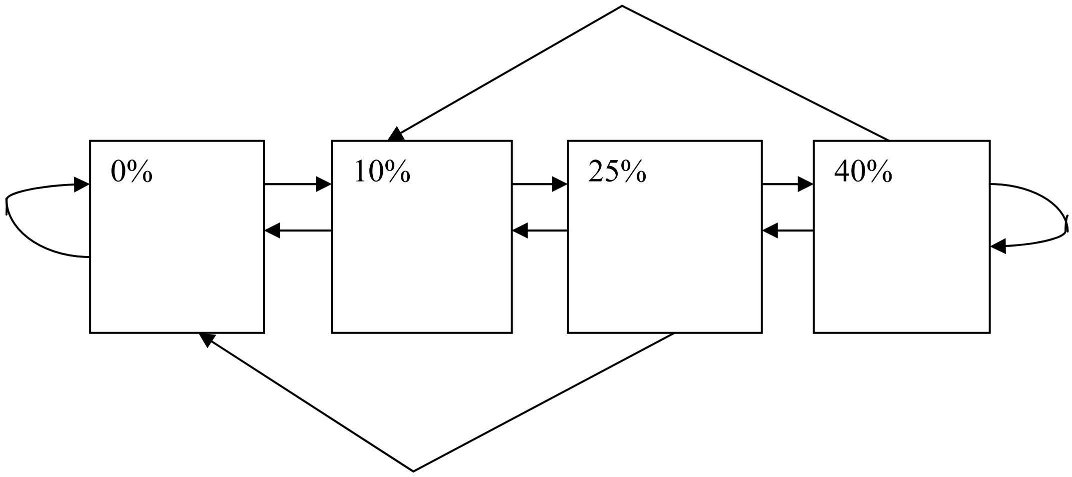
The value of "time" can be represented by many factors, for example the time of year, age or duration.

An example might be a No Claims Discount scheme where the probability of a claim reflects trends in accident frequency over time.

(ii) Both boundaries are mixed as policyholders can either stay in that state for consecutive periods or move back to another state.

E.g. When at the maximum 40% level, a policyholder who makes no claim will stay there the next year, whereas one who makes one claim will drop to the 25% level and one who makes more than one claim will drop to the 10% level.

(iii) Four states are required: 0%, 10%, 25% and 40%.



(iv) Prob [no claims in year] = $0.96^{12} = 0.6127$
Prob [exactly 1 claim in year] = 0.96^{11} (0.04) 12 = 0.3064
Prob [more than one claim in a year] = 1 – (0.6127 + 0.3064) = 0.0809

$$\pi\begin{pmatrix} 0.3873 & 0.6127 & 0 & 0 \\ 0.3873 & 0 & 0.6127 & 0 \\ 0.0809 & 0.3064 & 0 & 0.6127 \\ 0 & 0.0809 & 0.3064 & 0.6127 \end{pmatrix} = \pi$$

$$\pi_1 = 0.3873\pi_1 + 0.3873\pi_2 + 0.0809\pi_3 \qquad (1)$$
$$\pi_2 = 0.6127\pi_1 + 0.3064\pi_3 + 0.0809\pi_4 \qquad (2)$$
$$\pi_3 = 0.6127\pi_2 + 0.3064\pi_4 \qquad (3)$$
$$\pi_4 = 0.6127\pi_3 + 0.6127\pi_4 \qquad (4)$$
$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 . \qquad (5)$$

From (4) $\pi_4\left(1 - 0.6127\right) = 0.6127\pi_3$
so $\pi_4 = 1.5820\pi_3$

From (3) $\pi_2 = \pi_3\left(1 - 0.3064 \times 1.5820\right) / 0.6127$
so $\pi_2 = 0.8411\pi_3$

and (1) gives $\pi_1 = \pi_3(0.0809 + 0.3873 \times 0.8411)/(1 - 0.3837)$
so $\pi_1 = 0.6637\pi_3$

Using (5) we get $\pi_3\left(0.6637+0.8411+1+1.5820\right)=1$

so $\pi_3=0.2447$

so $\pi_4=1.5820\times 0.2447$ $\pi_4=0.3871$

In the long run 24.47% of policyholders are at the 25% level.

(v) Equal probability of an accident in every month is pretty unlikely.

Perhaps more accidents in winter when driving conditions are worse, or in summer, when mileage is higher.

The probability of a second claim may differ from the first and may be dependent upon the level the person is at (e.g. does it make a difference to the future premium?)

Claim probability may depend upon policyholder age/sex or car size/age, and on many other factors (occupation, geographical area, marital status, mileage, where car is stored, etc.)

Claim levels may be affected by the past history of a person's claims (so the process is no longer Markov).

Unrealistic to assume at most one claim per month.

Parts (i), (iii) and (v) of this question were well answered, though in part (i) it was not often clear how the examples given operated in discrete time. Part (ii) was very poorly attempted. In part (iv) a common error was to assume that the 0.04 claim rate is annual. This gave the answer that just under 4 per cent of policyholders were at the 25 per cent level. Candidates who made this error were penalised for using incorrect probabilities, but were given full credit for solving the equations to obtain the steady-state probabilities. In part (v) sensible suggestions other than those listed were given credit.

END OF EXAMINERS' REPORT