The Actuarial Profession
making financial sense of the future

Calibrating Dependencies
Case Study based on Market Returns

October 2010

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Workshop Overview

- Calibrating Tail Correlations
- Calibrating Copulas
- Multiple comparisons
- Input Consistency
- Conclusions and Questions

This presentation is based on **Measurement and modelling of dependencies in economic capital** by Richard Shaw, Andrew Smith & Grigory Spivak (2010)

http://www.actuaries.org.uk/sites/all/files/documents/pdf/sm20100510.pdf

Calibrating Tail Correlations

UK and Denmark Monthly Equity Returns 1970-2009

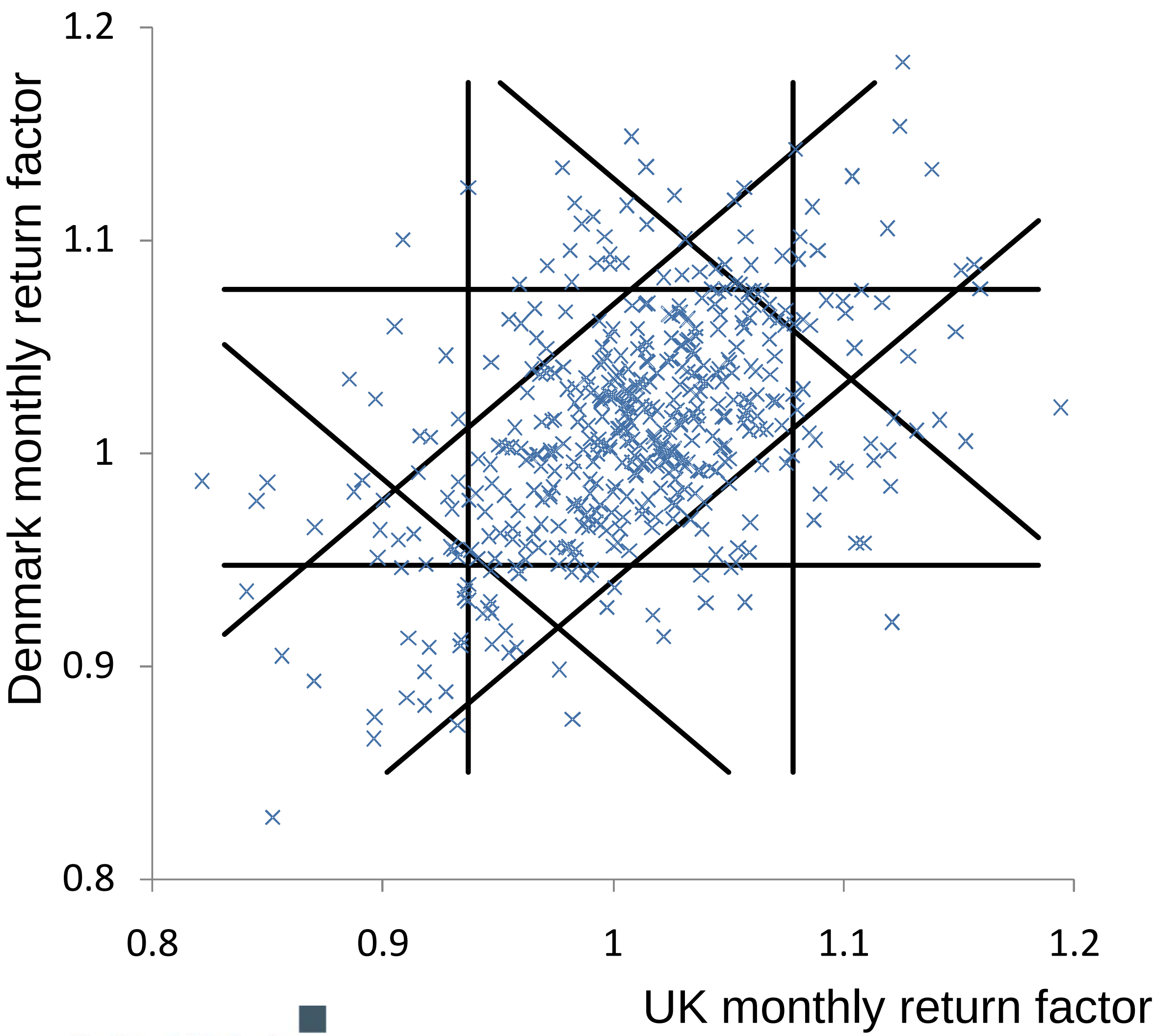
Is this a simple linear relationship of the form

y = mx + c + error

Errors independent of x.

Source: Datastream

90%-ile lines

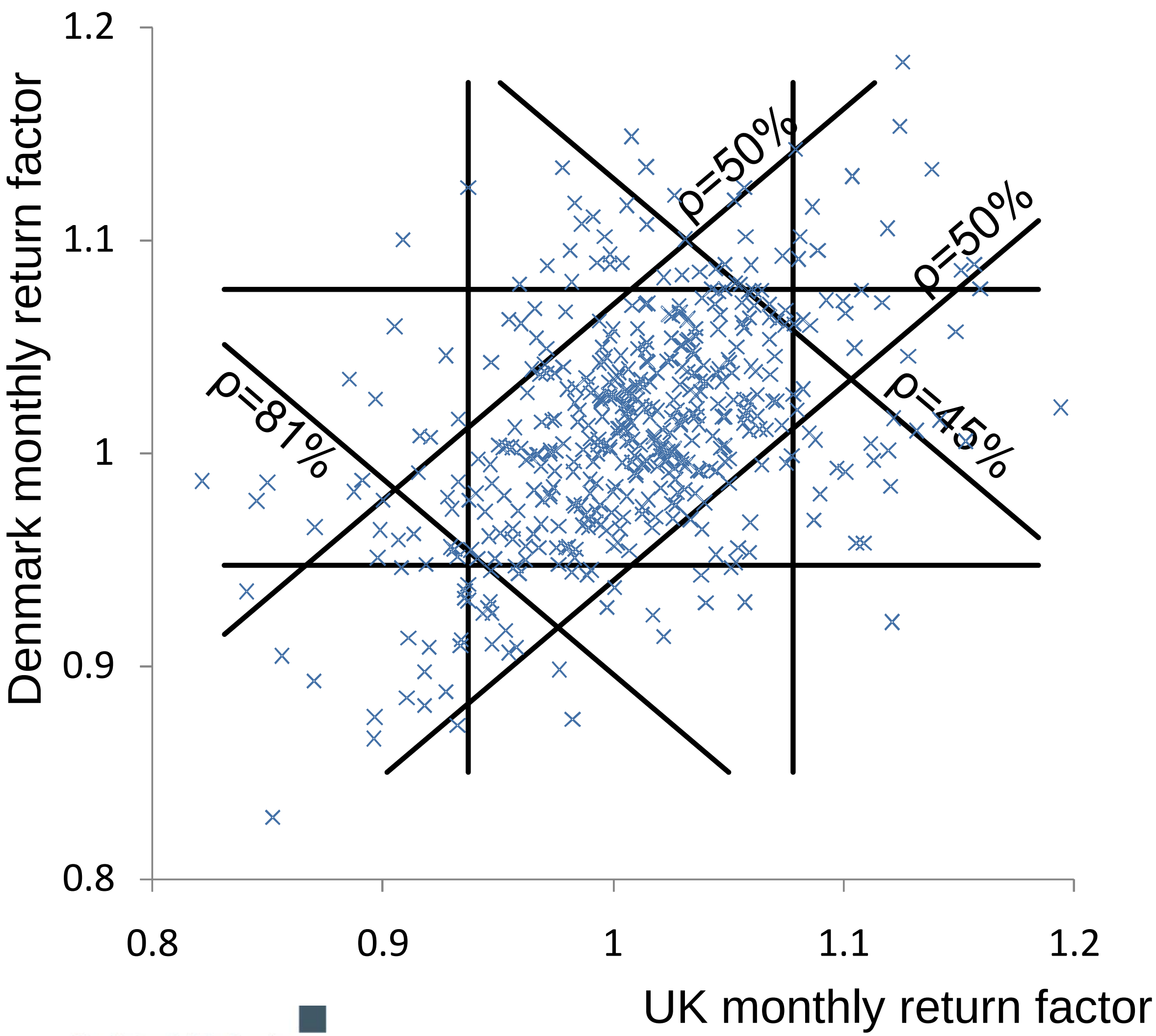


Each line separates 90% of the data from 10% of the data.

Aggregation Formula

- Consider a confidence level α with $\frac{1}{2} < \alpha < 1$ (eg $\alpha = 0.995$)
- Denote quantiles by $q_{1-\alpha}$ and q_{α}
- Let X and Y be risk drivers
  - With $q_{1-\alpha}(X) = q_{1-\alpha}(Y) = -1$
  - And $q_{\alpha}(X) = q_{\alpha}(Y) = 1$, without loss of generality by scaling
- Then, for elliptical distributions with correlation ρ:
  - Sums: $q_{\alpha}(X+Y) = -q_{1-\alpha}(X+Y) = \sqrt{(2+2\rho)}$
  - Differences: $q_{\alpha}(X-Y) = -q_{1-\alpha}(X-Y) = \sqrt{(2-2\rho)}$
- This gives four ways (for any α) for estimating correlation ρ
- Let's call these "tail correlations"

Solved Values of Tail Correlations



These are the correlations which, when substituted into a "correlation and square root" aggregation formula, gives the correct capital requirement.

Note the higher correlation in the South-West corner. Some would interpret this as correlations increasing in adverse situations, ie equity market falls.

For context, the Pearson correlation between UK and Denmark returns is 46%.

Tail Correlations
Strengths and Weaknesses

Strengths

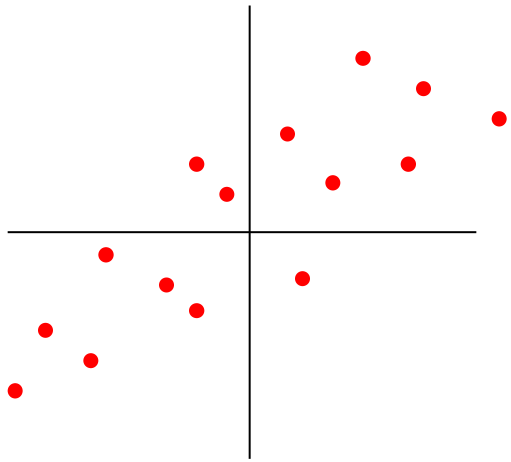
- Measures regions that likely correspond to insurers' ruin regions (half-spaces rather than quadrants)
- Captures different correlation effects in four quadrants
- Formulas easy to calculate
- Visual representation assists communication

Weaknesses

- Relies on firm's risks exposures being linear
  - This is about a firm's asset and liability valuation function, and is nothing to do with linear/nonlinear dependency in risk drivers
- Extension to other linear combinations or multiple risks involves interpolation
- Not easy to implement with Monte Carlo

Calibrating Copulas

Recall – The Gauss Copula Idea



Bivariate Normal Z
sims = N

Transform to uniform

Transform to desired marginal distribution

Transform to uniform. Replace n^{th} smallest by $u = n/(N+1)$

Transform to desired marginal distribution

For d dimensions, specify
- Marginal distributions, d times
- Correlations: d d matrix

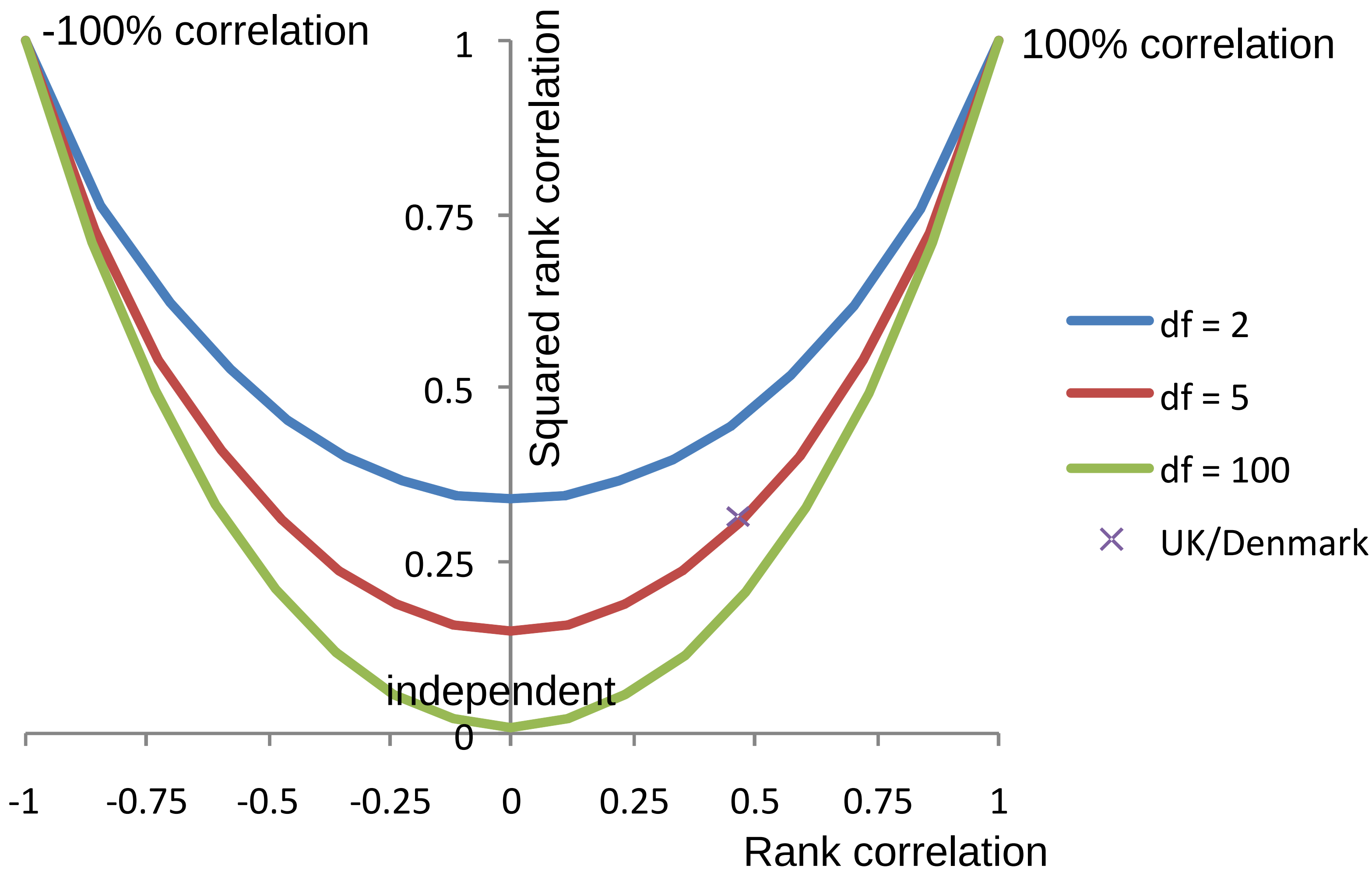
This is a Gauss copula. Generalisations include:
- T copula
  - Divide all Z's by a common $\sqrt{\chi^2_{df}}$
- Individuated T copula
  - Divide each Z_i by its own $\sqrt{\chi^2_{df(i)}}$
  - Divisors are increasing functions of each other

Rank Correlations and Squared Rank Correlations

- Suppose we have 2 dimensions, $N = 480$ observations
- In each dimension, replace n^{th} smallest by $u = n/(N+1)$

Correlation matrix	U_{dk}	U_{uk}	$(2U_{dk}-1)^2$	$(2U_{uk}-1)^2$
U_{dk}	1	Rank correlation	0	
U_{uk}	Rank correlation	1		0
$(2U_{dk}-1)^2$	0		1	Squared rank correlation
$(2U_{uk}-1)^2$		0	Squared rank correlation	1

Rank Correlation and Squared Rank Correlation: UK and Denmark



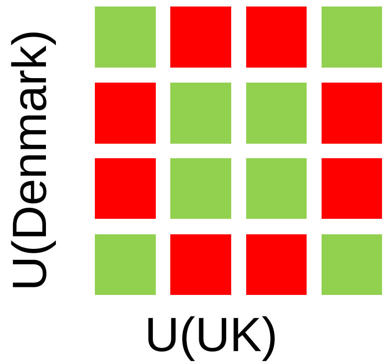
Cross Correlations: Denmark & UK

Correlation matrix	U_{dk}	U_{uk}	$(2U_{dk}-1)^2$	$(2U_{uk}-1)^2$
U_{dk}	1	46.2%	0	-12.0%
U_{uk}	46.2%	1	-12.5%	0
$(2U_{dk}-1)^2$	0	-12.5%	1	31.4%
$(2U_{uk}-1)^2$	-12.0%	0	31.4%	1

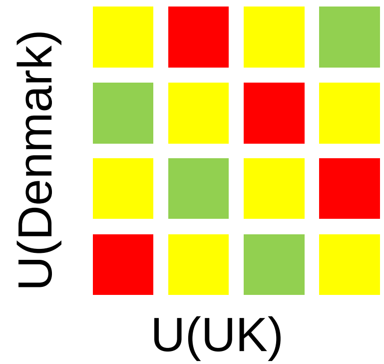
Empirical estimates for UK/Danish equity returns

Interpreting Cross Correlations

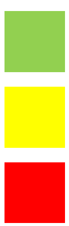
Positive rank correlation



Positive squared rank correlation



Positive cross correlation



Increase density

Reduce density

Notes:
Extreme case of squared rank correlation= 1 is attained for spider copula (mixture of increasing and decreasing copulas)
Individuated T copula (and so gauss and T copula) imply cross correlations are zero.

Copula Approaches: Strengths and Weaknesses

Strengths

- Invariant under increasing transforms of x and y (for example, taking logs)
- Captures all the information in the dependency structure without reference to marginal distributions
- Allows unconstrained choice of marginal distributions
- Suitable for Monte Carlo

Weaknesses

- May be difficult to find copula functions to capture specific data features
- For example, negative cross terms
- Seldom amenable to analytical calculations

Multiple Comparisons

Purpose of Multiple Comparisons

- We have observed specific features of the UK / Danish data set
  - Are these real features of the underlying distribution or sampling artefact?
  - Difficult to analyse mathematically because of the need to start with a hypothesis about the "true" copula
- An alternative is to test consistency across multiple economies in the search for "stylised facts".
  - We could also test robustness across different time periods
  - Recognise that the economies are not independent, so feature seen in all economies could still be statistical fluke

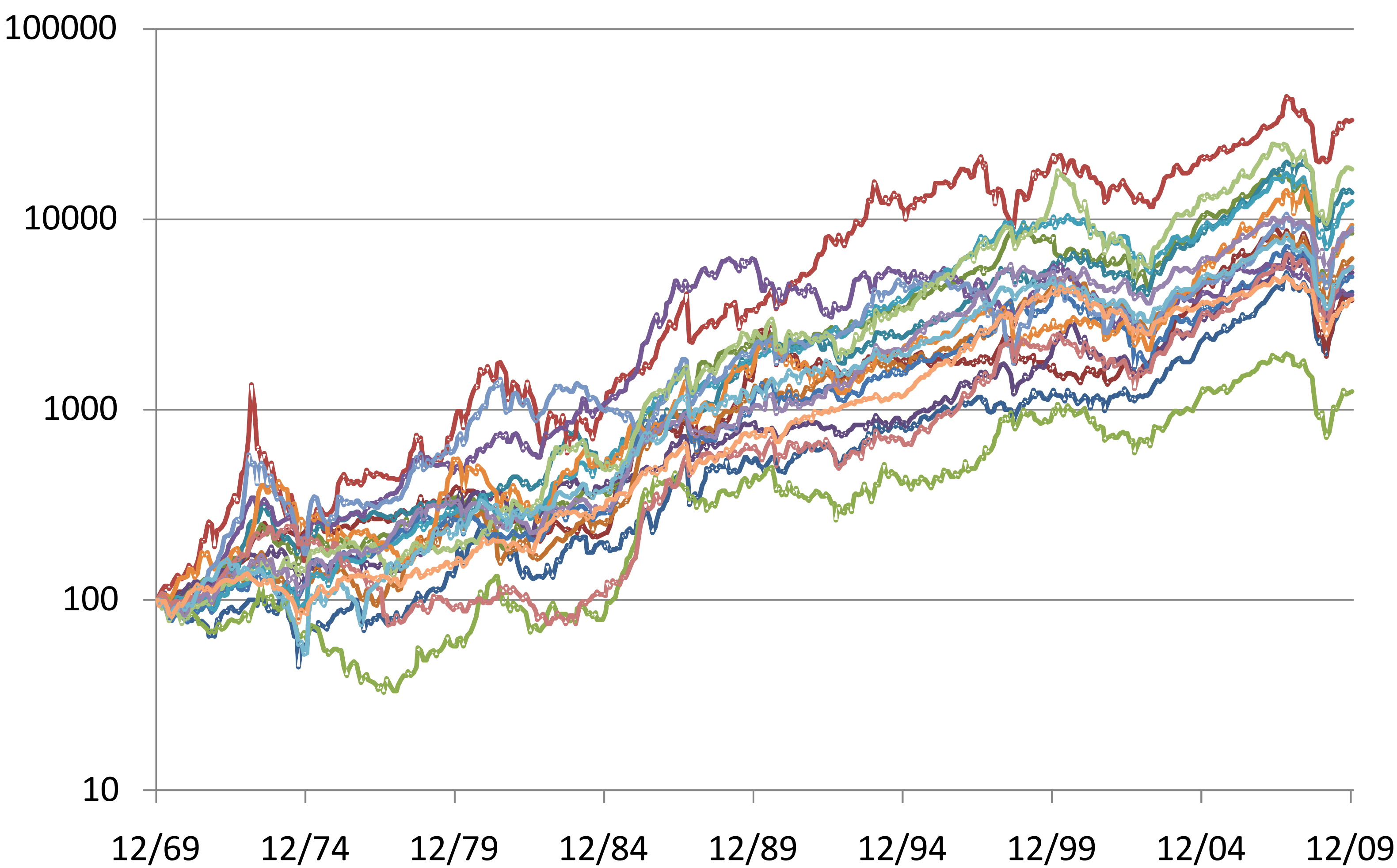
Our Chosen Data Set

- MSCI equity indices
- 31/12/1969 – 31/12/2009
- Monthly total return indices, coverage for 480 months
- In US Dollars
- 18 series representing different countries

  Countries represented: Australia, Austria, Belgium, Canada, Denmark, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Singapore, Spain, Sweden, Switzerland, UK, US

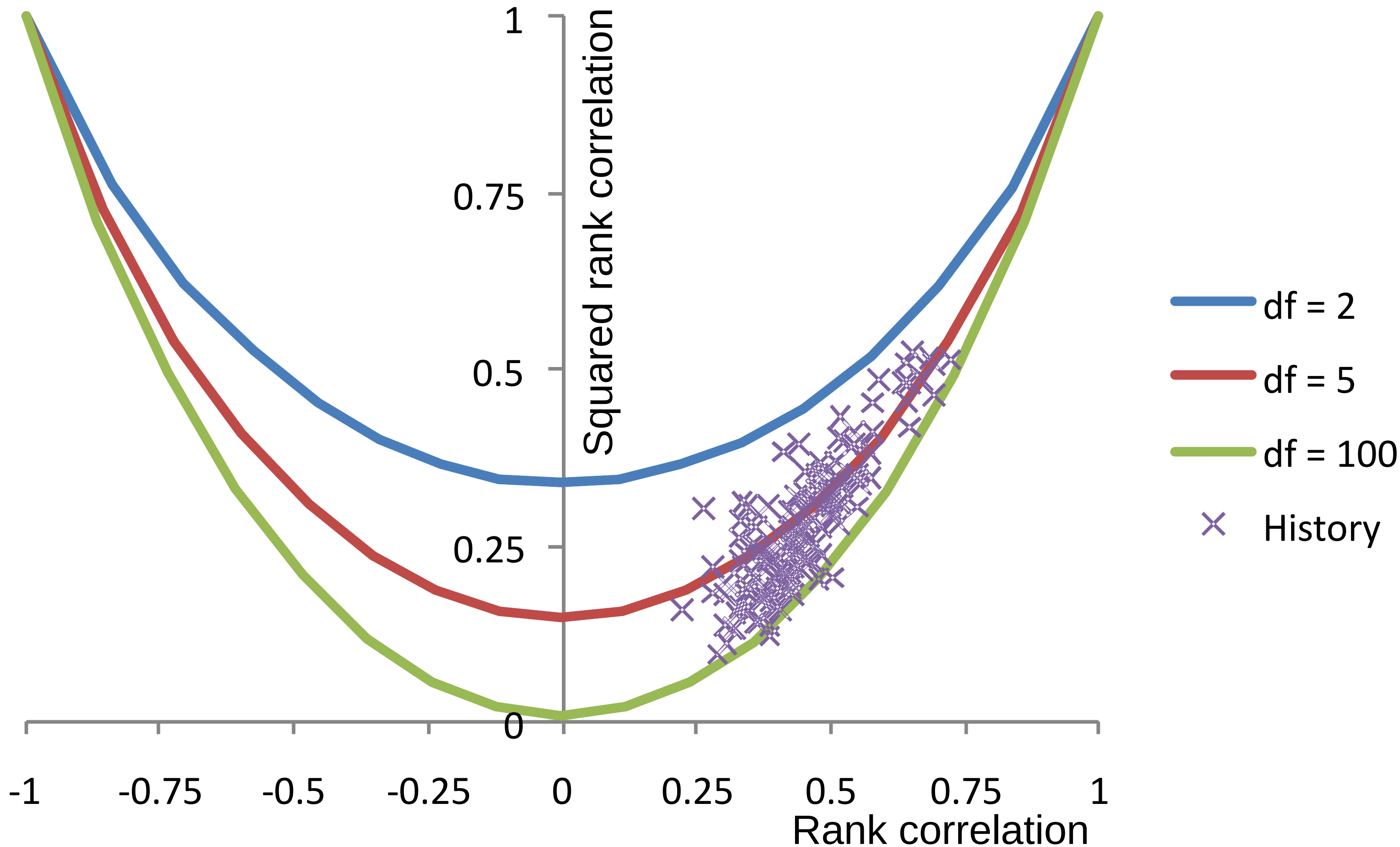
- In this presentation we analyse only two-dimensional dependency. There are 153 pairs of countries for which this can be analysed. In the charts that follow, each country pair is represented by one point.

Equity Total Return Data 1970-2010

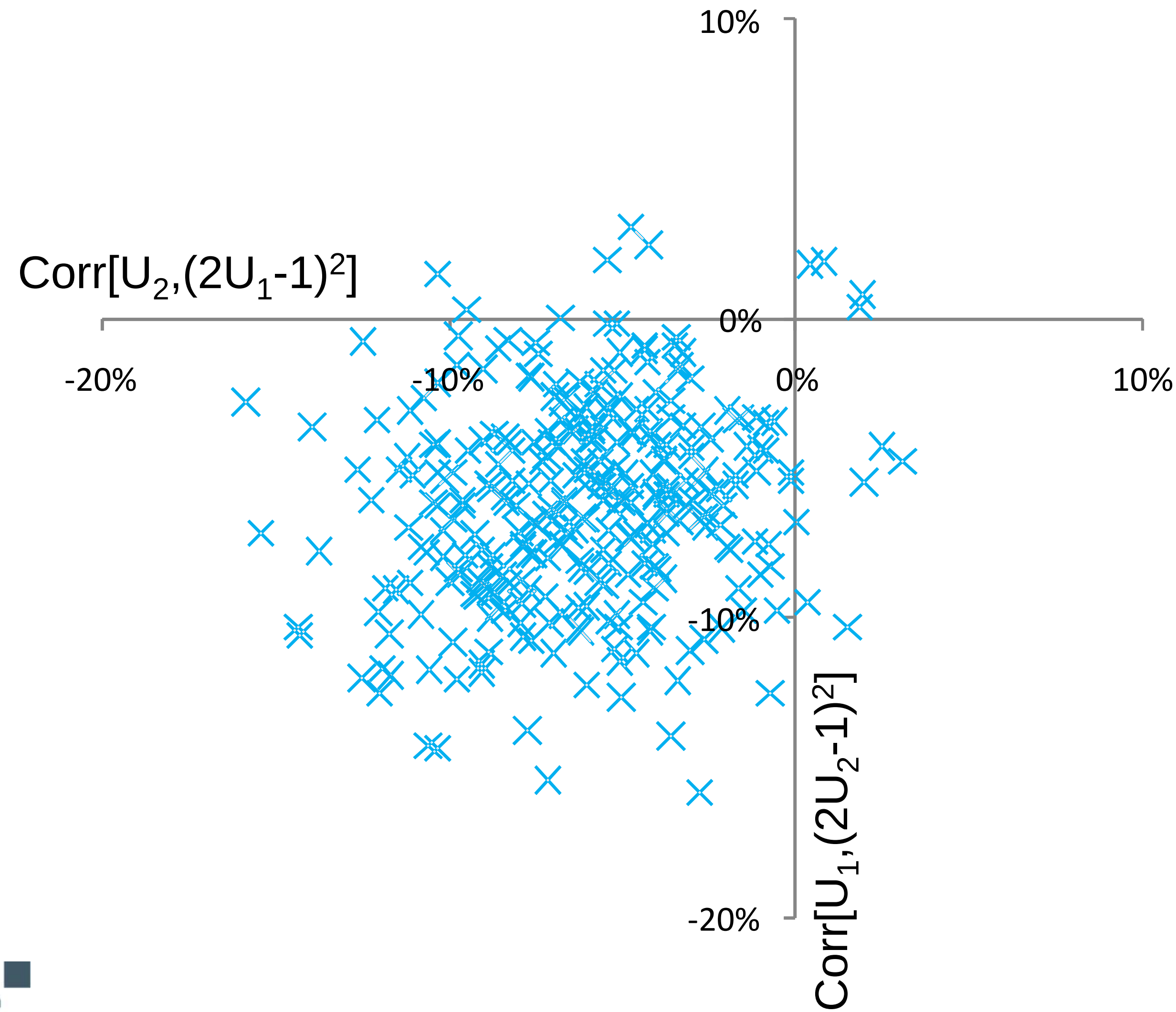


Source: Datastream

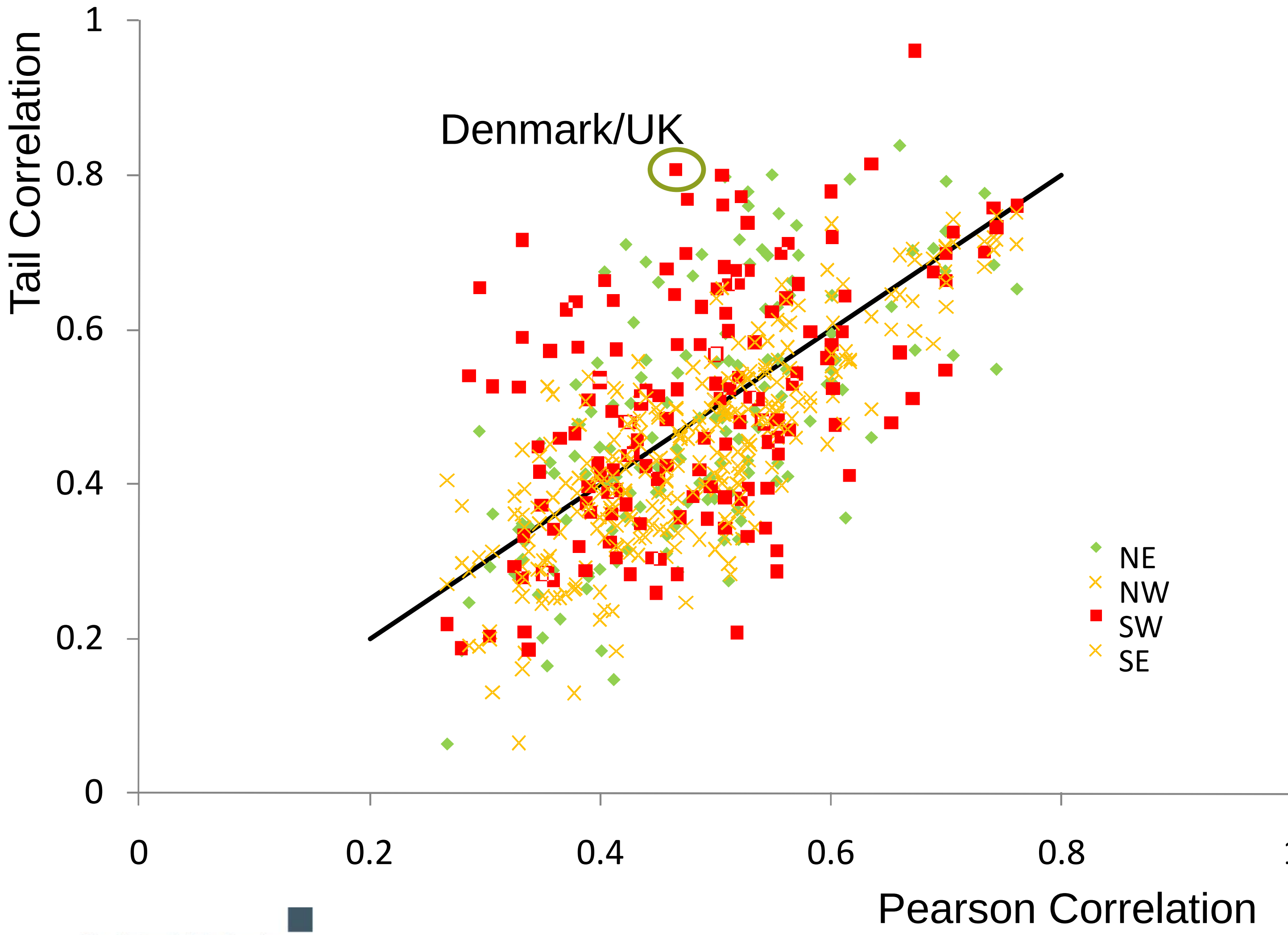
Fitting a T Copula: 5 df is typical
Significant Rejection of Gauss Copula



Negative Cross Correlations: Systematic Feature Rejection of T copula (standard or individuated)



Tail Correlations (at 90% confidence)



Denmark/UK is an outlier. In general this plot offers little evidence that correlations increase specifically in bad outcomes.

Are Pearson correlations a good enough guide to tail correlations for risk aggregation?

Assumption Consistency

The Mystery of the Missing Correlation

$$R = \begin{pmatrix} 1 & 0.5 & 0 & 0.5 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & r_{34} \\ 0.5 & 0 & r_{34} & 1 \end{pmatrix}$$

What are the feasible values for r_{34}?
Triangle inequality: $\cos^{-1}(r)$ is a metric
Positive definite condition

Values and Tests

Minimum value of r_{34}	Maximum value of r_{34}	Description
-1.000	-0.866	Not PSD, triangle inequality fails
-0.866	-0.833	Not PSD even though triangle inequality satisfied
-0.833	-0.810	Positive definite, but cannot be a rank correlation matrix for gauss nor spider copulas
-0.810	-0.500	Can be a rank correlation matrix for a Gauss copula but not a spider copula
-0.500	0.428	Can be a rank correlation matrix for both Gauss and spider copulas
0.428	0.500	Can be a rank correlation matrix for a spider copula but not a Gauss copula
0.500	0.866	Not PSD even though triangle inequality satisfied. This range provides interesting examples of matrices that are not PSD even though all elements are positive and the triangle properties hold.
0.866	1.000	Not PSD, triangle inequality fails.

Conclusions and Questions

Conclusions

- Simple graphical analysis of historic data and analytical aggregation may be as robust as sophisticated Monte Carlo models for capital calculations
- There is a need to document stylised facts about dependencies in historic data and develop classes of models to capture these
  - Negative cross correlations may be a bigger issue than tail correlations
- Care in analysis of bivariate data is no guarantee of a consistent multivariate model. This generates difficult calibration trade-offs so firms need a “calibration plan B”

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and are encouraged.

The views expressed in this presentation are solely those of the presenters.

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