INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

28 April 2014 (am)

Subject CT4 – Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a new page.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 State the benefits of modelling in actuarial work. [4]

2 (i) Explain why data are subdivided into homogeneous groups when mortality investigations are conducted. [2]

(ii) List four factors, other than age and sex, by which mortality statistics are often subdivided. [2]

[Total 4]

3 Explain what a stochastic model is and how it differs from a deterministic model. [4]

4 (i) State the principle of correspondence as it relates to mortality investigations. [1]

Two small countries conduct population censuses on an annual basis. Country A records its population on 1 February every year based on an age definition of age last birthday. Country B records its population on every 1 August using a definition of age nearest birthday. Each country records deaths as they happen based on age next birthday.

Below are some data from the last few years.

Country A

Age last birthday	*Population 1 February 2011*	*Population 1 February 2012*	*Population 1 February 2013*
44	382,000	394,000	401,000
45	374,000	381,000	385,000
46	354,000	372,000	375,000

Country B

Age nearest birthday	*Population 1 August 2011*	*Population 1 August 2012*	*Population 1 August 2013*
44	382,000	394,000	401,000
45	374,000	381,000	385,000
46	354,000	372,000	375,000

In the combined lands of Countries A and B in the calendar year 2012 there were 4,800 deaths of those aged 46 next birthday and 4,500 deaths of those aged 45 next birthday.

The two countries decide to form an economic union, after which it will be mandatory to offer the same rates for life insurance to residents of each country.

(ii) Estimate the death rate at age 45 years last birthday for the two countries combined. [6]

(iii) Explain the exact age to which your estimate relates. [1]

[Total 8]

5 An investigation has been performed into risk factors for liver disease in persons currently resident in the United Kingdom (UK) and aged over 50 years. It considered the impact of three covariates: age at the start of the investigation, weekly alcohol consumption and previous residence in a tropical country.

The investigation used a Cox regression model for the hazard of developing the disease, $h(t)$, with three parameters, β_A, β_C, and β_T, as follows:

$$h(t) = h_0(t)\exp(\beta_A A + \beta_C C + \beta_T T).$$

A was defined as exact age at the start of the investigation less 50 years.

C represented weekly alcohol consumption, and took the value 1 if the person consumed more than the recommended maximum per week (a heavy drinker) and 0 otherwise.

T represented previous residence in a tropical country, and took the value 1 if the person had lived in a tropical country for more than 12 months and 0 otherwise.

(i) State the characteristics of a person to whom the baseline hazard, $h_0(t)$, applies. [1]

The results of the investigation revealed that the hazard was:

- twice as high for a heavy drinker aged 60 years exact at the start of the investigation than for a person aged 50 years exact at the start of the investigation who was not a heavy drinker, where neither had previously lived in a tropical country.
- four times as high for a heavy drinker who had previously lived in a tropical country for more than 12 months than for a non-heavy drinker of the same age who had not previously lived in a tropical country.
- three times as high for a person who had lived in a tropical country for more than 12 months than for a person of the same age and drinking habits who had always lived in the UK.

(ii) Calculate β_A, β_C, and β_T. [5]

The probability of a person aged 50 years exact at the start of the investigation, who does not drink heavily and has always lived in the UK remaining free of the disease for 10 years is 0.8.

(iii) Show that the probability of a person of the same age and drinking habits, who has lived for more than 12 months in a tropical country, remaining free of the disease for 10 years is slightly over one half. [4]

[Total 10]

6 In the Poisson model, if the average number of events occurring to each member of a population in a given period of time is λ, then the probability of observing exactly d events occurring to any one individual in the same period of time is:

$$\Pr[D = d] = \frac{\exp(-\lambda)\lambda^d}{d!}.$$

(i) Derive the maximum likelihood estimator under the Poisson model of the average rate at which events occur, μ, in a population where the exposed to risk for each person i is E. [4]

A university runs a bus service between its teaching campus and its student halls of residence. Traffic conditions mean that the arrival of buses at the bus stop on the teaching campus can be considered to follow a Poisson process.

The university decided to commission a study of how long students typically have to wait at the bus stop for a bus to arrive. Students were asked to record the time they arrive at the stop, and the time the next bus arrived. Students who became tired of waiting at the stop and left before the next bus arrived were asked to record the time they left. Below are given data from 10 students.

Student	*Time arrived*	*Time left (if left before next bus arrived)*	*Time next bus arrived*
1	4.00 p.m.		4.05 p.m.
2	4.10 p.m.		4.35 p.m.
3	4.20 p.m.	4.30 p.m.	
4	4.30 p.m.		4.35 p.m.
5	4.40 p.m.		4.50 p.m.
6	4.45 p.m.		4.50 p.m.
7	4.55 p.m.	5.05 p.m.	
8	5.00 p.m.	5.20 p.m.	
9	5.10 p.m.	5.40 p.m.	
10	5.10 p.m.		6.10 p.m.

(ii) Calculate the maximum likelihood estimate of the hourly rate at which buses arrive at the bus stop, using the Poisson estimator, and assuming that only one bus arrived at any given time. [3]

(iii) Comment on the use of the Poisson model for this investigation. [3]

[Total 10]

7 A team of medical researchers is interested in assessing the effect of a certain condition on mortality. The condition is, in itself, non-fatal and is curable, but is believed to increase the risk of death from heart disease. The team proposes to use a model with four states: (1) "Alive, without condition", (2) "Alive, with condition", (3) "Dead from heart disease" and (4) "Dead from other causes".

(i) Draw a diagram showing the possible transitions between the four states. [2]

Let the transition intensity between state i and state j at time $x+t$ be μ_{x+t}^{ij}. Let the probability that a person in state i at time x will be in state j at time $x+t$ be ${}_t p_x^{ij}$.

(ii) Show, from first principles, that

$$\frac{d}{dt}({}_t p_x^{24}) = {}_t p_x^{21}\mu_{x+t}^{14} + {}_t p_x^{22}\mu_{x+t}^{24}.$$ [5]

An empirical investigation using data for persons aged between 60 and 70 years produces the following results:

Waiting time in state "Alive, without condition" is 2,046 person-years
Waiting time in state "Alive, with condition" is 1,139 person-years
10 deaths from heart disease to persons "Alive, without condition"
30 deaths from other causes to persons "Alive, without condition"
25 deaths from heart disease to persons "Alive, with condition"
20 deaths from other causes to persons "Alive, with condition"

(iii) Show that there is a statistically significant difference (at the 95% confidence level) between the death rates from heart disease for persons with and without the condition. [5]

[Total 12]

8 (i) Describe what is meant by censoring in the context of a mortality investigation. [1]

(ii) Explain what right-censoring, left-censoring and interval censoring are, giving an example of each. [3]

A toy manufacturer is testing the lifetime of its new electric children's toy. 500 are set going at 9 a.m. one morning on test rigs plugged into the electricity supply and are run until 5 p.m. the next day or until they fail, whichever comes first. Unfortunately the cleaner unplugged a test rig on which 17 toys were still working at 7 p.m. on the first evening in order to plug his floor polisher in. Then, as he left work three hours later, he took three of the still working toys for his children to play with. Of the other 480 toys it was found that 12 failed after four hours, 25 failed after 11 hours and a further 8 failed after 31 hours.

(iii) Explain which forms of censoring are present in this investigation. [2]

(iv) Calculate the Nelson-Aalen estimate of the survival function. [5]

(v) Sketch a graph of the Nelson-Aalen estimate of the survival function, labelling the axes. [2]

(vi) Comment on the length of time for which a new toy has a 60% probability of surviving. [1]

[Total 14]

9 (i) (a) State three features which are desirable when a graduation is performed.

(b) Explain why they are desirable. [3]

The actuary to a large pension scheme has attempted to graduate the scheme's recent mortality experience with reference to a table used for similar sized schemes in a different industry. He has calculated the standardised deviations between the crude and the graduated rates, z_x, at each age and has sent you a printout of the figures over a small range of ages. Unfortunately the dot matrix printer on which he printed the results was very old and the dots which would form the minus sign in front of numbers no longer function, so you cannot tell which of the standardised deviations is positive and which negative. Below are the data which you have.

Age	*Standardised deviation*
60	2.40
61	0.08
62	0.80
63	0.76
64	1.04
65	0.77
66	1.30
67	1.76
68	0.28
69	0.68
70	0.93

(ii) (a) Carry out an overall goodness-of-fit test on the data.

(b) Comment on your result. [5]

(iii) (a) List four defects of a graduation which the test you have carried out would fail to detect.

(b) Suggest, for each of the defects, a test which could be used to detect it. [4]

(iv) Carry out one of the tests suggested in part (iii)(b). [3]
[Total 15]

10 An industrial kiln is used to produce batches of tiles and is run with a standard firing cycle. After each firing cycle is finished, a maintenance inspection is undertaken on the heating element which rates it as being in Excellent, Good or Poor condition, or notes that the element has Failed.

The probabilities of the heating element being in each condition at the end of a cycle, based on the condition at the start of the cycle are as follows:

START	*END* Excellent	*Good*	*Poor*	*Failed*
Excellent	0.5	0.2	0.2	0.1
Good		0.5	0.3	0.2
Poor			0.5	0.5
Failed				1

(i) Write down the name of the stochastic process which describes the condition of a single heating element over time. [1]

(ii) Explain whether the process describing the condition of a single heating element is:

(a) irreducible.
(b) periodic. [2]

(iii) Derive the probability that the condition of a single heating element is assessed as being in Poor condition at the inspection after two cycles, if the heating element is currently in Excellent condition. [2]

If the heating element fails during the firing cycle, the entire batch of tiles in the kiln is wasted at a cost of £1,000. Additionally a new heating element needs to be installed at a cost of £50 which will, of course, be in Excellent condition.

(iv) Write down the transition matrix for the condition of the heating element in the kiln at the start of each cycle, allowing for replacement of failed heating elements. [2]

(v) Calculate the long term probabilities for the condition of the heating element in the kiln at the start of a cycle. [4]

The kiln is fired 100 times per year.

(vi) Calculate the expected annual cost incurred due to failures of heating elements. [2]

The company is concerned about the cost of ruined tiles and decides to change its policy to replace the heating element if it is rated as in Poor condition.

(vii) Evaluate the impact of the change in replacement policy on the profitability of the company. [6]

[Total 19]

END OF PAPER