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ON THE CALCULATION OF "REAL" INVESTMENT RETURNS

by

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INTRODUCTION

1. The rise in the general level of prices that has occurred in Britain at varying speeds over the last half century has caused many investments to appreciate considerably in money terms, but without this necessarily representing any increase in the purchasing power of the monetary proceeds. It has for some time been desirable to adjust for changes in the general purchasing power of money, and to calculate returns on investment " in real terms ". The issue by the British Government of Index-Linked Treasury Stocks provides investments whose future return is defined in terms of the Retail Prices Index, and it is desirable to calculate " real " redemption yields on these stocks. There has been some discussion about how such real yields should be calculated, and, as will be seen, it is necessary to make certain assumptions in order to produce explicit results. The Joint Investment and Index Committee of the Institute of Actuaries and the Faculty of Actuaries has asked me to prepare this note in order to explain the details of how real yields should be calculated, and what assumptions need to be made and stated. I am very grateful to the members of the Committee, particularly the Chairman, R. H. Pain, for their assistance in its preparation.

2. The expression " real " will be taken to mean that returns or yields are calculated in relation to the U.K. Retail Prices Index. There are many philosophical arguments against using the term to imply more than this. It is not necessary to claim any greater " reality " of return in a wider sense, and no such claim is made here.

3. It would clearly be possible to choose some other index as the basis for calculation, such as an index of earnings. There are no differences in the method of calculation, but the Committee recommends that any such real return or real yield should be described as being calculated in relation to the particular index used. Without any qualification the term " real " would mean in relation to the Retail Prices Index.

The Retail Prices Index

4. The Retail Prices Index (RPI) is calculated by the Central Statistical Office once a month, on a specific applicable date in that month (usually the second or third Tuesday), and it is published in the Employment Gazette (of the Department of Employment) and elsewhere usually on the Friday of the second or third week of the following month. The value of the Index for month M, which is usually published in month M+1, will in this note be denoted Q(M). For example Q(March 1983) = 327·9; this applied to 15 March 1983, and was published on 22 April 1983. Values of the index from January 1980 are given in the Appendix.

5. For many purposes it is necessary to postulate a value of the RPI for any intervening day. Such a value can be calculated by interpolation between the neighbouring published values. The interpolated value for exact date t will be denoted $q(t)$. It is reasonable to assume a uniform compound growth of the Index between two neighbouring values. It is therefore appropriate to calculate $q(t)$ by linear interpolation on the logarithms of the neighbouring RPI values. Thus if date t is $d1$ days after the applicable date in month M, and $d2$ days before the applicable date in month M+1, with $d1+d2 = d$, we get

$$q(t) = \exp\left\{\frac{d2}{d}\log \mathrm{Q(M)} + \frac{d1}{d}\log \mathrm{Q(M+1)}\right\}. \tag{1}$$

6. However, the RPI usually changes by only a small proportion each month, so there will normally be only a small difference between $q(t)$ calculated by (1) and $q^*(t)$ calculated by linear interpolation between the Index values, as

$$q^*(t) = \frac{d2}{d}\mathrm{Q(M)} + \frac{d1}{d}\mathrm{Q(M+1)}. \tag{2}$$

7. The largest monthly proportionate rise in the RPI in recent years was between June and July 1979, when it rose by 4·3%, from 219·6 to 229·1. At a point half way between these dates formula (1) gives 224·30, whereas formula (2) gives 224·35, an overestimate of 0·02%. In these circumstances the difference is trivial. But if inflation rates were to be at a much higher level then formula (1) would be preferable.

Observed real returns

8. We first discuss the calculation of observed real returns over past periods, when all the relevant values are known. Assume first

that a single investment of amount A is made at time t, when the (interpolated) value of the RPI is $q(t)$. The investment is realised for an amount B at time u, when the value of the RPI is $q(u)$. The value of the proceeds, in terms of purchasing power at t, is

$$\mathrm{B}q(t)/q(u)$$

and the return on the investment is given by

$$\mathrm{R} = \frac{\mathrm{B}q(t)}{\mathrm{A}q(u)}. \tag{3}$$

Alternatively, the value of the invested amount, in terms of purchasing power at u, is

$$\mathrm{A}q(u)/q(t)$$

giving the same formula (3) for the value of R.

9. This return, R, is equivalent to a uniform compound real yield per unit of time, j, where

$$(1+j)^{u-t} = \mathrm{R} \tag{4}$$

or

$$j = \mathrm{R}^{1/(u-t)} - 1.$$

10. The money return on the investment is given by

$$\mathrm{B}/\mathrm{A}$$

and the uniform compound money yield, i is given by

$$(1+i)^{u-t} = \mathrm{B}/\mathrm{A} \tag{5}$$

or

$$i = \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^{1/(u-t)} - 1.$$

11. The uniform rate of growth of the RPI over the time period, the rate of inflation, r, is given by

$$(1+r)^{u-t} = q(u)/q(t). \tag{6}$$

or

$$r = \left(\frac{q(u)}{q(t)}\right)^{1/(u-t)} - 1.$$

12. Formulae (3), (4), (5) and (6) can be combined to give

$$(1+j)^{u-t} = \mathrm{R} = \frac{\mathrm{B}}{\mathrm{A}} \cdot \frac{q(t)}{q(u)} = (1+i)^{u-t}/(1+r)^{u-t}$$

or

$$(1+j) = (1+i)/(1+r) \tag{7}$$

or

$$(1+i) = (1+j)(1+r).$$

Thus, in this case, the real yield, j, can be calculated from the money yield, i, and the rate of inflation, r. This relationship holds over any single time period; but it needs modification when multiple payments are involved.

13. Note that the rough approximation

$$i = j+r,$$

is not exact. The correct expression is

$$i = j+r+jr,$$

and the final term is usually significant.

14. In this case the intervening values of the RPI are of no relevance, any more than are the market values of the investment on any intervening date.

Example 1

15. As an example we calculate the real yield on the Financial Times-Actuaries All-Share Index (ignoring income) over the period from 31 December 1981 to 31 December 1982. The values of the All-Share Index on these dates were 313·12 (= A) and 382·22 (= B) respectively. The necessary values of the RPI were:

15 December 1981	308·8
12 January 1982	310·6
14 December 1982	325·5
11 January 1983	325·9.

To calculate the interpolated values of the RPI for 31 December 1981 we need to interpolate over 28 days from 15 December 1981 to 12 January 1982, divided into 16 days to 31 December and 12 days from then to 12 January.

Interpolation using logarithms of the RPI values gives

$$q(\text{31 December 1981}) = 309{\cdot}83.$$

Similarly, for 31 December 1982, apportioning the period into 17 days and 11 days we get

$$q(31 \text{ December } 1982) = 325{\cdot}74.$$

Linear interpolation on the values, in these cases, gives results identical to 2 decimal places.

The real return can then be calculated as

$$\frac{\mathrm{B}}{\mathrm{A}} \cdot \frac{q(31 \text{ December } 1981)}{q(31 \text{ December } 1982)} = \frac{382{\cdot}22}{313{\cdot}12} \times \frac{309{\cdot}83}{325{\cdot}74} = 1{\cdot}1611,$$

giving a real yield of 16·11% over the period. The corresponding money yield can be calculated in the usual way from the ratio

$$\frac{\mathrm{B}}{\mathrm{A}} = \frac{382{\cdot}22}{313{\cdot}12} = 1{\cdot}2207$$

giving a money yield of 22·07%.

The rate of inflation over the period can be calculated from the ratio

$$\frac{q(31 \text{ December } 1982)}{q(31 \text{ December } 1981)} = \frac{325{\cdot}74}{309{\cdot}83} = 1{\cdot}0514,$$

giving a rate of inflation of 5·14%.

It can also be confirmed that

$$(1+\text{money yield}) = (1+\text{real yield}) \times (1+\text{inflation rate}),$$

or

$$1{\cdot}2207 = 1{\cdot}1611 \times 1{\cdot}0514,$$

(subject to rounding).

The rough approximation referred to above would give a real yield equal to the money yield minus the rate of inflation, or

$$22{\cdot}07\% - 5{\cdot}14\% = 16{\cdot}93\%.$$

It can be seen that this is too inexact for many purposes.

An approximation

16. It may be convenient, rather than interpolating for the exact date of payments, to use the RPI for the month in which a payment falls due. Thus, in formula (3) we could use Q(M1) and Q(M2),

where M1 and M2 were the months in which dates $t1$ and $t2$ fell, in place of $q(t1)$ and $q(t2)$. The altered formula (3) now becomes

$$R^* = \frac{B}{A} \cdot \frac{Q(M1)}{Q(M2)}. \tag{8}$$

For the example above we would calculate

$$\frac{B}{A} \cdot \frac{Q(\text{December 1981})}{Q(\text{December 1982})} = \frac{382{\cdot}22}{313{\cdot}12} \times \frac{308{\cdot}8}{325{\cdot}5} = 1{\cdot}1581$$

giving an approximate real return of 15·81%.

It can be seen that this gives a result noticeably different from the exact calculation. In some circumstances such an approximation may be adequate, and there are further circumstances described below where it may b^e^ justified; but the exact calculation should, in general, be preferred.

Multiple payments

17. It is frequently desired to calculate the real yield on an investment transaction that has involved more than one purchase of assets or more than one receipt of proceeds. We thus need to generalise the situation of paragraph 8.

Let there be m purchases of amounts A_k at times t_k, $k = 1, 2, \ldots, m$; and n proceeds of B_k at times u_k, $k = 1, 2, \ldots, n$.

Let

$$w = 1/(1+j).$$

Formula (3) could be re-expressed as

$$\frac{A}{q(t)} \cdot w^t = \frac{B}{q(u)} \cdot w^u, \tag{9}$$

and w is found as the solution to this equation.

18. The corresponding "equation of real value" for multiple payments is

$$\sum_{k=1}^{m} A_k \, w^{t_k}/q(t_k) = \sum_{k=1}^{n} B_k \, w^{u_k}/q(u_k), \tag{10}$$

and w is found as the solution to this equation. (As with any compound interest equation of value there may be multiple real solutions for w, and there are normally multiple complex solutions; but normally there is only one real solution of relevance in each case). We should note that in formula (10) each payment is discounted to an arbitrary zero time, and is revalued to an arbitrary RPI base date. In practical calculations one would rearrange the equation as desired.

Example 2

19. The price of units in the FTA-ASI Unit Trust is always the same as the index value of the F.T.-Actuaries All-Share Index. On 31 December 1980 an investor bought 1,000 units at a price of 291·99p; on 31 December 1981 he bought a further 1,000 units at a price of 313·12p. On 31 December 1982 the unit price was 382·22p. What is his real return to that date for the transaction, ignoring any income? The relevant RPI values were:

16 December 1980	275·6
13 January 1981	277·3
15 December 1981	308·8
12 January 1982	310·6
14 December 1982	325·5
11 January 1983	325·9.

The interpolated values of the retail price index are:

q(31 December 1980)	276·51
q(31 December 1981)	309·83
q(31 December 1982)	325·74.

The investor's purchases cost £2,919·9 on 31 December 1980 and £3,131·2 on 31 December 1981. His investment was worth £7,644·4 on 31 December 1982. The equation of value, discounting to 31 December 1980 is

$$\frac{2{,}919{\cdot}9}{276{\cdot}51}+\frac{3{,}131{\cdot}2w}{309{\cdot}83}=\frac{7{,}644{\cdot}4w^2}{325{\cdot}74}$$

or

$$23{\cdot}4678w^2-10{\cdot}1062w-10{\cdot}5598=0.$$

The solutions to this equation are $w = 0{\cdot}9198$ and $-0{\cdot}4892$. The negative answer is irrelevant to the problem; thus we obtain

$$1+j = 1/w = 1/0{\cdot}9198 = 1{\cdot}0872,$$

or a real yield of 8·72%.

The money yield on the transaction can be calculated from the usual equation of value

$$2{,}919{\cdot}9+3{,}131{\cdot}2v = 7{,}644{\cdot}4v^2$$

where $v = 1/(1+i)$, the discounting factor for money yields. This gives the practical solution $v = 0{\cdot}8559$, or a money yield of 16·84%.

The equivalent uniform inflation rate, v, over the two-year period is given by

$$276{\cdot}52(1+r)^2 = 325{\cdot}74,$$

whence $r = 8{\cdot}54\%$.

It should be noted that, in this example, it is not true that $(1+j) = (1+i)/(1+r)$. The reason for this is that both i, the money yield, and j, the real yield, are " internal rates of return " or " money-weighted returns ", which take account of the money flows at different dates. The overall rate of inflation, r, is based only on the values of the RPI at the beginning and end of the whole period, and corresponds to a " time-weighted return ".

Accounting flows

20. It is often required to calculate approximate yields or " money-weighted returns " on the basis, not of exact records of the amounts and dates of each payment, but on accounting records that show the total flow of income and outgo for each month, or even longer period. Provided the data are available for each calendar month, and there is reason to assume that the flows are fairly even over the calendar month, then it is appropriate to use the RPI for the month in question, *i.e.*, to divide the flows in month M by Q(M). Approximate methods such as this have to assume, say, that all payments are due or received in the middle of the month and are discounted using factors appropriate to a mid-month date, and the inaccuracy in assuming that all payments in the month are devalued by the same price index is probably of a comparable order. Any specially large payments should however be separately treated, as should the value of the portfolio at the beginning and end of the period.

Future yields

21. When an investment is contemplated it is desirable to calculate the future yield that will be obtained on the transaction, on the basis of such assumptions as are necessary. When a fixed interest investment is concerned, it is usual to assume that the promised money payments will certainly be made, so one can avoid making any estimates of the amounts of future payments. In other circumstances one may make estimates of the amounts of future payments, and calculate a yield on these assumptions. Since the future values of the RPI are not known, it is necessary to make some appropriate assumptions in order to calculate real yields.

22. Let us assume that an investment of A is made at time t, " now ", which will result in n proceeds B_k at times u_k, $k = 1, 2, \ldots, n$. We may know the money amounts B_k, or we may need to estimate them; but we here assume that we have made such estimates as we

need. Let us assume a uniform rate of inflation, r, from time t onwards, so that the estimated RPI value at time u_k is given by

$$q^*(u_k) = q(t) \,.\, (1+r)^{u_k-t}$$

Let the money yield on the transaction be i, with $v = 1/(1+i)$, and the real yield be j, with $w = 1/(1+j)$.

The equation of value in real terms (10) can now be rewritten as

$$\frac{\mathrm{A}\ w^t}{q(t)} = \sum_{k=1}^{n} \frac{\mathrm{B}_k w^{u_k}}{q^*(u_k)} = \sum_{k=1}^{n} \frac{\mathrm{B}_k w^{u_k}}{q(t)(1+r)^{u_k-t}} \tag{11}$$

or

$$\mathrm{A} = \sum_{k=1}^{n} \frac{\mathrm{B}_k w^{u_k-t}}{(1+r)^{u_k-t}} \tag{12}$$

But the equation of value for the money yield, i, is

$$\mathrm{A} = \sum_{k=1}^{n} \mathrm{B}_k v^{u_k-t}$$

which is the same as (12) if we put

$$v = w/(1+r)$$

or

$$(1+j) = (1+i)/(1+r), \tag{13}$$

which is the same relationship as stated in (7). Thus, we can either solve for the money yield, i, and calculate j from (13), or we can revalue each of the B_k by dividing by

$$(1+r)^{u_k-t}$$

and solving for the real yield j.

23. Relationship (13) holds only because we are assuming a uniform future rate of inflation. We could alternatively assume a variable future inflation rate, giving a form for the estimated RPI value at u_k as

$$q^*(u_k) = q(t) \,.\, f(u_k - t).$$

In this case relation (13) would not hold, and we would have to solve for j directly, *i.e.*, by first solving for w in equation (11).

Example 3

24. Exchequer 10·5% 1988 is a British Government Stock that pays interest of £5·25 per £100 nominal on 10 May and 10 November

each year until 10 May 1988 inclusive, on which date it will be redeemed at par. Its price for settlement on 10 May 1983 was £98 per £100 nominal. What was the prospective real yield to redemption (*a*) assuming future inflation at 7% per annum, and (*b*) assuming inflation in successive years from 10 May 1983 of 9%, 8%, 7%, 6% and 5%?

(*a*) The money redemption yield is found from the usual equation of value

$$98 = 5{\cdot}25a_{\overline{10}|}+100v^{10},$$

where the unit of time is a half year. This gives a solution, $v = 0{\cdot}947728$, which corresponds to an interest rate of 5·55% per half year, 11·10% per year convertible half yearly, or 11·41% per year effective.

The effective yearly real yield can be calculated from formula (13) by

$$(1+j) = (1+i)/(1+r),$$

where i, the effective annual money yield, is 0·1141, and r, the assumed rate of inflation, is 0·07, giving $1+j = 1{\cdot}1141/1{\cdot}07 = 1{\cdot}0412$, whence the effective yearly real yield is 4·12%, corresponding to 4·08% per year convertible half-yearly.

(*b*) In order to answer the question with a non-uniform rate of future inflation we must use formula (11) and write down the equation of value in real terms. If we assume that the RPI has a value of 100 on 10 May 1983, its value on future dates will be:

10 November 1983	104·40	10 May 1984	109·0C
10 November 1984	113·28	10 May 1985	117·72
10 November 1985	121·77	10 May 1986	125·96
10 November 1986	129·68	10 May 1987	133·52
10 November 1987	136·82	10 May 1988	140·19

The above values have been calculated as follows:

10 November 1983	$100\times 1{\cdot}09^{1/2}$
10 May 1984	$100\times 1{\cdot}09$
10 November 1984	$100\times 1{\cdot}09\times 1{\cdot}08^{1/2}$
10 May 1985	$100\times 1{\cdot}09\times 1{\cdot}08$
etc.	

The real equation of value is then, putting $w = 1/(1+j)$, the real half-yearly discount rate,

$$\frac{98}{100} = \frac{5{\cdot}25\ w}{104{\cdot}40} + \frac{5{\cdot}25\ w^2}{109{\cdot}00} + \frac{5{\cdot}25\ w^3}{113{\cdot}28} + \frac{5{\cdot}25\ w^4}{117{\cdot}72} + \dots$$
$$+ \frac{5{\cdot}25\ w^{10}}{140{\cdot}19} + \frac{100\ w^{10}}{140{\cdot}19}$$

a solution to which can be found, after some calculation, to be $w = 0{\cdot}981228$, or a real yield of 3·83% p.a. convertible half-yearly.

Index-Linked National Savings Certificates

25. Since June 1975 the British Government has issued Index-Linked National Savings Certificates whose value is linked to the Retail Prices Index. The first (Retirement) Issue was available only to persons of retirement age and was replaced by the second Index-Linked Issue in November 1980, which is available to anyone. The repayment value within one year of purchase is the purchase price. For certificates cashed within five years the repayment value is equal to the purchase price multiplied by the RPI two months before repayment and divided by the RPI two months before purchase. However, the repayment value is never less than the purchase price. A supplement of 4% of the purchase price is payable if the certificate is held for a full five years, and this supplement is itself subsequently indexed. Further supplements may be and have been paid on certificates in issue from time to time. We shall ignore these further supplements in this note.

26. A certificate bought in month M at date t (measured in years) for £A is repaid in month N at date u for an amount £B where

$$\mathrm{B} = \begin{cases} \text{(i)}\ \mathrm{A} & \text{if } u<t+1 \\ \text{(ii)}\ \mathrm{A}\,.\,\dfrac{\mathrm{Q(N-2)}}{\mathrm{Q(M-2)}} & \text{if } t+1 \leqslant u<t+5 \\ \text{(iii)}\ \mathrm{A}\left\{\dfrac{\mathrm{Q(M+58)}}{\mathrm{Q(M-2)}}+0{\cdot}04\right\}.\,\dfrac{\mathrm{Q(N-2)}}{\mathrm{Q(M+58)}} & \text{if } t+5 \leqslant u \end{cases}$$

and where in every case B is not less than A.

The real rate of return, r, that has been earned on such a certificate can then be calculated in the usual way from

$$(1+j)^{u-t} = \mathrm{R} = \frac{\mathrm{B}}{\mathrm{A}}\,.\,\frac{q(t)}{q(u)}\,.$$

The period from month M−2 to month N−2 is roughly the same length as from t (in month M) to u (in month N), but not necessarily exactly so. The difference can be up to almost one month either way. The rate of inflation experienced over these two periods will have been similar, but not necessarily exactly equal, since they are out of step by about two months at each end. Even in case (ii) therefore the real rate of return is not necessarily exactly zero.

27. If we are about to purchase a certificate at date t in month M, with the intention, say, of redeeming it at date u in month N, where $t+1 \leqslant u < t+5$ (so that case (ii) applies), and where $u-t$ is an exact number of months (and therefore equal to (N−M)/12), we can calculate the expected return on the assumption of a future inflation rate r by putting

$$q^*(\mathrm{N}-2) = q(\mathrm{M}-2)\,.\,(1+r)^{(\mathrm{N}-\mathrm{M})/12},$$
$$q^*(u) = q(t)\,.\,(1+r)^{u-t},$$

so that the same uniform rate of inflation is assumed for the period from month M−2 through to date u. Note that we do not know the value of $q(t)$ at date t. Provided that the price index rises over the period, the assumed return is then given by

$$\begin{aligned}(1+j)^{u-t} &= \frac{\mathrm{B}}{\mathrm{A}}\cdot\frac{q(t)}{q^*(u)} = \frac{\mathrm{A}}{\mathrm{A}}\cdot\frac{q^*(\mathrm{N}-2)}{q(\mathrm{M}-2)}\cdot\frac{q(t)}{q^*(u)}\\ &= \frac{q(\mathrm{M}-2)\,.\,(1+r)^{(\mathrm{N}-\mathrm{M})/12}}{q(\mathrm{M}-2)}\cdot\frac{q(t)}{q(t)\,.\,(1+r)^{u-t}}\\ &= 1.\end{aligned}$$

Hence $j = 0$. The popular, and natural assumption that Index-Linked Savings Certificates held for more than one but less than five years give a nil real return is thus seen to be a reasonable assumption for the future, though it depends on the Certificate being held for an exact number of months, and on the experienced rates of inflation for the roughly two month periods from the RPI date in month M−2 to the purchase date t and from the RPI date in month N−2 to the sale date u being equal.

Example 4

28. A certificate for £100 was purchased in July 1981 and redeemed in August 1982 for £109·79. What was the real rate of return if the certificate was (*a*) bought on 1 July 1981 and sold on 1 August 1982; (*b*) bought on 1 July 1981 and sold on 31 August 1982; (*c*) bought on 31 July 1981 and sold on 1 August 1982; (*d*) bought on 31 July 1981 and sold on 31 August 1982. The relevant RPI values were:

19 May 1981	294·1
16 June 1981	295·8
14 July 1981	297·1
18 August 1981	299·3
15 June 1982	322·9
13 July 1982	323·0
17 August 1982	323·1
14 September 1982	322·9.

We can first confirm that the redemption value of £109·79 in fact equals

$$£100 \times \frac{Q(\text{June } 1982)}{Q(\text{May } 1981)} = £100 \times \frac{322{\cdot}9}{294{\cdot}1}.$$

By interpolation on the logarithms of the price indices we can calculate the intermediate RPI values as:

$$\begin{aligned} q(1 \text{ July } 1981) &= 296{\cdot}50 \\ q(31 \text{ July } 1981) &= 298{\cdot}17 \\ q(1 \text{ August } 1982) &= 323{\cdot}05 \\ q(31 \text{ August } 1981) &= 323{\cdot}00. \end{aligned}$$

The returns can then be calculated:

$$(a)\ \mathrm{R} = \frac{109{\cdot}79}{100} \frac{q(1 \text{ July } 1981)}{q(1 \text{ August } 1982)} = \frac{109{\cdot}79}{100} \times \frac{296{\cdot}50}{323{\cdot}05} = 1{\cdot}0077.$$

The period $(u-t)$ can be taken as 1 year and 31 days = 1·0849 years, so the real return per annum is

$$1{\cdot}0077^{1/1{\cdot}0849} - 1 = 1{\cdot}0071 - 1 = 0{\cdot}71\%$$

$$(b)\ \mathrm{R} = \frac{109{\cdot}79}{100} \cdot \frac{q(1 \text{ July } 1981)}{q(31 \text{ August } 1982)} = \frac{109{\cdot}79}{100} \times \frac{296{\cdot}50}{323{\cdot}00} = 1{\cdot}0078$$

The period $(u-t)$ can be taken as 1 year and 61 days = 1·1671 years, so the real return per annum is

$$1{\cdot}0078^{1/1{\cdot}1671} - 1 = 0{\cdot}67\%.$$

$$(c)\ \mathrm{R} = \frac{109{\cdot}79}{100} \cdot \frac{q(31 \text{ July } 1981)}{q(1 \text{ August } 1982)} = \frac{109{\cdot}79}{100} \times \frac{298{\cdot}17}{323{\cdot}05} = 1{\cdot}0133$$

The period $(u-t)$ can be taken as 1 year and 1 day = 1·0027 years, so the real return per annum is

$$1{\cdot}0133^{1/1{\cdot}0027} - 1 = 1{\cdot}33\%.$$

$$(d)\ \mathrm{R} = \frac{109{\cdot}79}{100} \cdot \frac{q(31\text{ July }1981)}{q(31\text{ August }1982)} = \frac{109{\cdot}79}{100} \times \frac{298{\cdot}17}{323{\cdot}05} = 1{\cdot}0135$$

The period $(u-t)$ can be taken as 1 year and 31 days $= 1{\cdot}0849$ years, so the real return per annum is

$$1{\cdot}0135^{1/1{\cdot}0849} - 1 = 1{\cdot}24\%.$$

Note that the real return varies with the dates of purchase and sale within the month: even though cases (*a*) and (*d*) are both for a 13-month period they show different real yields. In case (*b*) the certificate is held for a longer period, nearly 14 months, so the real yield may be expected to be smaller. In case (*c*) the certificate is held for just over 12 months, so may be expected to show the highest real yield (as it would the highest money yield).

Note also that the rate of inflation in the months following May 1981 was greater than that in the months following June 1982. Thus the real returns on the certificates were all positive, though small. If the reverse had been true, the real returns could have been negative and small.

Index-Linked Government Stocks

29. The British Government issued its first Index-Linked Treasury Stock in March 1981. It carried a coupon of 2% and is redeemable at " par " on 16 September 1996. Interest payments are due on 16 March and 16 September up to 16 September 1996. The stock was issued at a price of 100 per £100 nominal, payable in three instalments: £35 on 27 March 1981, with calls of £30 on 1 May 1981 and £35 on 26 May 1981. Each interest payment is indexed according to the ratio of the RPI for the month eight months before the due date to the RPI for July 1980, eight months before the issue date, the value of which was 267·9. Thus the interest payment per £100 nominal in month M is

$$£1 \times \frac{\mathrm{Q}(\mathrm{M}-8)}{\mathrm{Q}(\text{July }1980)}.$$

The redemption amount is indexed similarly and, per £100 nominal, will be

$$£100 \times \frac{\mathrm{Q}(\text{January }1996)}{\mathrm{Q}(\text{July }1980)},$$

since January 1996 is eight months before the redemption date of 16 September 1996. The first interest payment was a fractional one, prorated for the period from the dates of issue and calls to 16 September 1981, and ratioed by Q(January 1981) divided by Q(July 1980).

30. A number of similar stocks have been issued since the first one, with various redemption dates and coupons either of 2% or 2·5%. All have had essentially the same provisions, though they have differed in the exact method of calculating the interest and redemption payments (in the early issues these amounts per £100 nominal were calculated to 2 decimal places of £1 rounded down; in the later issues to 4 decimal places). None has had a spread of redemption dates, nor is redeemable at other than an indexed " par ". One, 2·5% Index-Linked Treasury Convertible 1999 includes an option to convert into a specific fixed money stock on specified terms.

31. The calculation of observed real returns for such stocks follows exactly along the lines described above, as the example below shows.

Example 5

32. An investor subscribed for a quantity of 2% Index-Linked Treasury Stock 1996 at the issue date, paying in instalments per £100 nominal of £35 on 27 March 1981, £30 on 1 May 1981 and £35 on 26 May 1981. He received interest payments per £100 nominal of £0·80 on 16 September 1981, £1·10 on 16 March 1982 and £1·15 on 16 September 1982. He sold the stock for £107·50 per £100 nominal on 31 December 1982. What was his real return over the period, ignoring tax? The relevant values of the RPI were:

15 July 1980	267·9
17 March 1981	284·0
14 April 1981	292·2
19 May 1981	294·1
16 June 1981	295·8
14 July 1981	297·1
15 September 1981	301·0
13 October 1981	303·7
12 January 1982	310·6
16 March 1982	313·4
14 September 1982	322·9
12 October 1982	324·5
14 December 1982	325·5
11 January 1983	325·9.

We can first confirm that the interest payments were correctly calculated as

$$1\times\frac{Q(\text{July 1981})}{Q(\text{July 1980})}=1\times\frac{297{\cdot}1}{267{\cdot}9}=1{\cdot}10 \text{ on 16 March 1982,}$$

and

$$1\times\frac{Q(\text{January 1982})}{Q(\text{July 1980})}=1\times\frac{310{\cdot}6}{267{\cdot}9}=1{\cdot}15 \text{ on 16 September 1982,}$$

in each case rounded down to two decimal places of £1.

The intermediate RPI values can be calculated by the usual interpolation on the logarithms as:

$$\begin{aligned}
&q(\text{27 March 1981}) &&= 286{\cdot}90\\
&q(\text{1 May 1981}) &&= 293{\cdot}12\\
&q(\text{26 May 1981}) &&= 294{\cdot}52\\
&q(\text{16 September 1981}) &&= 301{\cdot}10\\
&q(\text{16 March 1982}) &&= Q(\text{March 1982}) = 313{\cdot}4\\
&q(\text{16 September 1982}) &&= 323{\cdot}01\\
&q(\text{31 December 1982}) &&= 325{\cdot}74.
\end{aligned}$$

Note that we do not need to interpolate to obtain the value of the RPI on 16 March 1982.

It is convenient to work in time units of a half year between interest dates, and to measure from the issue date.

Thus we have:

27 March 1981 to 1 May 1981 = 35 days = 35/182·5 = 0·1918 units
27 March 1981 to 26 May 1981 = 60 days = 60/182·5 = 0·3288 units
27 March 1981 to 16 September 1981 = 173 days = 173/182·5 = 0·9479 units
27 March 1981 to 16 March 1982 = 173 days + 1 half-year = 1·9479 units
27 March 1981 to 16 September 1982 = 173 days + 2 half-years = 2·9479 units
27 March 1981 to 31 December 1982 = 173 days + 2 half-years + 106 days = 3·5288 units.

The equation of value in real terms, from formula (10), is then, per £100 nominal,

$$\frac{35}{286{\cdot}90}+\frac{30}{293{\cdot}12}w^{0{\cdot}1918}+\frac{35}{294{\cdot}52}w^{0{\cdot}3288}$$

$$=\frac{0{\cdot}80}{301{\cdot}10}w^{0{\cdot}9479}+\frac{1{\cdot}10}{313{\cdot}4}w^{1{\cdot}9479}+\frac{1{\cdot}15}{323{\cdot}01}w^{2{\cdot}9479}+\frac{107{\cdot}5}{325{\cdot}74}w^{3{\cdot}5288},$$

where the payments made are on the left hand side and the receipts on the right hand side. This, after some calculation, gives the solution $w = 1{\cdot}003038$, or a real return of $-0{\cdot}30\%$ per half year, $-0{\cdot}60\%$ per annum convertible half-yearly, or $-0{\cdot}60\%$ per annum effective.

Future yields on index-linked stocks

33. In order to calculate future prospective real yields to redemption we must, as described above, make some explicit assumptions about future inflation, both in order to estimate the future interest and redemption amounts, and to estimate the future RPI values at the payment dates. These, however, are two different calculations, as they have been shown to be for the Index-Linked Savings Certificates.

34. We assume purchase of an index-linked stock for settlement at date t in month M for a price per £100 nominal of 1. This price should include any accrued interest payments due by or to the purchaser, as happens when a stock is less than five years to redemption; since the money amount of the next coupon payment is always known, the amount of this accrued interest adjustment can be calculated.

35. The money amounts of any future calls due are known. Let the call amounts be B1 and B2, due at fractions of a half year from the settlement date of $b1$ and $b2$.

36. The money amount of the interest due on the next due date after settlement is also known; this may be the first irregular interest payment, a full indexed payment, or if the stock is purchased " ex interest " it will be zero. Let the amount of this payment be C0, due a fraction of a half year from the settlement date of f.

37. Let us first assume that the value of the RPI is actually published at the end of the month following its applicable date. Thus, for settlements in month M the latest known RPI is Q(M−2). This is true for most of the month, but not true for about ten days at the end of the month, when the value of Q(M−1) is actually known; we shall consider this case later.

38. Consider now the next but one interest payment, due in $f+1$ half years from the settlement date, in month N. Month N may be between 6 months later than month M and 12 months later than month M (the month of the settlement date). Let K = 12+M−N, so that as M increases, K also increases from 0, when M = N−12, to 6, when M = N−6.

The interest payment due in month N will be based on the value of the RPI for month N−8, viz. Q(N−8); this is (usually) not known; but if we assume a level half-yearly rate of inflation of r, we can estimate Q(N−8) from the last known RPI, Q(M−2), as

$$
\begin{aligned}
Q^*(N-8) &= Q(M-2) \,.\, (1+r)^{(N-M-6)/6} \\
&= Q(M-2) \,.\, (1+r) \,.\, (1+r)^{-K/6}.
\end{aligned}
$$

When M = N−6, so that K = 6,

$$Q^*(N-8) = Q(M-2),$$

i.e. the actual RPI value used to calculate the next but one interest payment is already known, and is (usually) the latest one published.

The amount of the interest payment is

$$C \,.\, \frac{Q^*(N-8)}{QB} = C \,.\, \frac{Q(M-2)}{QB} \,.\, (1+r) \,.\, (1+r)^{-K/6},$$

where C is the nominal coupon per half year, and QB is the RPI for the base month for the stock.

39. Following the same argument we can estimate the amount of each future interest payment, and the amount of the redemption payment. Thus the interest payment due in $f+u$ half years from settlement (u integral) is estimated as

$$C \,.\, \frac{Q(M-2)}{QB} \,.\, (1+r)^u \,.\, (1+r)^{-K/6},$$

and the redemption payment due in $f+n$ half years from settlement (n integral) is estimated as

$$100 \,.\, \frac{Q(M-2)}{QB} \,.\, (1+r)^n \,.\, (1+r)^{-K/6}.$$

40. We can now write down the money equation of value, with v as the half-yearly money discount factor, as

$$
\begin{aligned}
&P+B1 \,.\, v^{b1}+B2 \,.\, v^{b2} = C0 \,.\, v^f \\
&+C \,.\, \frac{Q(M-2)}{QB} \,.\, (1+r)^{-K/6} \,.\, v^f \,.\, \{(1+r)v+(1+r)^2v^2+\ldots+(1+r)^nv^n\} \\
&+100 \,.\, \frac{Q(M-2)}{QB} \,.\, (1+r)^n \,.\, (1+r)^{-K/6} \,.\, v^{f+n}, \qquad (14)
\end{aligned}
$$

and if we substitute the half-yearly real discount factor $w = v(1+r)$, assuming the same level future inflation rate r per half year, we get

$$P+B1\left(\frac{w}{1+r}\right)^{b1}+B2\left(\frac{w}{1+r}\right)^{b2}$$

$$=\left(\frac{w}{1+r}\right)^{f}\left\{C0+C\cdot\frac{Q(M-2)}{QB}\cdot(1+r)^{-K/6}\quad(w+w^2+\ldots+w^n)\right.$$

$$\left.+100\cdot\frac{Q(M-2)}{QB}\cdot(1+r)^{-K/6}\cdot w^n\right\}, \tag{15}$$

which is a convenient formula to solve to obtain w.

41. Note that the factor $(1+r)$ involving the estimated future inflation rate only enters the expression for durations of up to one half year. The value of w and hence the real yield j (convertible half-yearly) $= 2(1/w-1)$, is not very greatly affected by changes in the assumed value of r, but nevertheless the effect of the assumed value of r is by no means negligible. See further paragraph 46.

42. We now consider the case where the latest known RPI is that for month $M-1$, which is normally the case for the last ten days or so of each month. In most months we can take $Q(M-1)$ into account simply by replacing the expressions

$$\frac{Q(M-2)}{QB}\cdot(1+r)^{-K/6}$$

in formula (15) by

$$\frac{Q(M-1)}{QB}\cdot(1+r)^{-(K+1)/6}$$

However, if M is a month in which a payment date for the stock falls we know the amounts, both of the payment due that month (C0), and of that due in six months' time, which will be exactly

$$C\cdot\frac{Q(M-2)}{QB}.$$

In this case we need to replace the appropriate part of formula (15) by

$$C0+C\cdot\frac{Q(M-2)}{QB}\cdot\frac{w}{1+r}+C\cdot\frac{Q(M-1)}{QB}\cdot(1+r)^{-(K+1)/6}\,(w^2+w^3+\ldots$$

$$+w^n)+100\,\frac{Q(M-1)}{QB}\cdot(1+r)^{-(K+1)/6}\cdot w^n.$$

In fact $K = 6$ in this case.

Yet a further modification is necessary in the last few months before the redemption date, when the actual amount of redemption payment is known. But by this stage the payments on the stock are wholly fixed in money terms, and one would normally calculate a fixed money redemption yield rather than an estimated real yield.

43. One possible assumption for r is to put

$$(1+r) = \frac{Q(M-2)}{Q(M-8)},$$

i.e. inflation is assumed to continue indefinitely at the same rate as over the last known six months. Since f decreases uniformly from 1 to 0 as the settlement dates move from one interest date to the next, and K/6 increases stepwise from 0 to 1 by steps of 1/6 over the same period, the value of $(f+K/6)$ is approximately 1, or say $(1+e)$, where e may range from about $-1/6$ to $+1/6$, depending on the actual due dates.

We then rewrite formula (15) as

$$P+B1\left(\frac{w}{1+r}\right)^{b1}+B2\left(\frac{w}{1+r}\right)^{b2}$$

$$=\left(\frac{w}{1+r}\right)^{f} C0+w^{f}\left\{C \cdot \frac{Q(M-2)}{QB} \cdot \frac{1}{(1+r)^{1+e}}(w+w^{2}+\ldots+w^{n})\right.$$

$$\left.+100 \cdot \frac{Q(M-2)}{QB} \cdot \frac{1}{(1+r)^{1+e}} w^{n}\right\}$$

or substituting $Q(M-2) = (1+r)\,Q(M-8)$,

$$=\left(\frac{w}{1+r}\right)^{f} C0+w^{f}\left\{C \cdot \frac{Q(M-8)}{QB} \cdot \frac{1}{(1+r)^{e}}(w+w^{2}+\ldots+w^{n})\right.$$

$$\left.+100 \cdot \frac{Q(M-8)}{QB} \cdot \frac{1}{(1+r)^{e}} w^{n}\right\}. \tag{16}$$

Since, after a short initial period, the calls B1 and B2 fall away, we can rewrite (15) to give:

$$P \cdot \frac{QB}{Q(M-8)}$$

$$=\left(\frac{w}{1+r}\right)^{f} \frac{QB}{Q(M-8)} C0+\frac{w^{f}}{(1+r)^{e}}\left\{C(w+w^{2}+\ldots+w^{n})+100w^{n}\right\}.$$

On an exact payment date $f = e = 0$ and $C0 = 0$, so we get

$$P \cdot \frac{QB}{Q(M-8)} = C(w+w^{2}+\ldots+w^{n})+100w^{n},$$

which is the formula that would be appropriate for a corresponding fixed interest stock with coupon C per half year, where the price was "deflated" to the base date according to the latest RPI value. Although this expression is simple and thus superficially attractive, the above development shows that it is accurate only on an exact payment date, and not otherwise, and that it depends on the assumption that inflation will continue at the same rate as over the last known six months. This assumption may not be appropriate to the circumstances at any particular time.

Example 6

44. Calculate the expected real yield to redemption on 2% Index-Linked Treasury Stock 1996, the details of which are given in Example 5, assuming purchase on the issue date, 27 March 1981, at £35 per £100 nominal, and assuming future inflation at the rates of 0%, 4%, 7%, 10% and 13%. The latest known RPI value is to be taken as that for:

13 January 1981 277·3.

In the notation of the preceding paragraphs we have, per £100 nominal:

Immediate purchase price = P = £35 (at duration 0)
First call = B1 = £30, due at duration $b1 = 0{\cdot}1918$
Second call = B2 = £35, due at duration $b2 = 0{\cdot}3288$
First (irregular) interest = C0, due at duration $f = 0{\cdot}9479$
Month of purchase = M = March 1981
Month of last known RPI = M−2 = January 1981
Regular nominal interest per half year = C = £1
Next (regular) interest = C×Q(July 81)/QB, due at duration $1+f = 1{\cdot}9479$
Month of next interest = N = March 1982
K = 12+M−N = 12+March 1981−March 1982 = 0
Last interest and redemption due at duration $n+f = 30{\cdot}9479$

It is convenient to state the future payments and receipts in a schedule (see page 103). See also the notes at the foot of that schedule.

We then sum the last column of the schedule, equate the answer to zero, choose the required value of $(1+r)$ (noting that this should be the rate of inflation per half year), and solve for w. The formula, which corresponds to formula (15), can be somewhat simplified to

$$-35-30\left(\frac{w}{1+r}\right)^{0\cdot1918}-35\left(\frac{w}{1+r}\right)^{0\cdot3288}$$
$$+\left(\frac{w}{1+r}\right)^{0\cdot9479}\left\{0{\cdot}80+\frac{QC}{QB}\,(w+w^2+\ldots+w^{30})+100\,\frac{QC}{QB}w^{30}\right\}=0.$$

We can also calculate the money discounting factor, v, from the formula $v = w/(1+r)$, and obtain a money return per annum; we could equally well obtain this by summing the second last column of the schedule, equating the answer to zero to give a formula equivalent to (14), and solving for v; this method of calculation gives identical results.

For the various inflation rates assumed we get, after some calculation:

Inflation rate p.a. (R) %	$(1+r) = (1+R/100)^{\frac{1}{2}}$	w	Real return p.a. convertible half-yearly %	v	Money return p.a. convertible half-yearly %
0	1·0	0·988817	2·26	0·988817	2·26
4	1·0198	0·989383	2·14	0·970170	6·15
7	1·0344	0·989793	2·06	0·956869	9·01
10	1·0488	0·990192	1·98	0·944111	11·84
13	1·0630	0·990580	1·90	0·931860	14·62

Further considerations

45. Instead of assuming a uniform future rate of inflation, we could assume a rate that varied in some specified way, as in Example 3 above. This would require us to specify a function for the value of the RPI applying to any date after the applicable date of the last known RPI, to use first for calculating the future interest and redemption amounts, and then for deflating the assumed money payments. In order to maintain consistency between the two uses one must assume specific applicable dates for future monthly RPI values. Since the prospective real yield is comparatively insensitive to changes in the assumed rate of future inflation, such complications are seldom worth while.

46. The sensitivity of the real return to changes in the assumed rate of future inflation can be investigated algebraically. Let us first define the inflation rate per cent per year as R (as in Example 6 above), so that

$$(1+r) = (1+R/100)^{1/2},$$

and put

$$g = 1/(1+r) = 1/(1+R/100)^{1/2}.$$

Then define the real return per cent per year convertible half-yearly as J, so that

$$(1+j) = (1+J/200),$$

and

$$w = 1/(1+j) = 1/(1+J/200).$$

Payment	Date	Amount	Estimate	Duration from 27.3.81	Money discounting factor	Present value in terms of v	Present value in terms of w
Issue price	27.3.81	$-£35$	—	0	1	-35	-35
Call 1	1.5.81	$-£30$	—	0·1918	$v^{0\cdot1918}$	$-30\times v^{0\cdot1918}$	$-30\left(\frac{w}{1+r}\right)^{0\cdot1918}$
Call 2	26.5.81	$-£35$	—	0·3288	$v^{0\cdot3288}$	$-35\times v^{0\cdot3288}$	$-35\left(\frac{w}{1+r}\right)^{0\cdot3288}$
Interest 0	16.9.81	$+£0\cdot80$	—	0·9479	$v^{0\cdot9479}$	$0\cdot80\times v^{0\cdot9479}$	$0\cdot80\left(\frac{w}{1+r}\right)^{0\cdot9479}$
Interest 1	16.3.82	$\frac{1\times Q(7{:}81)}{QB}$	$\frac{QC}{QB}(1+r)$	1·9479	$v^{1\cdot9479}$	$\frac{QC}{QB}(1+r)v^{1\cdot9479}$	$\frac{QC}{QB}\frac{w^{1\cdot9479}}{(1+r)^{0\cdot9479}}$
...	...	...	...	...	...	...	...
Interest u	16.Month U	$\frac{1\times Q(U-8)}{QB}$	$\frac{QC}{QB}(1+r)^{u}$	$u\cdot9479$	$v^{u\cdot9479}$	$\frac{QC}{QB}(1+r)^{u}v^{u\cdot9479}$	$\frac{QC}{QB}\frac{w^{u\cdot9479}}{(1+r)^{0\cdot9479}}$
...	...	...	...	...	...	...	...
Interest 30	16.9.96	$\frac{1\times Q(1.96)}{QB}$	$\frac{QC}{QB}(1+r)^{30}$	30·9479	$v^{30\cdot9479}$	$\frac{QC}{QB}(1+r)^{30}v^{30\cdot9479}$	$\frac{QC}{QB}\frac{w^{30\cdot9479}}{(1+r)^{0\cdot9479}}$
Redemption	16.9.96	$\frac{100\times Q(1.96)}{QB}$	$100\frac{QC}{QB}(1+r)^{30}$	30·9479	$v^{30\cdot9479}$	$100\frac{QC}{QB}(1+r)^{30}v^{30\cdot9479}$	$100\frac{QC}{QB}\frac{w^{30\cdot9479}}{(1+r)^{0\cdot9479}}$

Notes: (1) Payments are marked as negative, receipts as positive.
(2) QB = Q for Base month = Q(July 1980) = 267·9.
(3) QC = Q for Current month = Q(March 1981 − 2 months) = Q(January 1981) = 277·3.
(4) r is the assumed rate of inflation per half year.
(5) Duration is calculated in half years on 16 March and 16 September.
(6) v is the money discount factor per half year.
(7) w is the real discount factor per half year $= v/(1+r)$.

Now rewrite formula (15) as

$$F = -P - B1\left(\frac{w}{1+r}\right)^{b1} - B2\left(\frac{w}{1+r}\right)^{b2}$$
$$+\left(\frac{w}{1+r}\right)^{f}\left\{C0 + C\,.\,\frac{Q(M-2)}{QB}\,.\,(1+r)^{-K/6}\,(w+w^2+\ldots+w^n)\right.$$
$$\left.+100\,.\,\frac{Q(M-2)}{QB}\,.\,(1+r)^{-K/6}w^n\right\} = 0 \qquad (18)$$

and substituting for w and r re-express F as a function of J and R,

$$F = F(J, R) = 0. \qquad (19)$$

Given a particular value of R, we can find the value of J that satisfies this equation. Thus equation (19) may be regarded as defining J as a function of R; this is what we have done in Example 6 above.

However, if we differentiate equation (19) with respect to R we get

$$\frac{dF}{dR} = \frac{\partial F}{\partial J}\cdot\frac{dJ}{dR}+\frac{\partial F}{\partial R} = 0,$$

whence

$$\frac{dJ}{dR} = -\frac{\partial F}{\partial R}\Big/\frac{\partial F}{\partial J},$$

for the values of R and J that make $F = 0$.

The derivative dJ/dR for these values gives the change in J per unit change in R at this point.

It is convenient to derive the derivative as follows:

$$\frac{\partial F}{\partial J} = \frac{\partial F}{\partial w}\cdot\frac{dw}{dJ}$$

and

$$\frac{\partial F}{\partial R} = \frac{\partial F}{\partial g}\cdot\frac{dg}{dR}$$

$$\frac{\partial F}{\partial w} = -\frac{B1\,.\,b1}{w}\left(\frac{w}{1+r}\right)^{b1} - \frac{B2\,.\,b2}{w}\left(\frac{w}{1+r}\right)^{b2}$$
$$+\left(\frac{w}{1+r}\right)^{f}\left\{\frac{f}{w}C0 + C\frac{Q(M-2)}{QR}\,(1+r)^{-K/6}\left[\frac{f}{w}(w+w^2+\ldots+w^n)\right.\right.$$
$$\left.\left.+\frac{1}{w}\,(w+2w^2+\ldots+nw^n)\right] + 100\frac{Q(M-2)}{QB}(1+r)^{-K/6}(n+f)w^{n-1}\right\} \qquad (20)$$

$$\frac{dw}{dJ} = \frac{-1}{200}\,1/(1+J/200)^2 = -w^2/200. \qquad (21)$$

Putting

$$F = -P - B1(wg)^{b1} - B2(wg)^{b2} + (wg)^{f}\left\{C0 + C\frac{Q(M-2)}{QB}g^{K/6}(w+w^2+\ldots+w^n) + 100\frac{Q(M-2)}{QB}g^{K/6}w^n\right\},$$

we get

$$\frac{\partial F}{\partial g} = -\frac{B1\,.\,b1}{g}(wg)^{b1} - \frac{B2\,.\,b2}{g}(wg)^{b2} + \frac{(wg)^f}{g}\left\{f\,C0 + C\frac{Q(M-2)}{QB}\left(f+\frac{K}{6}\right)g^{K/6}\,(w+w^2+\ldots+w^n) + 100\frac{Q(M-2)}{QB}\left(f+\frac{K}{6}\right)g^{K/6}w^n\right\}$$

$$= -(1+r)b1\,.\,B1\left(\frac{w}{1+r}\right)^{b1} - (1+r)b2\,.\,B2\left(\frac{w}{1+r}\right)^{b2} + (1+r)\left(\frac{w}{1+r}\right)^{f}\left\{f\,C0 + C\frac{Q(M-2)}{QB}\left(f+\frac{K}{6}\right)(1+r)^{-K/6}(w+w^2+\ldots+w^n) + 100\frac{Q(M-2)}{QB}\left(f+\frac{K}{6}\right)(1+r)^{-K/6}w^n\right\} \qquad (22)$$

$$\frac{dg}{dR} = \frac{-1}{200}(1/(1+R/100^{1\cdot5}) = -1/200(1+r)^3 \qquad (23)$$

whence

$$\frac{dJ}{dR} = -\frac{\partial F}{\partial R}\Big/\frac{\partial F}{\partial J} = -\frac{(22)(23)}{(20)(21)} = G(w, r) \qquad (24)$$

This rather complicated function can be evaluated for any chosen r and for the value of w that gives the solution at this value of r.

Example 7

47. Evaluate dJ/dR for the data of Example 6, and also calculate the real yields and the values of dJ/dR on the assumptions that the stock was redeemable in 1986, 1991, 2001 and 2006 instead of in 1996.

Some calculations solving formula (15) and using formula (24) give

	1986		1991		1996		2001		2006	
R%	J%	dJ/dR	J%	dJ/dR	J%	dJ/dR	J%	dJ/dR	J%	dJ/dR
0	2·68	−0·077	2·37	−0·042	2·26	−0·030	2·21	−0·023	2·17	−0·020
4	2·38	−0·074	2·21	−0·040	2·15	−0·028	2·12	−0·022	2·10	−0·019
7	2·16	−0·071	2·09	−0·039	2·06	−0·028	2·05	−0·022	2·04	−0·018
10	1·95	−0·069	1·97	−0·038	1·98	−0·027	1·98	−0·021	1·99	−0·018
13	1·75	−0·067	1·86	−0·037	1·90	−0·026	1·92	−0·021	1·93	−0·017

By taking differences of the values of J and R, one can confirm that the differential coefficients are of the expected size.

It is of interest also to note that the value of dJ/dR is fairly constant as R alters within each redemption date, and reduces in absolute value with an increase in the period to redemption. Thus the yield on a long stock is fairly insensitive to the assumed rate of inflation, while the yield on a shorter stock is more sensitive, though, of course, nothing like so sensitive as the money yield.

48. When the real yield on an index-linked stock is being quoted it is necessary to give also the assumed rate of inflation, R in the above formulae. It is also helpful to quote either the real yield at a different value of R, or the value of dJ/dR above, which could be called the sensitivity of J to changes in R. The approximately linear relationship of J and R over the relevant range allows others to estimate J for some other desired value of R.

APPENDIX

Values of the Retail Prices Index from January 1980

Applicable date	Value	Date published	Applicable date	Value	Date published
	1980			1982	
15 Jan.	245·3	15 Feb.	12 Jan.	310·6	12 Feb.
12 Feb.	248·8	14 Mar.	16 Feb.	310·7	19 Mar.
18 Mar.	252·2	18 Apr.	16 Mar.	313·4	23 Apr.
15 Apr.	260·8	16 May	20 Apr.	319·7	21 May
13 May	263·2	13 June	18 May	322·0	18 June
17 June	265·7	18 July	15 June	322·9	16 July
15 July	267·9	15 Aug.	13 July	323·0	13 Aug.
12 Aug.	268·5	12 Sept.	17 Aug.	323·1	17 Sept.
16 Sept.	270·2	17 Oct.	14 Sept.	322·9	15 Oct.
14 Oct.	271·9	14 Nov.	12 Oct.	324·5	12 Nov.
18 Nov.	274·1	19 Dec.	16 Nov.	326·1	17 Dec.
16 Dec.	275·6	16 Jan. 1981	14 Dec.	325·5	21 Jan. 1983
	1981			1983	
13 Jan.	277·3	13 Feb.	11 Jan.	325·9	11 Feb.
17 Feb.	279·8	20 Mar.	15 Feb.	327·3	18 Mar.
17 Mar.	284·0	16 Apr. (Thurs.)	15 Mar.	327·9	22 Apr.
14 Apr.	292·2	22 May	12 Apr.	332·5	20 May
19 May	294·1	19 June	17 May	333·9	17 June
16 June	295·8	17 July	14 June	334·7	15 July
14 July	297·1	14 Aug.	12 July	336·5	12 Aug.
18 Aug.	299·3	18 Sept.	16 Aug.	338·0	16 Sept.
15 Sept.	301·0	16 Oct.	13 Sept.	339·5	14 Oct.
13 Oct.	303·7	13 Nov.	11 Oct.	340·7	11 Nov.
17 Nov.	306·9	18 Dec.	15 Nov.	341·9	16 Dec.
15 Dec.	308·8	15 Jan. 1982	13 Dec.	342·8	20 Jan. 1984

Source: Employment Gazette.