

**ASSET MODELLING - EMPIRICAL TESTS OF
YIELD CURVE GENERATORS**

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ASSET MODELING — EMPIRICAL TESTS OF YIELD CURVE GENERATORS

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Some empirical tests for stochastic yield curve generators are proposed. The basis of the tests is to compare history to the model for the conditional distributions of various yield-curve statistics given the three-month rate. The term structure of US Treasury instruments will be used as an example.

ASSET MODELING — EMPIRICAL TESTS OF YIELD CURVE GENERATORS

Based on arbitrage arguments, yield curves can be produced from stochastic generators of the movements of just the short-term rates. There are a number of criteria used to evaluate short-term rate generators, usually focussing on how well they capture the dynamics of the short-term process. Here a series of tests is proposed for the resulting yield curves.

These proposed tests all ask the same question: what is the conditional distribution of the yield curve given the three-month rate? There is more than one test because the yield curve is not described by a single parameter. In this paper the shape of the yield curve will be measured by yield spreads – like 3-year rate minus 1-year rate – and differences between these spreads. Thus the tests will look at the distribution of the spreads and differences given the three-month rate.

The idea is that the distributions produced by many simulations from the model should compare well with historical distributions. This might not be reasonable for short-range simulations, however, as present conditions would be likely to dominate the outcomes. Thus the tests are done for ending distributions from longer-range simulations. The residual effect of current conditions might make the dispersion less than historical, nonetheless, so a perfect match would not be anticipated. Requiring only a reasonable approximation to historical conditional distributions turns out to be enough to disqualify various models.

INTRODUCTION

A good deal of the work in yield-curve simulation is done for the purpose of pricing or evaluating the pricing of interest-rate options. For this purpose it is important that the model captures the current yield curve and its short-term dynamics as precisely as possible. These models may have many parameters, so that every bond option can be fit, and the parameters might be updated daily. This approach could be important to insurers who are actively trading bond options. However, the usual emphasis in DFA modeling is a little different. Know-

ing the risks inherent in holding various investment mixes over a longer time frame is an area of particular focus. A wide variety of yield curves should be produced to test this, but the model producing the widest variety is not necessarily the most useful – the different yield curves should be produced in relative proportion to their probability of occurring. It would be nice if in addition the short-term forecasts were very close for the current yield curve and option prices, but this is less important for DFA than it is for option trading.

Historical data on the distribution of yield curves can be used to test the reasonability of the distribution of curves being produced by any given model. However, it is not reasonable to expect that the probability of yield curves in a small given range showing up in the next two or three years is the same as their historical appearance. Some recognition needs to be given to the current situation and the speed at which changes in the curve tend to occur. Care also needs to be exercised in the selection of the historical period to which comparisons are to be made. The years 1979-81 exhibited dramatic changes in the yield curve, and the analyst needs to consider how prominent these years will be in the history selected. It seems reasonable that using a period beginning in the 1950's will give this unusual phase due recognition without over-emphasizing it.

The following are proposed as general criteria that a model of the yield curve should meet:

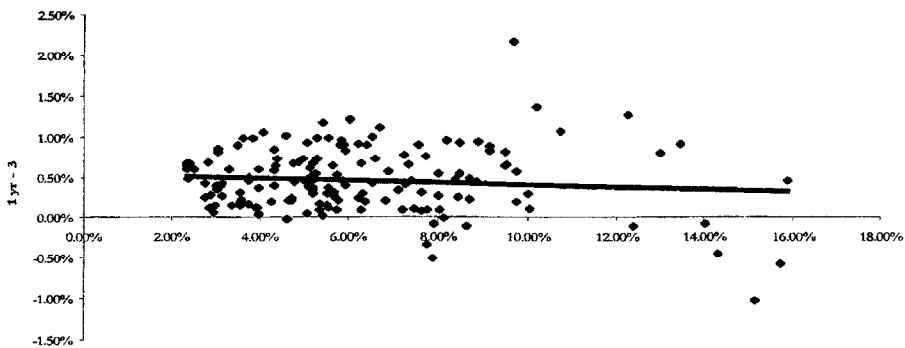
- It should closely approximate the current yield curve.
- It should produce patterns of changes in the short-term rate that match those produced historically.
- Over longer simulations, the ultimate distributions of yield curve shapes it produces, given any short-term rate, should match historical results.

This last criterion looks at the contingent distributions of yield curve shapes given the short-term rate. Thus it allows for the possibility that the distribution of short-term rates simulated even after several years will not match the diversity of

historical rates. But it does require that for any given short-term rate the distribution of yield curves should be as varied as seen historically for that short-term rate. It could be argued that somewhat less variability would be appropriate, and this may be so. How much less would be a matter of judgment, but too little variation in this conditional distribution would seem ill-advised when generating scenarios to test investment strategies against.

To measure the distribution of yield curve shapes, some shape descriptors are needed. The ones used here are based on differences of interest rates of different maturities. The first measures are just the successive differences in yield rates for 3-month, 1-year, 3-year, and 10-year instruments. Then the differences in these differences are taken, and finally the differences of those second differences. The first differences quantify the steepness of different parts of the yield curve. These would be zero for a flat curve. The second differences quantify the rate of change in the steepness as you move up the curve. These would be zero for a linearly rising curve. The third difference would be zero for a quadratic curve, and so quantifies the degree to which the curve is not quadratic.

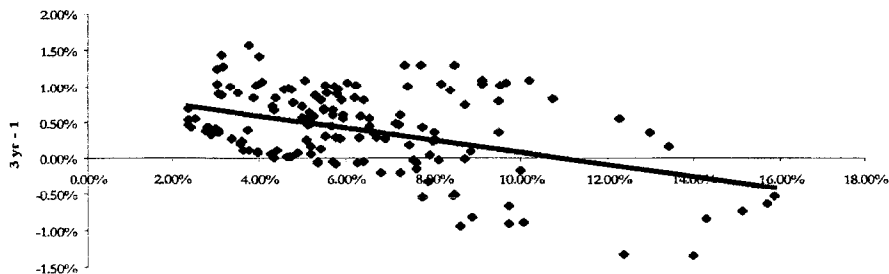
These shape measures will be reviewed historically as a function of the 3-month rate. The patterns for these six measures are graphed below along with the regression lines against the 3-month rate. It is interesting to note that the 1-year / 3-month yield spread appears to be independent of the 3-month rate, but the longer-term spreads appear to decline slightly with higher 3-month rates. At least in the US economy, when the short-term rates are high, the long-term rates tend to show less response, perhaps because investors expect the short-term rates to come down, and so the yield curve flattens out or even shows reversals (i.e., short-term rates higher than long-term). It might be argued that the slopes of the regression lines are small enough compared to the noise that they should not be considered significant. It turns out, however, that in testing models against this data the non-significance of the slope is a most significant issue – most models tend to produce more steeply falling slopes than the data shows.



3 mo



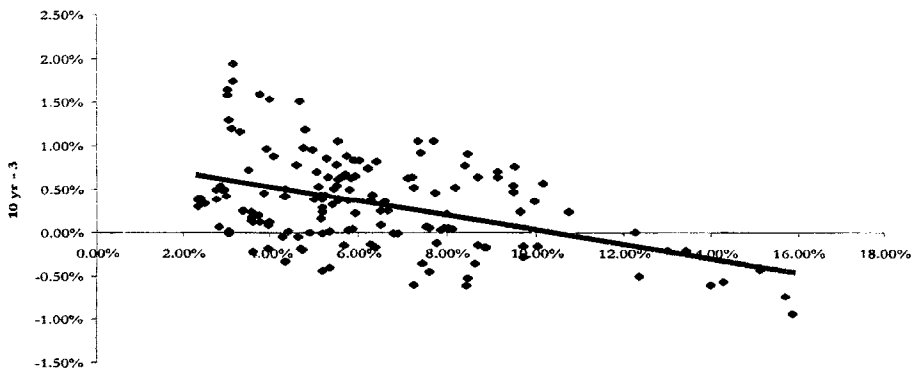
Historical 3 Year - 1 Year



3 mo



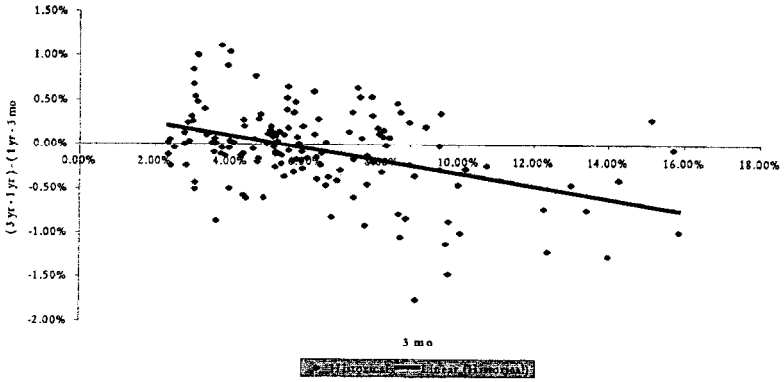
Historical 10 Year - 3 Year



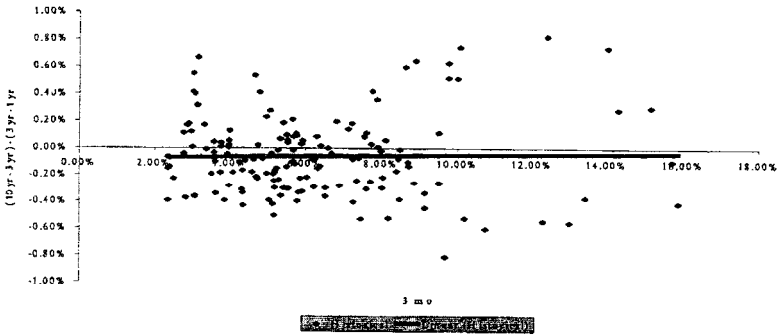
3 mo



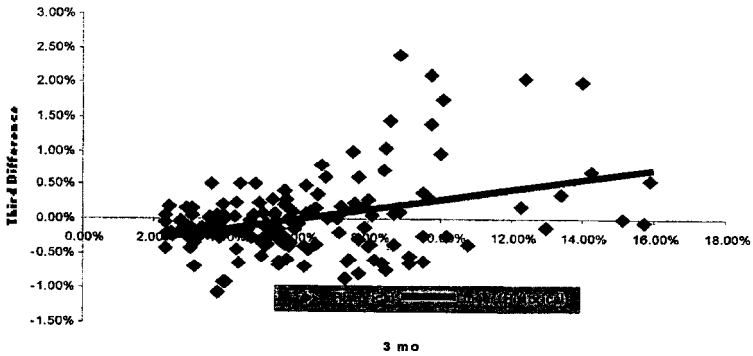
Historical Second Difference - Short



Historical Second Difference - Long



Historical Third Difference



YIELD CURVE MODELS

Typically the short-term interest rate, denoted as r , is modeled directly, and longer-term rates are inferred from the implied behavior of r , along with market considerations. The modeling of r is usually done as a continuously fluctuating diffusion process. This is based on Brownian motion. A continuously moving process is hard to track, and processes with random elements do not follow a simple formula. These processes are usually described by the probability distribution for their outcomes at any point in time. A Brownian motion has a simple definition for the probabilities of outcomes: the change from the current position between time zero and time t is normally distributed with mean zero and variance $\sigma^2 t$ for some σ . If r is the short-term interest rate and it follows such a Brownian motion, it is customary to express the instantaneous change in r by $dr = \sigma dz$. Here z represents a Brownian motion with $\sigma=1$. If r also has a trend of bt during time t , this could be expressed as $dr = bdt + \sigma dz$.

Cox, Ingersoll and Ross (*A Theory of the Term Structure of Interest Rates* Econometrica 53 March 1985) provided a model of the motion of the short-term rate that has become widely studied. In the CIR model r follows the following process:

$$dr = a(b - r)dt + sr^{1/2}dz.$$

Here b is the level of mean reversion. If r is above b , then the trend component is negative, and if r is below b it is positive. Thus the trend is always towards b . The speed of mean reversion is expressed by a , which is sometimes called the half-life of the reversion. Note that the volatility depends on r itself, so higher short-term rates would be associated with higher volatility. The period 1979-81 had high rates and high volatility, and studies that emphasize this period have found that the power of $1/2$ on r is too low. It appears to be about right in longer studies however.

Nonetheless, the CIR model fails to capture other elements of the movement of short-term rates. There have been periods of high volatility with low interest rates, and the rates sometimes seem to gravitate towards a temporary mean for a while, then shift and go towards some other. One way to account for these features is to let the volatility parameter s and the reversion mean b both be stochastic themselves.

Andersen and Lund (Working Paper No. 214, Northwestern University Department of Finance) give one such model:

$$dr = a(b - r)dt + sr^k dz_1 \quad k > 0$$

$$d \ln s^2 = c(p - \ln s^2)dt + v dz_2$$

$$db = j(q - b)dt + wb^{1/2} dz_3$$

Here there are three standard Brownian motion processes, z_1 , z_2 , and z_3 . The rate r moves subject to different processes at different times. It always follows a mean-reverting process, with mean b . But that mean itself changes over time, following a mean-reverting process defined by k , q , and w . The volatility parameter s^2 also varies over time via a mean reverting geometric Brownian motion process (i.e., Brownian motion on the log). In total there are eight parameters: a , c , j , k , p , q , v , and w and three varying factors r , b , and s .

Models of the short-term rate can lead to models of the whole yield curve. This is done by modeling the prices of zero-coupon bonds with different maturities all paying \$1. If $P(T)$ is the current price of such a bond for maturity T , the implied continuously compounding interest rate can be shown to be $-\ln[P(T)]/T$. $P(T)$ itself is calculated as the risk adjusted discounted expected value of \$1. Here "discounted" means continuously discounted by the evolving interest rate r , and "expected value" means that the mean discount is calculated over all possible paths for r . This can be expressed as:

$$P(T) = E^*[\exp(-\int_0^T r_t dt)]$$

Where r_t is the interest rate at time t , the integral is over the time period 0 to T , and E^* is the risk-adjusted expected value of the results of all such discounting processes.

If E were not risk adjusted, $P(T)$ could be estimated by many instances of simulating the r process to time T over small increments and then discounting back over each increment. The risk-adjusted expected value is obtained by using a risk-adjusted process to simulate the r 's. This process is like the original process except that it tends to produce higher r 's over time. These higher rates provide a reward for bearing the longer-term interest rate risk. Increasing the trend portion of the diffusion process produces the adjusted process. In the CIR model it is increased by λr , where λ is called "the market price of risk." Andersen and Lund add λrs , and also add a similar risk element to the b diffusion.

However, in the case of the CIR model a closed form solution exists which simplifies the calculation. The yield rate for a zero coupon bond of maturity T is given by $Y(T) = A(T) + rB(T)$ where:

$$A(T) = -2(ab/s^2T)\ln C(T) - 2aby/s^2$$

$$B(T) = [1 - C(T)]/yT$$

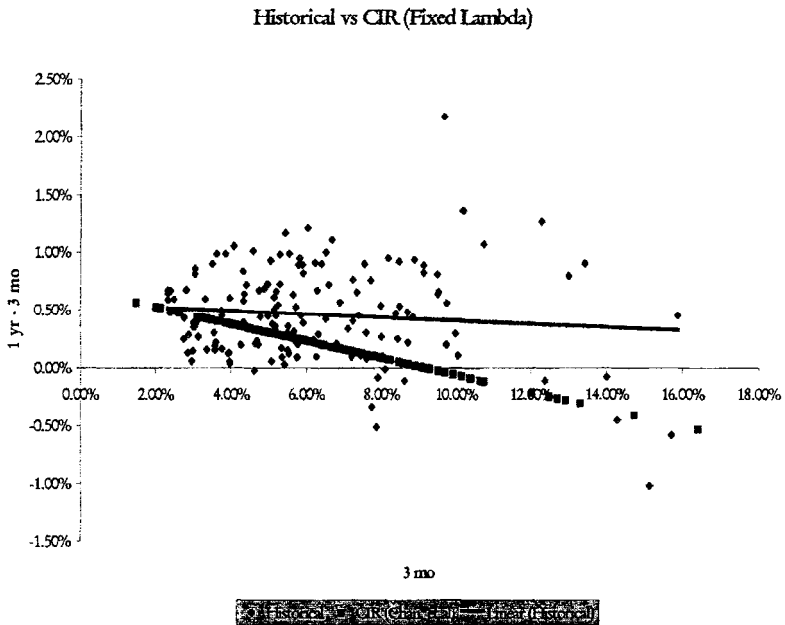
$$C(T) = (1 + xye^{T/x} - xy)^{-1}$$

$$x = [(a - \lambda)^2 + 2s^2]^{-1/2}$$

$$y = (a - \lambda + 1/x)/2.$$

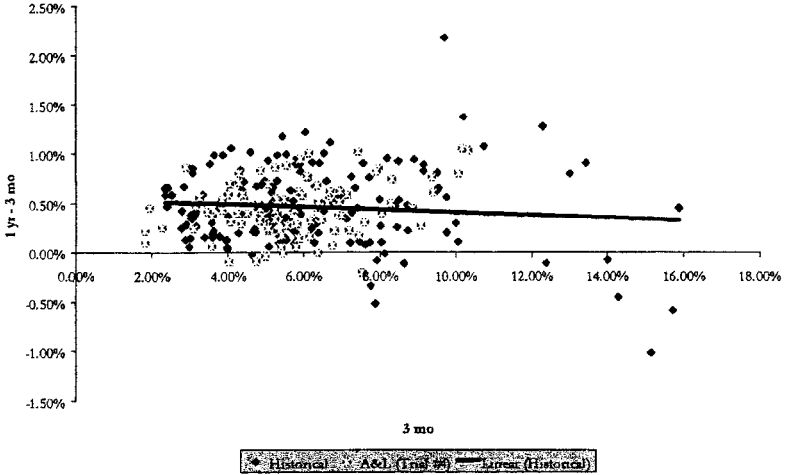
Note that neither A nor B is a function of r , so Y is a linear function of r (but not of T of course). Thus for the CIR model, all the yield curve shape measures defined above are linear functions of r , and as the three-month rate is as well, the shape measures are strictly linear in the three-month rate. This is in contrast to the historical data, which shows a dispersion of the shape measures around a perhaps linear relationship. The graph below as an example shows the historical and CIR implied 1 year less 3 month spread as a function of the 3-month rate, along with the historical trend line.

The parameters used here for the CIR model, from Chan et al. (1992) are: $a=.2339$; $b=.0808$; $s=.0854$, with λ set to .03. Different parameter values could possibly get the slope closer to that of the historical data, but the dispersion around the line cannot be achieved with this model. Experimentation with different parameter values suggests that even getting the slopes to match historical for all three of the first-difference measures may be difficult as well.



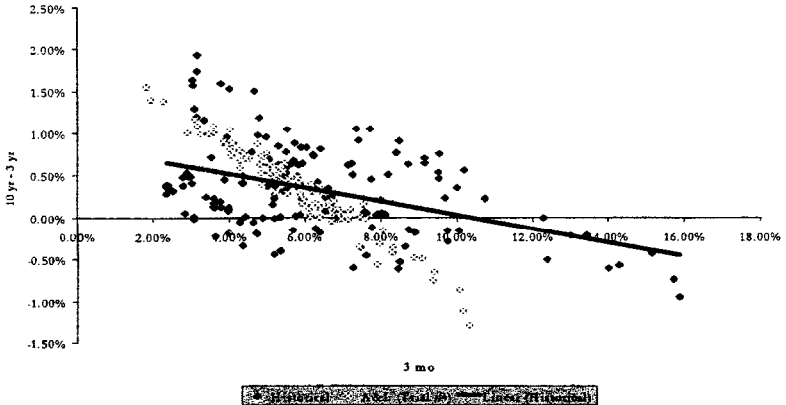
Another potential problem with the CIR model is that the very long-term rates do not vary with r at all, but it's not clear how long the rates have to be for this.

Historical vs A&L (Fixed Lambda)



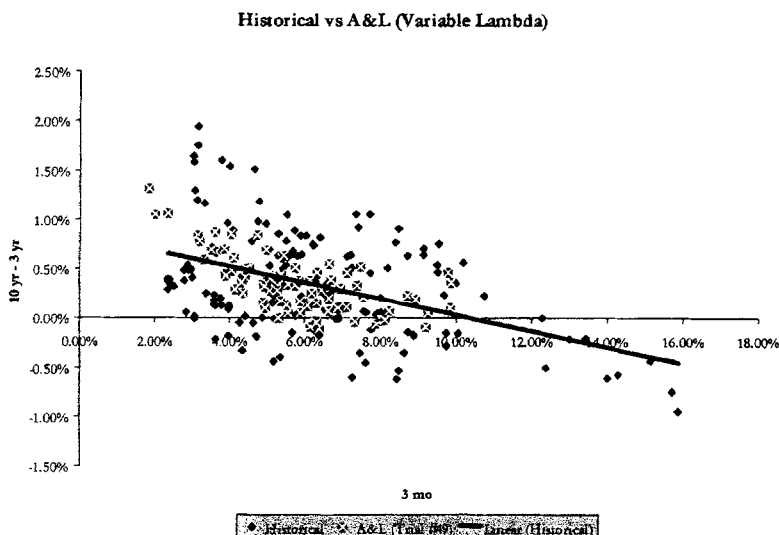
The Andersen-Lund model does provide more dispersion around the trend line, as also has about the right slope for the 3-month to 1-year spread, as the graph above shows. It does not do as well with the 3-year to 10-year spread in either

Historical vs A&L (Fixed Lambda)



slope or dispersion, as shown here.

One approach that seems to give a degree of improvement is to let the market price of risk vary as well, through its own stochastic process. This would allow the same short-rate process to generate different yield curves at different times due to different market situations. This approach is capable of fixing the slope and dispersion problem for the long spread, as shown below.



Allowing stochastic market price of risk may improve the CIR model's performance on these tests as well, but it's not clear how to do this and still maintain a closed-form yield curve, which is the main advantage of CIR.

The table below summarizes some of the comparisons of model and historical results discussed above. For each of the models and each of the yield spreads, the linear relationship between the yield spread and the three-month rate is summarized by three statistics: the slope of the regression line of the spread on the three-month rate, the value on that line for $r = .06$, and the standard deviation of the points around the line. The value at $r = .06$ was compared instead of the in-

tercept of the line to show how the model matched historical values for a typical interest rate.

The values were based on simulations of rates about three years beyond the initial values. Thus perhaps less variability of the residuals might be justifiable than in the historical data, which were quarterly values from 1959 through 1997. The whole variety of yield curve shapes from this nearly forty-year period may not be likely in just three years. A longer simulation period would thus give a better test of these models, and a somewhat lower residual standard deviation than historical may be acceptable for the test actually performed.

1 yr - 3 mo	Historical	CIR	AL Fixed	AL Variable
Slope	1.17%	-7.31%	2.49%	1.66%
Predicted @ 6%	0.48%	0.23%	0.44%	0.43%
Std Dev of Residuals	0.35%	0.00%	0.25%	0.43%

3 yr - 1 yr				
Slope	-8.41%	-16.58%	-5.68%	-2.57%
Predicted @ 6%	0.42%	0.49%	0.39%	0.40%
Std Dev of Residuals	0.52%	0.00%	0.12%	0.36%

10 yr - 3 yr				
Slope	-8.17%	-34.23%	-29.22%	-9.86%
Predicted @ 6%	0.35%	0.89%	0.29%	0.32%
Std Dev of Residuals	0.48%	0.00%	0.12%	0.50%

All the models tested had a lower residual standard deviation for the 3-year to 1-year spread than seen historically, but not unreasonably so for the variable price of risk model. The slopes of the 10-year to 3-year spread were all steeper than historical, but again the variable model was best.

This methodology gives an indication of a method of testing interest rate generators. There are quite a few of these in the finance literature, so none of the generators tested above can be considered optimal. In addition some refinement of the testing methodology may be able to tighten the conclusions discussed above.