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Calibration and Communication of Dependencies with a Case Study based on Market Returns

November 2010

Richard Shaw, Andrew Smith & Grigory Spivak

Workshop Overview

- Calibrating Copulas (Andrew Smith)
  - Using more than one measure of tail dependence
  - Rank correlation and arachnitude
  - Calibrating the T-copula
  - Equity Return Case Study
- Communicating Dependencies (Richard Shaw)
  - Why it is important
  - Economic capital aggregation, joint probability density function, scatter plot, joint excess probability, tail concentration function, Kendall tau correlation, coefficient of tail dependence, implied Gaussian correlation
- Conclusions and Questions
- This presentation is based on **Measurement and modelling of dependencies in economic capital** by Richard Shaw, Andrew Smith & Grigory Spivak (2010)

http://www.actuaries.org.uk/sites/all/files/documents/pdf/sm20100510.pdf

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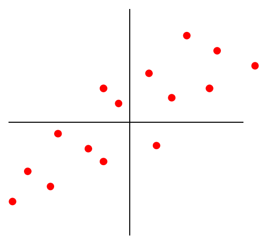
Calibrating Copulas

Recall – The Gauss Copula Idea



Transform to uniform. Replace n^{th} smallest by $u = n/(N+1)$

Transform to desired marginal distribution

Bivariate Normal Z
sims = N

Transform to uniform

Transform to desired marginal distribution

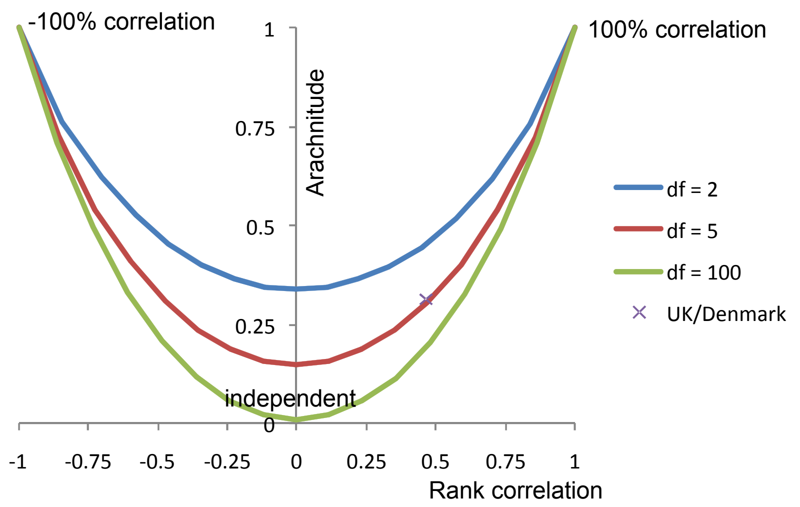
For d dimensions, specify
•Marginal distributions, d times
•Correlations: $d \times d$ matrix
This is a Gauss copula. Generalisations include:
•T copula
  •Divide all Z's by a common $\sqrt{\chi^2_{df}}$
•Individuated T copula
  •Divide each Z_i by its own $\sqrt{\chi^2_{df(i)}}$
  •Divisors are increasing functions of each other

Rank Correlation and Arachnitude

- Suppose we have 2 dimensions, N = 480 observations
- In each dimension, replace n^{th} smallest by $u = n/(N+1)$

Correlation matrix	U_{dk}	U_{uk}	$(2U_{dk}-1)^2$	$(2U_{uk}-1)^2$
U_{dk}	1	Rank correlation	0	
U_{uk}	Rank correlation	1		0
$(2U_{dk}-1)^2$	0		1	Arachnitude
$(2U_{uk}-1)^2$		0	Arachnitude	1

Rank Correlation and Arachnitude: UK and Denmark

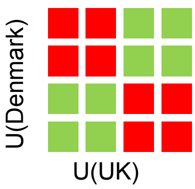


Cross Correlations: Denmark & UK

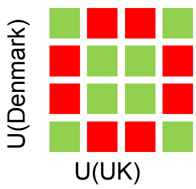
Correlation matrix	U_{dk}	U_{uk}	$(2U_{dk}-1)^2$	$(2U_{uk}-1)^2$
U_{dk}	1	46.2%	0	-12.0%
U_{uk}	46.2%	1	-12.5%	0
$(2U_{dk}-1)^2$	0	-12.5%	1	31.4%
$(2U_{uk}-1)^2$	-12.0%	0	31.4%	1

Empirical estimates for UK/Danish equity returns

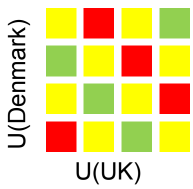
Interpreting Cross Correlations



Positive rank correlation



Positive arachnitude



Positive cross correlation

Increase density

Reduce density

Notes:
Extreme case of arachnitude = 1 is attained for spider copula (mixture of increasing and decreasing copulas)
Individuated T copula (and so gauss and T copula) imply cross correlations are zero.

Copula Approaches: Strengths and Weaknesses

Strengths
- Invariant under increasing transforms of x and y (for example, taking logs)
- Captures all the information in the dependency structure without reference to marginal distributions
- Allows unconstrained choice of marginal distributions
- Suitable for Monte Carlo

Weaknesses
- May be difficult to find copula functions to capture specific data features
- For example, negative cross terms
- Seldom amenable to analytical calculations

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Multiple Comparisons

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Purpose of Multiple Comparisons

- We have observed specific features of the UK / Danish data set
  - Are these real features of the underlying distribution or sampling artefact?
  - Difficult to analyse mathematically because of the need to start with a hypothesis about the “underlying” copula
- An alternative is to test consistency across multiple economies in the search for “stylised facts”.
  - We could also test robustness across different time periods
  - Recognise that the economies are not independent, so feature seen in all economies could still be statistical fluke

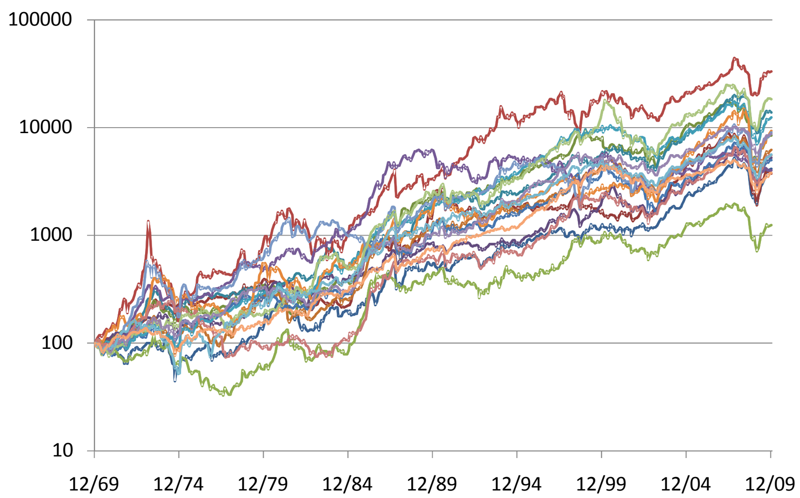
Our Chosen Data Set

- MSCI equity indices
- 31/12/1969 – 31/12/2009
- Monthly total return indices, coverage for 480 months
- In US Dollars
- 18 series representing different countries

  Countries represented: Australia, Austria, Belgium, Canada, Denmark, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Singapore, Spain, Sweden, Switzerland, UK, US

- In this presentation we analyse only two-dimensional dependency. There are 153 pairs of countries for which this can be analysed. In the charts that follow, each country pair is represented by one point.

Equity Total Return Data 1970-2010



Source: Datastream

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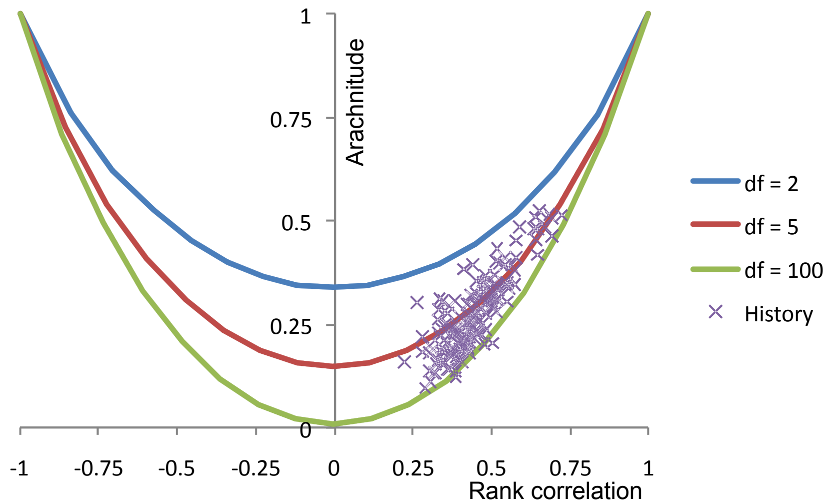
Fitting a T Copula: 5 df is typical Significant Rejection of Gauss Copula



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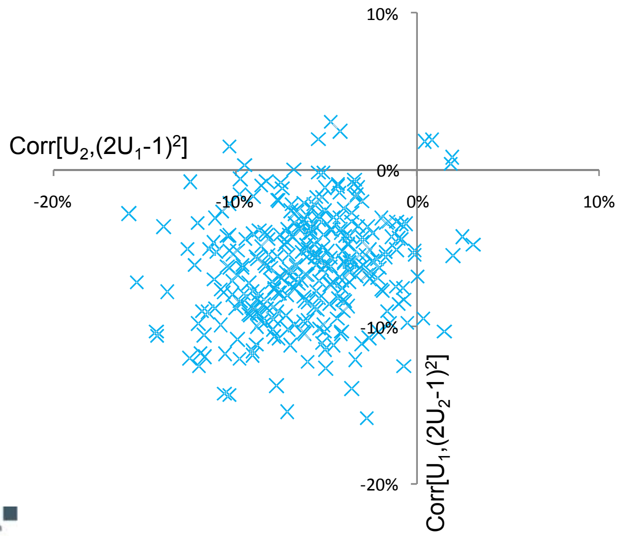
Negative Cross Correlations: Systematic Feature Rejection of T copula (standard or individuated)



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Dependency Communication

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Different Ways of thinking about Dependency

- Solvency II IMAP (Use Test) requires Senior Management to:
  - Demonstrate that they understand the internal model and its fit with the business model and risk management framework
  - Demonstrate understanding of the limitations of the internal model and take account of these in their decision-making
- This section describes some simple measures that could be adopted by firms in the communication of dependency:

  Economic Capital Aggregation / Joint Probability Density Function / Scatter Plot / Joint Excess Probability / Tail Concentration Function / Kendall Tau Correlation / Coefficient of Tail Dependence / Implied Gaussian Correlation

Economic Capital Aggregation

Economic Capital - 25% Correlation

Percentile	Return	Gaussian	t - 10 df	t - 5 df	t - 2 df	V CV
75.0%	4	1,760	1,685	1,578	1,421	1,658
90%	10	3,688	3,610	3,582	3,418	3,763
95%	20	4,928	4,906	5,004	4,889	5,182
99%	100	7,423	7,916	8,177	9,049	8,212
99.5%	200	8,391	9,087	10,031	11,052	9,455
99.95%	2,000	11,082	13,926	14,929	18,544	13,468

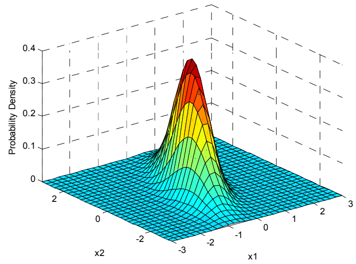
% change cf Gaussian Copula

Percentile	Return	Gaussian	t - 10 df	t - 5 df	t - 2 df	V CV
75.0%	4	0.0%	-4.2%	-10.3%	-19.3%	-5.8%
90%	10	0.0%	-2.1%	-2.9%	-7.3%	2.0%
95%	20	0.0%	-0.4%	1.6%	-0.8%	5.2%
99%	100	0.0%	6.6%	10.2%	21.9%	10.6%
99.5%	200	0.0%	8.3%	19.5%	31.7%	12.7%
99.95%	2,000	0.0%	25.7%	34.7%	67.3%	21.5%

Lognormal distribution:
E(X) = 2,000; SD(X) = 500

- (A) It is relatively simple to understand
- (A) It is possible to directly measure the financial impact on a company
- (D) No information of what is happening at an individual risk category level at each percentile of interest
- (D)The calculations are more computer intensive than those that will be discussed in the following sections

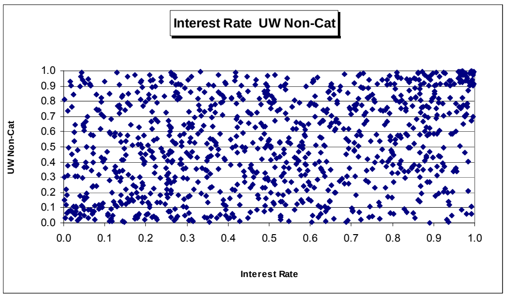
Joint Probability Density Function



The Joint Probability Density function is a 3 dimensional representation of the plot of the values from two risk factor distributions, in this case risks X1 and X2. A greater density of points represented by a larger value of the PDF

- (A) It is relatively simple to understand
- (A) The exhibits are relatively easy to create
- (D) Simulation error may distort the presence or otherwise of ‘tail’ dependency strength
- (D) There is no numerical measure that reflects the degree of dependency between risks
- (D) One can only use this method for a pair of risks at a time

Scatter Plot



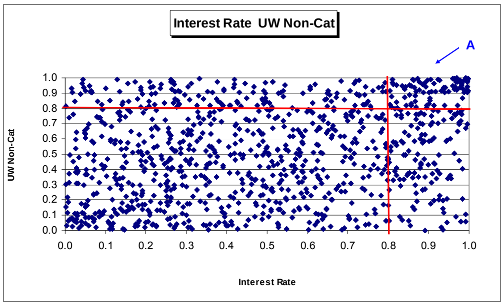
A scatter plot involves a plot of the joint values simulated from two risk distributions. In this example the values (u,v) corresponding to amounts x and y from risk distributions X and Y have been plotted.

Furthermore, u and v are defined by the relationships $F_X(x) = u$ and $F_Y(y) = v$, where u and v are values on the interval [0,1].

The extent of the clustering of points in the region of (1,1) indicates the level of ‘tail’ dependency between two risks

- (A) It is relatively simple to understand
- (A) The exhibits are very easy to create
- (D) Simulation error may distort the presence or otherwise of ‘tail’ dependency strength
- (D) There is no numerical measure that reflects the degree of dependency between risks
- (D) It may be difficult to distinguish a pair or risks with higher tail dependence from a pair of risks with higher correlation but lower tail dependence. One can only use this method for a pair of risks at a time

Joint Excess Probability



For a pair of risks, the Joint Excess Probability is the joint probability that 2 risks are either greater or lower than some threshold

RJEP(z) = P(u>z, v>z)
LJEP(z) = P(u<z, v<z)
where: by $F_X(x) = u$ and $F_Y(y) = v$

For independence the values of RJEP(z) and LJEP(z) are $(1-z)^2$ and z^2 respectively

RJEP(0.8) = No. of Points in A / Total No. of Points (in this case 1,000)

- (A) It is relatively simple to understand and practical
- (A) The calculation is relatively easy to perform
- (A) It allows the quantification of the level of dependence at a given percentile in a way which is both mathematically tractable, and simple to understand
- (A) It provides a consistent methodology for comparing the relative strength of dependency between two or more risks whether the dependence between them is expressed using copulas or correlations

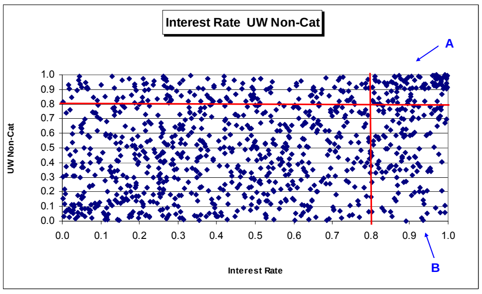
Joint Excess Probability

RJEP(Z): t Copula 5 d.f. z 95.0%

	No.	1	2	3	4	5	6	7	8	9	10
Equity	1		0.87%	1.08%	1.07%	1.04%	0.98%	1.00%	0.89%	0.87%	0.98%
Property	2			1.04%	0.98%	0.97%	0.95%	0.98%	1.00%	0.95%	0.96%
Interest Rate	3				1.12%	1.06%	1.03%	1.10%	1.09%	1.04%	1.10%
Credit Spread	4					1.02%	0.99%	1.19%	1.01%	1.07%	1.17%
Credit Default	5						0.97%	1.05%	1.02%	1.00%	1.00%
UW - Cat	6							0.93%	0.91%	0.96%	1.07%
UW Non-Cat	7								0.97%	0.98%	1.09%
Reserve	8									1.04%	1.05%
Expenses	9										0.98%
Operational	10										
									Independence		0.25%

- (D) For most of practitioners used to linear correlations this would be a new concept and some confusion between the two numbers is possible. In particular, it could be mistaken to be a 'tail correlation', i.e. the level of correlation in the tail. In fact, the RJEP(z) and LJEP(z) functions are probabilities, i.e. take values between 0 and 1 whereas a correlation coefficient takes values between -1 and 1.
- (D) It is difficult to translate a value of RJEP(z) or LJEP(z) into a number that is commonly understood e.g. linear correlation, or its equivalent at the 'tails'.
- (D) Simulation error may distort the presence or otherwise of 'tail' dependency strength

Tail Concentration Function



For a pair of risks, the strength of 'tail' dependence between risk factors can be defined using the Right and Left Tail Concentration Functions R(z) and L(z) respectively as follows:

Right Tail Concentration Function: R(z) = P(v>z / u>z) = P(v>z, u>z) / P (u>z)
Left Tail Concentration Function: L(z) = P(v<z / u<z) = P(v<z, u<z) / P (u<z)

R(0.8) = No. of Points in A / (Total Points (A + B))

- (A) It is practical and the concept is relatively easy to understand
- (A) The calculation is relatively easy to perform
- (A) It allows the quantification of dependence at a given percentile which is mathematically tractable
- (A) It is closely linked to another important copula parameter: "Coefficient of Tail Dependence" which is a limiting case of the tail concentration function
- (A) It provides a consistent methodology for comparing the relative strength of dependency

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Tail Concentration Function

R(Z): t Copula 5 d.f. z 95.0%

	No.	1	2	3	4	5	6	7	8	9	10
Equity	1		17.58%	21.85%	21.61%	20.89%	19.68%	20.16%	17.98%	17.58%	19.68%
Property	2			21.18%	20.03%	19.87%	19.38%	20.03%	20.36%	19.38%	19.71%
Interest Rate	3				21.20%	19.98%	19.45%	20.82%	20.67%	19.68%	20.89%
Credit Spread	4					19.06%	18.46%	22.20%	18.91%	19.96%	21.82%
Credit Default	5						19.13%	20.71%	20.16%	19.61%	19.76%
UW - Cat	6							18.87%	18.46%	19.35%	21.62%
UW Non-Cat	7								19.27%	19.51%	21.74%
Reserve	8									21.06%	21.38%
Expenses	9										19.48%
Operational	10										
									Independence		5.0%

- (D) For most of practitioners used to linear correlations this would be a new concept and some confusion between the two numbers is possible. In particular, it could be misunderstood to be a 'tail correlation', i.e. the level of correlation in the tail. In fact, the tail concentration functions are different mathematical objects: they are probabilities, i.e. take values between 0 and 1 whereas correlation takes values between -1 and 1
- (D) It is difficult to translate R(z) or L(z) into a number that is commonly understood i.e. linear correlation
- (D) Simulation error may distort the presence or otherwise of 'tail' dependency strength

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Kendall Tau Correlation

Kendall Tau: t Copula 5 d.f.

	No.	1	2	3	4	5	6	7	8	9	10
Equity	1		17.62%	16.72%	17.74%	18.31%	16.29%	17.73%	16.72%	16.49%	17.50%
Property	2			15.38%	15.92%	16.39%	15.72%	17.20%	16.43%	16.29%	17.25%
Interest Rate	3				16.02%	16.67%	17.29%	17.66%	16.68%	16.71%	17.02%
Credit Spread	4					16.27%	17.17%	16.70%	15.94%	15.96%	16.02%
Credit Default	5						16.35%	17.22%	15.75%	15.68%	17.25%
UW - Cat	6							15.61%	15.16%	16.71%	17.90%
UW Non-Cat	7								15.56%	15.86%	17.63%
Reserve	8									16.73%	17.26%
Expenses	9										16.19%
Operational	10										

- The Kendall Tau correlation measures dependency as the tendency of two variables, *X* and *Y*, to move in the same (opposite) direction. Let (X_i, Y_i) and (X_j, Y_j) be a pair of observations of *X* and *Y*
- If $(X_j - X_i)$ and $(Y_j - Y_i)$ have the same sign, then we say that the pair is concordant, if they have opposite signs, then we say that the pair is discordant
- Let *C* (number of concordant pairs) and *D* (number of discordant pairs). A simple intuitive way to measure the strength of a relationship is to compute *S=C-D*, a quantity known as Kendall *S*.
- The normalised value of *S* is known as the Kendall Tau correlation coefficient, or Kendall Tau. $\tau = \dfrac{S}{\frac{1}{2}n(n-1)}$

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Coefficient of Tail Dependence

T d.f.	λ
10	2.6%
5	10.7%
2	27.2%

- The Coefficient of Tail Dependence between two risks is an asymptotic measure of the dependence in the tails of the bivariate distribution (X,Y).
- For a multivariate distribution with a Gaussian copula, the tail dependence between any pair of risks is always zero. This is one of the important deficiencies of the Gaussian copula for modelling dependence.
- For continuously distributed random variables with the t Copula the Coefficient of Tail Dependence is:

$$\lambda = 2t_{n+1}(-(n+1)^{0.5}(1-\rho)^{0.5}/(1+\rho)^{0.5})$$

- where ρ is the pairwise correlation coefficient between two risks
- Note: In this example ρ = 25%

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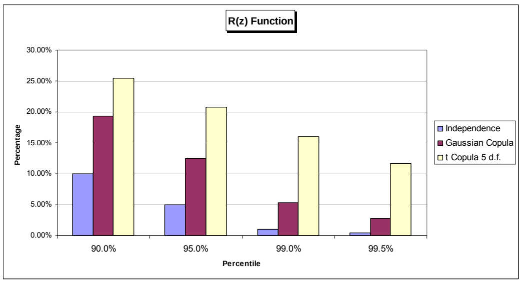
Implied Gaussian Correlation



- The approach can be used to determine a so-called ‘Implied’ Gaussian correlation between risk pairs
- For example, at 99.0% the value of R(z) = 15.96% for the t copula (5 df) and 5.31% for the Gaussian copula assuming a correlation of 25%.
- If the linear correlation is increased from 25% to 54% then the value of R(z) at 99.0% with the Gaussian copula now equals the same value of R(z) = 15.96% as before.
- The values of these so-called ‘Implied’ Gaussian correlations can be used to compare model outputs

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and are encouraged.

The views expressed in this presentation are solely those of the presenters.

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