AN AUTOREGRESSIVE YIELD CURVE MODEL - WITH NO FREE LUNCH

A report by the *Stochastic Investment Models Working Party*

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1. Introduction

Autoregressive models, such as the Wilkie model, CAP:Link, the TY model and the Falcon model, are very widely used for ALM and risk management. In general, these models describe inefficient markets, providing low-risk profits from dynamic switching between asset classes. It is fair to ask whether these models can also be used for pricing, alongside more conventional banking models such as Heath-Jarrow-Morton.

A crucial initial question is whether these models are arbitrage-free. To put this a different way, we can ask whether the trading opportunities are low-risk or no-risk. In the absence of dealing costs, if trading opportunities provide risk-free profits, then there is no way the model can produce coherent prices consistent with the market. However, in the absence of arbitrage, we produce pricing laws which are consistent with the model dynamics, even when markets are inefficient.

This paper focuses on the most challenging aspects of these models - yield curves. We use a stylised auto-regressive model, adjusted to eliminate some obvious arbitrages. We then invent a pricing law consistent with an existing probability model, using the machinery of deflators.

2 What are autoregressive models?

2.1 Introduction

An autoregressive model of a random variable (for example an interest rate or a price) is one where the random variable is assumed to exhibit a tendency to revert back to some long run mean value or distribution.

A number of questions arise:

- Which random variables exhibit autoregressive features?
- How can one detect autoregression?
- Are there different functional forms of autoregression and how should one choose between these?
- What are the implications for market efficiency and arbitrage, and for risk and return?
- How does mean reversion relate to volatility?
- How do market prices allow for mean reversion?
- Is there a relation between real world and risk neutral mean reversion?

2.2 Which random variables exhibit mean reversion?

In practice one can readily imagine that many variables exhibit this feature. Given a long history of data on real interest rates and divided yields, there is a perception that there are reasonableness bounds on their levels. The simplest form of model one might adopt could be:

$$r_{t+h} = r_t * a^h + (1 - a^h) * m + \boldsymbol{e}$$

where 1-a is the rate of reversion towards a long term mean level of m. Mean reversion might also apply to nominal interest rates, for example if there are pressures on Governments and central banks to keep inflation rates to within certain bounds.

In practice, many practitioners also believe that prices (equity index levels, exchange rates, etc), possibly adjusted (including dividends or interest possibly expressed in real terms, or relative to another index such as a rolled-up money market account), also exhibit mean reversion. Clearly the price of a bond exhibits mean reversion, since the price of a default-free bond must drift towards the redemption value over time.

2.3 Detecting auto-regression

In the simplest case, where the error terms are independent and normally distributed, one can perform a simple regression to determine the value of the mean reversion co-efficient a.

In practice, there are complications, because:

- The error term may not be normally distributed – the data may exhibit skewness, jumps etc
- The error terms may not be independent, so that mean reversion estimated from monthly data will differ from that if annual data is analysed
- For the purpose of long-term model estimation, there may be an insufficient amount of long-term data
- The fact that the *x*-data are not independent introduces biases into the analysis, which typically result in an overstatement of the degree of mean reversion (that is, *a* is too far below 1).
- The data may exhibit regime shifts (real interest rates of 3% or 4% p.a. in the 19th century, and again since 1980, but approximately 0% p.a. in much the rest of the 20th century), or trending.

Are current dividend yields of 2% set to mean revert to the long run average of 5.5% p.a., is 2%p.a. the new long run average, or perhaps there is no long-run level towards which yields revert?

2.4 Model functional form and purpose of modelling

Allowing for the issues raised in 2.3 will result in a need to modify our model to allow for non normality, a structure of error terms, regime shifts or a distribution of possible future means. In practice, the level of complexity selected will often depend on practical considerations such as:

- Tractability
- Cost, availability of alternatives
- Purpose of modelling

For example, pricing of an exotic interest rate derivative will require a rather complicated model whereas for a simple bond option a straightforward model will suffice. Sometimes, complexities that appear crucial at first sight turn out to not require significant attention. The bond option provides a good example of this. The options trader has available as data inputs information on forward interest rates and their volatility from a range of sources and hedging instruments. For the "Black" model he needs to feed in three main inputs:

- The price of a bond delivering one unit of currency at the exercise date, that is, a zero coupon bond price
- The level of forward interest rates at option expiry, that is the forward price of the underlying bond
- The volatility of the forward interest rates.

Since there is much market information available about these inputs, the trader may not be very interested in analysing the information to a greater depth, but may be content with the summary of market views which it represents. For example, the variance of interest rate innovations given by our simple model is:

$$v_h = \frac{1-a^{2h}}{1-a^2} s^2$$

However, the trader has access to market data which gives v_h for each h. For the single bond options with exercise maturity t, all that he needs is the value v_t and any model which replicates this value will suffice (irrespective of whether or not or how much mean reversion it incorporates). If he needs to hedge or risk manage his position with instruments of term other than t, he will be more concerned to ensure that the value of "a" more or less closely generates the market price term structure of v_h. Indeed, this information will likely not suffice because interest rates for bonds of different term will have a different volatility structure, and also non-zero correlation between each other.

On the other hand, the actuary analysing mis-match positions between insurance or pensions assets or liabilities may require a model which is in some sense more robust, i.e. allows for a number of observed market features approximately but simultaneously. This is because there may be less emphasis on precise hedging and more on applicability of the model in a range of circumstances. The need for very accurate calibration to a set of traded instruments may be less evident.

2.4 Market efficiency, the no-arbitrage condition and equilibrium

It should be clear from the above discussion that auto-regressive properties of a variable do not necessarily imply that markets are inefficient. There could be many reasons why an interest rate or a price differs from an assumed long-term average. In principle, if the speed of mean reversion is high enough, one could envisage exceptionally large profits arising over a short period. However, these profits are not risk free, so the resulting inefficient market could still exclude arbitrage.

A brief discussion of the concept of a market in equilibrium may also be in order. Broadly the concept of equilibrium is where markets clear at a point where investors choose between the various available risk profiles and rationally elect to hold the available supply of those securities (and would not prefer to hold more or less of any given security at the available price). A model exhibiting mean reversion might imply market in disequilibrium if specific securities or risks carry very high or very low expected returns compared with those for other "similar" levels of risk, or where expected return requirements vary dramatically from time to time. In physics, the concept of equilibrium applies when particles are subject to net forces of zero. A financial analogy might occur when all financial variables are at their long run average position - a state which Wilkie refers to as "Neutral". This concept of neutrality is not the same as financial equilibrium. For example, it is possible (even usual) for financial equilibrium to hold even when state variables are far away from their long run average values.

Once a model had been constructed, it may be possible to test for reasonableness contingent on the assumption that markets are not inefficient or that they are in broad equilibrium. For example it is possible to postulate a utility function (to allow judgmental trade offs between risk and return), or to construct a market implied utility function, and to then optimise asset allocation for an investor who believes in the model mean reversion characteristics. If the result of this optimisation is to generate very significant trading profits relative to a simple buy/hold strategy, one may question whether this is realistic in a large efficient capital markets setting. If not, the implication may be that the model mean reversion dynamics are poorly specified.

In financial economic as distinct from actuarial stochastic asset models, a no-arbitrage constraint is usually imposed because of the widespread belief that, significant arbitrage opportunities are rare or short-lived. In other words, the imposed constraint is that the model will not generate opportunities for riskless profit. This condition will typically impact the specification of the functional form for any interest rate or price process in a technical way rather than impact the overall characteristics of the process. However, where two assets are linked, the no arbitrage condition may limit the extent to which completely different mean reversion characteristics may be imposed for the two series, because of the way that bonds of different terms are related to a small number (for example 1) of mean reversion parameters.

2.5 How do market prices reflect mean reversion?

Eduardo Schwartz examined futures prices in the oil, copper and gold markets, and noted mean reversion characteristics in the first two, through estimation of the parameters in the following equation:

$$dx = k * (a - x) * dt + s * dz$$

where x is the log of the spot commodity price, k is the speed of mean reversion, a is the long run mean log price, and s*dz is a volatility term for a standard Brownian motion.

Schwartz also tested for mean reversion in interest rates and for the instantaneous convenience yields on oil, copper and gold. He found significant mean reversion effects in the oil and copper convenience yields and in short-term (3 month U.S. treasury bills) interest rates, but not in the convenience yield for gold. Mean reversion in the interest rate (real and nominal) markets is a common finding from many studies.

In the equity markets, evidence for long-run mean reversion is less clear cut. Indeed some studies suggest evidence of mean aversion rather than mean reversion! The analyses required are quite complicated and for the sake of brevity are not discussed further here.

2.6 Mean reversion, volatility and risk neutrality

We have already noted, in 2.4, the connection between mean reversion and volatility in the case of an interest rate or yield process. Uncertainty in future interest rates is lower if the rate of mean reversion increases. Tests of mean reversion in equity markets are often based on the observation that variance of equity market returns should increase linearly with holding period in the absence of mean reversion.

Another issue relates to risk neutrality. In the "real" world (for example, in relation to historic actual data) an equity total return index is expected to outperform a cash return index. The relative return index may or may not exhibit mean reversion (the empirical evidence being mixed, as noted above). The "risk neutral" world relates to today's markets in which the risk premium on equities cancels with the risk cost of holding equities. When specifying the model dynamics to price and risk manage a portfolio of equity risks, a bank will use a risk neutral model. In the model the net effect of cancelling out the risk premium and risk cost is an "expected" return on an equity portfolio equal to the risk-free rate of interest. Almost by definition, under the risk neutral law, there is no mean reversion in relative returns (it should be noted that this has no implication as to whether or not mean reversion applies in the real world). The dividend yield process can still exhibit mean reversion even when working in the risk neutral world (again with very little implication as to the real world process).

3 A Time Series Model based on Wilkie's Model

3.1 Autoregressive Models

We start off by defining a particular class of autoregressive models, motivated by Wilkie's 1995 model but generalising it considerably. We consider Markov models specifying two interest rates, a long rate A and a long rate B. We assume that distributions are conditionally lognormal. Then, we have a model of the form:

$$A_{t+1} = \mu_A \exp\left[\sigma_A \varepsilon_A + \tfrac{1}{2}\sigma_A^2\right]$$
$$B_{t+1} = \mu_B \exp\left[\sigma_B \varepsilon_B - \tfrac{1}{2}\sigma_B^2\right]$$

where ε_A and ε_B are two unit normal variables with correlation ρ. The parameters μ_A, μ_B and possibly σ_A, σ_B and ρ, could all be functions of the current (time t) state variables A_t and B_t.

Our choice of signs for the σ^2 terms was deliberate. We will later see why this makes sense. For the moment, we need observe only the conditional expectations:

$$\mu_A^{-1} = \mathbf{E}_t(A_{t+1}^{-1})$$
$$\mu_B = \mathbf{E}_t(B_{t+1})$$

3.2 Rate Definitions

To apply deflators we need to more specific about our rates A and B. In what follows, we take B as a one year, simply compounded rate. We take A to be the one-year deferred rate on a perpetual swap, with annual payments. This means that the price of a one-year zero coupon bond is

$$\frac{1}{1+B}$$

In the same vein, the price of a perpetual bond, paying 1 per annum in arrears is then given by

$$\frac{1}{1+B}\left(1+\frac{1}{A}\right)$$

Here the rate A has been used to discount all flows back to time t+1, followed by the final years' discounting at rate B.

3.3 Deflator Formulas

At this point, we depart from the traditional statistical approach and posit some formulas for deflators. Our proposed formula follows several steps. We will first state the formulas, and then seek to justify them.

We now define some formulas for deflators based on this specification of an autoregressive model. We define two functions ψ_A and ψ_B as follows:

Case (i). $A_t \leq 1$

$$\psi_B = \exp\left[\frac{\left[\rho\frac{\sigma_A\sigma_B}{1+\mu_A}+\frac{\mu_B\sigma_B^2}{1+\mu_B}\right]\log\left(\frac{A_t(1+\mu_A^{-1})}{1+\mu_B}\right)}{\frac{\sigma_A^2}{(1+\mu_A)^2}+2\rho\frac{\sigma_A}{1+\mu_A}\frac{\mu_B\sigma_B}{1+\mu_B}+\frac{\mu_B^2\sigma_B^2}{(1+\mu_B)^2}}\right]$$

$$\psi_A = \frac{A_t}{(1-A_t+\psi_B\mu_B)\mu_A}$$

Case (ii). $A_t > 1$

$$\psi_A = \exp\left[\frac{\left[\frac{\sigma_A^2}{1+\mu_A}+\rho\frac{\mu_B\sigma_A\sigma_B}{1+\mu_B}\right]\log\left(\frac{A_t(1+\mu_A^{-1})}{1+\mu_B}\right)}{\frac{\sigma_A^2}{(1+\mu_A)^2}+2\rho\frac{\sigma_A}{1+\mu_A}\frac{\mu_B\sigma_B}{1+\mu_B}+\frac{\mu_B^2\sigma_B^2}{(1+\mu_B)^2}}\right]$$

$$\psi_B = \frac{(A_t-1+A_t\psi_A^{-1}\mu_A^{-1})}{\mu_B}$$

In each case, we define two more parameters λ_A and λ_B by:

$$\lambda_A = \frac{\log\psi_A}{\sigma_A\left(1-\rho^2\right)}-\frac{\rho\log\psi_B}{\sigma_B\left(1-\rho^2\right)}$$

$$\lambda_B = \frac{\log\psi_B}{\sigma_B\left(1-\rho^2\right)}-\frac{\rho\log\psi_A}{\sigma_A\left(1-\rho^2\right)}$$

Finally, we define a deflator by the recurrence relations:

$$D_{t+1} = \frac{(1+B_{t+1})D_t}{(1+B_t)(1+\psi_B\mu_B)}\exp\left[\begin{array}{l}\lambda_A\varepsilon_A+\lambda_B\varepsilon_B \\ -\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\end{array}\right]$$

We now have to justify the choice of deflator. This involves two steps - first checking the deflator works, then arguing that of the possible deflator choices, our choice approximately solves an inefficiency minimisation problem.

3.4 Proving that the Deflator Works - One Year Bonds

To prove the deflator works we need to demonstrate that it correctly prices a zero coupon bond and a perpetual bond. Let us look first at the one-year bond, whose price should be given by

$$\frac{1}{D_t}\mathbf{E}_t(D_{t+1}) = \frac{\mathbf{E}_t\left[(1+B_{t+1})\exp(\lambda_A\varepsilon_A+\lambda_B\varepsilon_B)\right]}{(1+B_t)(1+\psi_B\mu_B)\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]}$$

Multiplying out the expectation in the numerator, we see firstly that

$$\mathbf{E}_t\left[\exp(\lambda_A\varepsilon_A+\lambda_B\varepsilon_B)\right] = \exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]$$

The second expectation is

$$\begin{aligned}\mathbf{E}_t\left[B_{t+1}\exp(\lambda_A\varepsilon_A+\lambda_B\varepsilon_B)\right] &= \mathbf{E}_t\left[\mu_B\exp(\lambda_A\varepsilon_A+(\lambda_B+\sigma_B)\varepsilon_B-\tfrac{1}{2}\sigma_B^2)\right]\\ &= \mu_B\exp\left[\tfrac{1}{2}\left(\lambda_A^2+(\lambda_B+\sigma_B)^2+2\rho\lambda_A(\lambda_B+\sigma_B)-\sigma_B^2\right)\right]\\ &= \mu_B\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]\exp\left[\lambda_B\sigma_B+\rho\lambda_A\sigma_B\right]\\ &= \mu_B\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]\psi_B\end{aligned}$$

where the last line is obtained on substitution for λ_A and λ_B. Putting these terms together, we have verified that

$$\frac{1}{D_t}\mathbf{E}_t(D_{t+1}) = \frac{1+\psi_B\mu_B}{(1+B_t)(1+\psi_B\mu_B)} = \frac{1}{1+B_t}$$

so the deflator correctly prices one-year bonds.

3.5 Proving that the Deflator Works - Perpetual Bonds

Our next step is to show that the deflator correctly prices perpetual bonds. In other words, we need to show that the value of a perpetual bond is the price of a one year deferred perpetual bond one year earlier. The relevant deflator expression is:

$$\frac{1}{D_t}\mathbf{E}_t\left[\frac{D_{t+1}}{1+B_{t+1}}\left(1+\frac{1}{A_{t+1}}\right)\right] = \frac{\mathbf{E}_t\left[(1+A_{t+1}^{-1})\exp\left(\lambda_A\varepsilon_A+\lambda_B\varepsilon_B\right)\right]}{(1+B_t)(1+\psi_B\mu_B)\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]}$$

As before, we multiply the expectation out .The first term is precisely as in the previous section. But the second term arising is:

$$\begin{aligned}\mathbf{E}_t\left[A_{t+1}^{-1}\exp\left(\lambda_A\varepsilon_A+\lambda_B\varepsilon_B\right)\right] &= \mathbf{E}_t\left[\mu_A^{-1}\exp\left((\lambda_A-\sigma_A)\varepsilon_A+\lambda_B\varepsilon_B-\tfrac{1}{2}\sigma_A^2\right)\right]\\ &= \mu_A^{-1}\exp\left[\tfrac{1}{2}\left((\lambda_A-\sigma_A)^2+\lambda_B^2+2\rho(\lambda_A-\sigma_A)\lambda_B-\sigma_A^2\right)\right]\\ &= \mu_A^{-1}\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]\exp\left[-\lambda_A\sigma_A-\rho\sigma_A\lambda_B\right]\\ &= \mu_A^{-1}\exp\left[\tfrac{1}{2}\left(\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right)\right]\psi_A^{-1}\end{aligned}$$

where in the last line we have substituted for λ_A and λ_B. Putting these terms together, we find:

$$\frac{1}{D_t}\mathbf{E}_t\left[\frac{D_{t+1}}{1+B_{t+1}}\left(1+\frac{1}{A_{t+1}}\right)\right] = \frac{1+\psi_A^{-1}\mu_A^{-1}}{(1+B_t)(1+\psi_B\mu_B)}$$

Finally, we notice that each of cases (i) and (ii) of our original algorithm, we had

$$\frac{1+\psi_B \mu_B}{1+\psi_A^{-1}\mu_A^{-1}} = A_t$$

Thus our calculation has shown that

$$\frac{1}{D_t}\mathbf{E}_t\left[\frac{D_{t+1}}{1+B_{t+1}}\left(1+\frac{1}{A_{t+1}}\right)\right] = \frac{1}{(1+B_t)A_t}$$

so confirming that under our model, the price of a perpetuity is a deferred perpetuity one year earlier.

3.6 Remaining Degrees of Freedom

Our proofs of the deflator property so far have relied only on three equations:

$$\psi_A = \exp\left(\lambda_A \sigma_A + \rho \lambda_B \sigma_A\right)$$
$$\psi_B = \exp\left(\lambda_B \sigma_B + \rho \lambda_A \sigma_B\right)$$
$$A_t = \frac{1+\psi_B \mu_B}{1+\psi_A^{-1}\mu_A^{-1}}$$

The first two equations imply the following lambdas:

$$\lambda_A = \frac{\log \psi_A}{\sigma_A\left(1-\rho^2\right)} - \frac{\rho \log \psi_B}{\sigma_B\left(1-\rho^2\right)}$$
$$\lambda_B = \frac{\log \psi_B}{\sigma_B\left(1-\rho^2\right)} - \frac{\rho \log \psi_A}{\sigma_A\left(1-\rho^2\right)}$$

Although our choices of ψ_A and ψ_B plainly satisfy this, they are far from being unique. The answer is that we chose ψ_A and ψ_B to maximise market inefficiency, that is, to minimise the variability of the deflator. However, the exact minimisation is a tricky mathematical task, so we settle for approximate minimisation.

Our first approximation is to neglect the term in $(1+B_{t+1})$ in the definition of D_{t+1}, so that we minimise the variance of $\lambda_A\varepsilon_A+\lambda_B\varepsilon_B$, that is:

$$\min\left\{\lambda_A^2 + \lambda_B^2 + 2\rho\lambda_A\lambda_B\right\}$$

By substitution into the last of our three equations, the locus of λ_A and λ_B is given by:

$$A_t = \frac{1+\mu_B \exp\left(\lambda_B\sigma_B + \rho\lambda_A\sigma_B\right)}{1+\mu_A^{-1}\exp\left(-\lambda_A\sigma_A - \rho\lambda_B\sigma_A\right)}$$

We can rewrite this locus alternatively in logarithmic terms:

$$\log A + \log\left[1+\mu_A^{-1}\exp\left(-\lambda_A\sigma_A - \rho\lambda_B\sigma_A\right)\right] = \log\left[1+\mu_B\exp\left(\lambda_B\sigma_B+\rho\lambda_A\sigma_B\right)\right]$$

In our next stage, we suppose that our optimisation has been moderately successful in reducing $\left\{\lambda_A^2+\lambda_B^2+2\rho\lambda_A\lambda_B\right\}$, so that both λ_A and λ_B are small. In this case, we can expand the logarithms and exponentials to first order, to give:

$$\begin{aligned}\log\left[1+\mu_A^{-1}\exp\left(-\lambda_A\sigma_A-\rho\lambda_B\sigma_A\right)\right] &\approx \log\left[1+\mu_A^{-1}-\mu_A^{-1}\left(\lambda_A\sigma_A+\rho\lambda_B\sigma_A\right)\right] \\ &\approx \log\left[1+\mu_A^{-1}\right] - \frac{\lambda_A\sigma_A+\rho\lambda_B\sigma_A}{1+\mu_A}\end{aligned}$$

In the same way, the last term is:

$$\log\left[1+m_B \exp\left(l_B s_B + r\, l_A s_B \right)\right] \approx \log\left[1+ m_B + m_B\left(l_B s_B + r\, l_A s_B \right)\right]$$
$$\approx \log\left[1+ m_B\right] + \frac{m_B}{1+m_B}\left(l_B s_B + r\, l_A s_B \right)$$

The approximate locus is therefore given by

$$\log\left(\frac{A(1+m_A^{-1})}{1+m_B} \right) = \frac{l_A s_A + r\, l_B s_A}{1+m_A} + \frac{m_B}{1+m_B}\left(l_B s_B + r\, l_A s_B \right)$$

and the constrained optimum is

$$l_A = \frac{\dfrac{s_A}{1+m_A}\log\left(\dfrac{A(1+m_A^{-1})}{1+m_B} \right)}{\dfrac{s_A^2}{(1+m_A)^2} + 2r\dfrac{s_A}{1+m_A}\dfrac{m_B s_B}{1+m_B} + \dfrac{m_B^2 s_B^2}{(1+m_B)^2}}$$

$$l_B = \frac{\dfrac{m_B s_B}{1+m_B}\log\left(\dfrac{A(1+m_A^{-1})}{1+m_B} \right)}{\dfrac{s_A^2}{(1+m_A)^2} + 2r\dfrac{s_A}{1+m_A}\dfrac{m_B s_B}{1+m_B} + \dfrac{m_B^2 s_B^2}{(1+m_B)^2}}$$

and finally, we deduce the solution to our approximate problem

$$y_A = \exp\left(l_A s_A + r\, l_B s_A \right) = \exp\left[\frac{\left[\dfrac{s_A^2}{1+m_A} + r\dfrac{m_B s_A s_B}{1+m_B} \right]\log\left(\dfrac{A(1+m_A^{-1})}{1+m_B} \right)}{\dfrac{s_A^2}{(1+m_A)^2} + 2r\dfrac{s_A}{1+m_A}\dfrac{m_B s_B}{1+m_B} + \dfrac{m_B^2 s_B^2}{(1+m_B)^2}} \right]$$

$$y_B = \exp\left(l_B s_B + r\, l_A s_B \right) = \exp\left[\frac{\left[r\dfrac{s_A s_B}{1+m_A} + \dfrac{m_B s_B^2}{1+m_B} \right]\log\left(\dfrac{A(1+m_A^{-1})}{1+m_B} \right)}{\dfrac{s_A^2}{(1+m_A)^2} + 2r\dfrac{s_A}{1+m_A}\dfrac{m_B s_B}{1+m_B} + \dfrac{m_B^2 s_B^2}{(1+m_B)^2}} \right]$$

As these have come from an approximation, they do not exactly satisfy the original constraint: $A_t = \dfrac{1+y_B m_B}{1+y_A^{-1} m_A^{-1}}$. We therefore choose one of $ø_A$ or $ø_B$ and substitute back into the identity to deduce the other. We make our choice according to whether $A_t > 1$ or <1, ensuring that the substitution will always provide a positive value for the second ø