Chain Ladder and Bornhuetter/Ferguson – Some Practical Aspects

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- Consider one single accident year
- Paid claims after k years of development
- Expected cumulative payout pattern

  $p_1, p_2, ..., p_k, ..., p_n = 1$ e.g.

  10%, 30%, 50%, 70%, 85%, 95%, 100%

- U_0 = prior est. of ultimate claims amount

  $R_{BF} = q_k U_0$ with $q_k = 1\text{-}p_k$ Bornhuetter/F.

- C_k = claims amount paid up to now (completely ignored by BF)
- $U_{BF} = C_k + R_{BF}$ posterior estimate ($\neq U_0$)
- $U = C_k + R$ (axiomatic relationship)
- $U_{CL} = C_k / p_k$ Chain Ladder ult. claims
- $R_{CL} = U_{CL} - C_k = q_k U_{CL}$ CL reserve (ignores U_0 completely)

Comparison:

- With CL, different actuaries usually come to similar results
- With BF, there is no clear way to U_0
- U_0 can be manipulated:
  If you want to have reserve X,
  simply put $U_0 = X / q_k$

- Gunnar Benktander's proposal (1976):

$$R_{GB} = p_k R_{CL} + (1-p_k) R_{BF}$$
$$= p_k q_k C_k / p_k + q_k R_{BF}$$
$$= q_k \,(C_k + R_{BF}) \;=\; q_k U_{BF}$$

- Iterated Bornhuetter/Ferguson

- The more the claims develop, the higher the weight p_k of R_{CL}.

Ultimate U(R)	Connection	Reserve R(U)
U_0	$\xrightarrow{*q_k}$	
	$\xleftarrow{C_k +}$	$R_{BF} = q_k U_0$
$U_1 = U_{BF} = C_k + q_k U_0$		
$= (1-q_k)U_{CL} + q_k U_0$	$\xrightarrow{*q_k}$	
		$R_1 = q_k U_1 = q_k U_{BF} = R_{GB}$
	$\xleftarrow{C_k +}$	$= (1-q_k)R_{CL} + q_k R_{BF}$
$U_2 = U_{GB}$		
$= (1-q_k^2)U_{CL} + q_k^2 U_0$		

Ultimate U(R)	Connection	Reserve R(U)

$U_n = (1-q_k{}^n)U_{CL} + q_k{}^n\, U_0$

$\searrow {}^{*q_k}$

$C_k + \swarrow$ $\qquad R_n = (1-q_k{}^n)R_{CL} + q_k{}^n R_{BF}$

$U_{n+1} = (1-q_k{}^{n+1})U_{CL} + q_k{}^{n+1}\, U_0$

$\searrow$

....

$U_\infty = U_{CL} \longleftrightarrow R_\infty = R_{CL}$

R_{GB} is a <u>credibility mixture</u> of R_{CL} and R_{BF}:

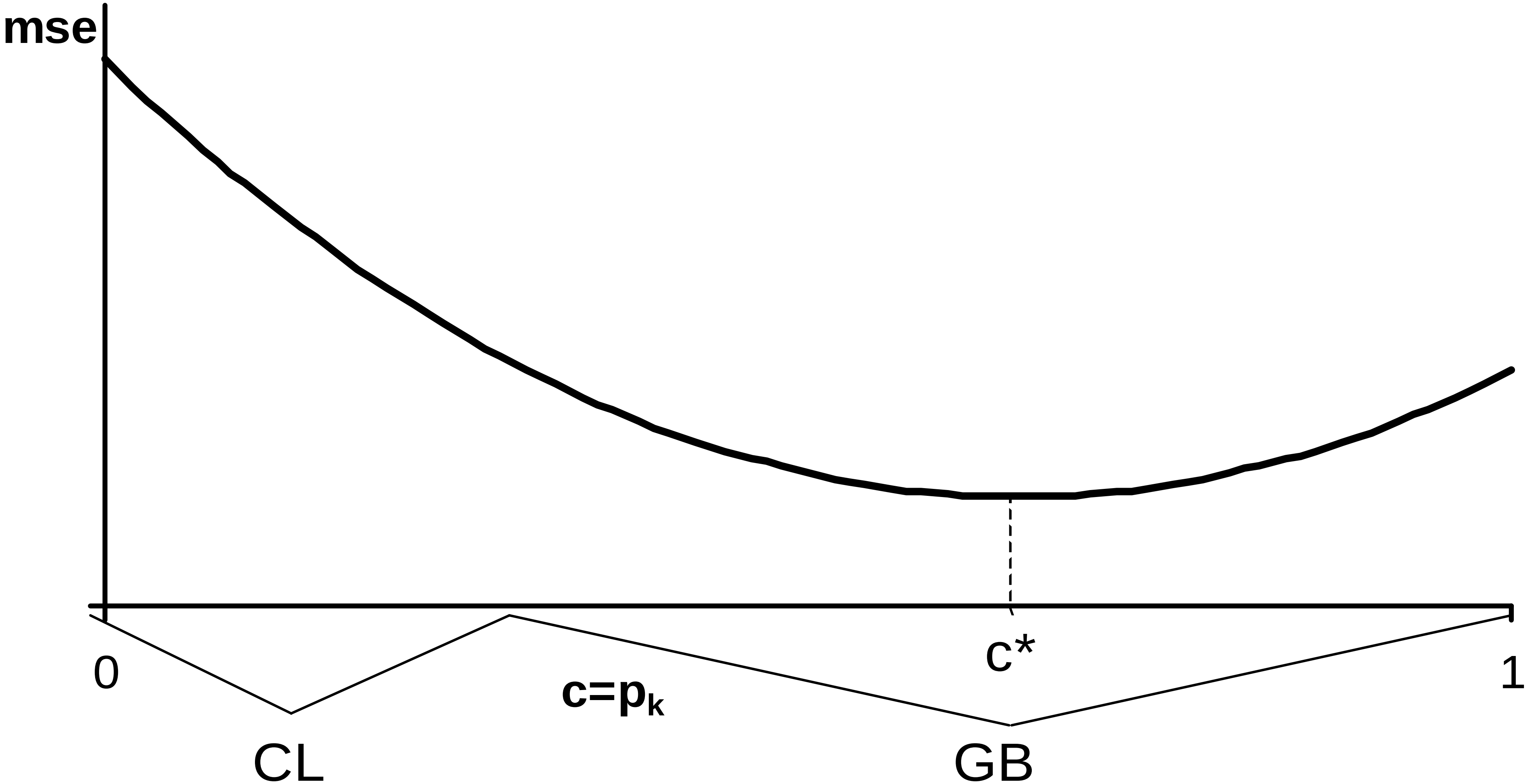
$$R_c = c\, R_{CL} + (1-c)\, R_{BF} \quad \text{with} \quad c = p_k \in [0; 1]$$

It gives R_{BF} for $c = 0$ and R_{CL} for $c = 1$.

<u>Best mixture</u>

if mean squared error is minimized:

$$\mathrm{mse}(R_c) = E(R_c - R)^2 = \min \qquad (\Rightarrow c^*)$$


mse
0
c*
1
c=p_k
CL
GB

- R_{c*} is always better than $R_0 = R_{BF}$, $R_1 = R_{CL}$
- R_{GB} is not always better but mostly

- How to determine c* ?
- How to decide
  which of R_{GB}, R_{CL}, R_{BF} is best
  at a given data set ?

$$R_c = cR_{CL} + (1\text{-}c)R_{BF} = c(R_{CL}\text{-}R_{BF}) + R_{BF}$$

$$E(R_c - R)^2 = E[c(R_{CL}\text{-}R_{BF}) + (R_{BF}\text{-}R)]^2$$

$$= c^2E(R_{CL}\text{-}R_{BF})^2 + 2cE[(R_{CL}\text{-}R_{BF})(R_{BF}\text{-}R)] +$$

$$+ E(R_{BF}\text{-}R)^2$$

$$c^* = \frac{E\big((R_{CL} - R_{BF})(R - R_{BF})\big)}{E(R_{CL} - R_{BF})^2}$$

$$= \frac{p_k}{q_k} \cdot \frac{\mathrm{Cov}(C_k, R) + p_k q_k \mathrm{Var}(U_0)}{\mathrm{Var}(C_k) + p_k^2 \mathrm{Var}(U_0)}$$

So far, we have not used any assumptions.

But for $Var(C_k)$, $Cov(C_k, R)$ we need a model.

Model A: (with $U = C_n$)

$$E(C_k|U) = p_k U, \quad Var(C_k|U) = p_k q_k \alpha^2(U)$$

$$\Rightarrow \quad Var(C_k) = p_k q_k E(\alpha^2(U)) + p_k^2 Var(U)$$

$$Cov(C_k, R) = p_k q_k \,(\, Var(U) - E(\alpha^2(U))\,)$$

But $E(\alpha^2(U))$ is difficult to estimate.

B: Increments $S_j = C_j - C_{j-1}$, $m_j = p_j - p_{j-1}$

$E(S_j/m_j|\Theta) = \mu(\Theta)$, $S_j|\Theta$ independent,

$Var(S_j/m_j|\Theta) = \sigma^2(\Theta)/m_j$, (Bühlmann/S.)

Θ indicates the "quality" of the acc.year

=> $Var(C_k) = p_k q_k E(\sigma^2(\Theta)) + p_k^2 Var(U)$

$$Cov(C_k,R) = p_k q_k \, (\underbrace{Var(U) - E(\sigma^2(\Theta))}_{Var(\mu(\Theta))})$$

Both models are math. equivalent and lead to

$$c^* = \frac{p_k}{p_k + t} \qquad t = \frac{E(\sigma^2(\Theta))}{Var(\mu(\Theta)) + Var(U_0)}$$

$E(\sigma^2(\Theta))$ = inner variab. — random error

$Var(\mu(\Theta))$ = level variab. — $Var(U)$

$Var(U_0)$ = estimation error

to be est. by actuary

An actuary who presumes

to establish a point estimate U_0

should also be able

to estimate its uncertainty $Var(U_0)$

and the variability $Var(U)$

of the underlying claims process.

For $E(\sigma^2(\Theta))$, we have an <u>unbiased estimate</u>

based on the data observed:

$$\frac{1}{k-1}\sum_{j=1}^{k} m_j\left(\frac{S_j}{m_j}-\frac{C_k}{p_k}\right)^2=\frac{p_k}{k-1}\sum_{j=1}^{k}\frac{m_j}{p_k}\left(\frac{S_j}{m_j}-U_{CL}\right)^2$$

Note that $\sum_{j=1}^{k} m_j = p_k$ and $\sum_{j=1}^{k}\frac{S_j}{m_j}=U_{CL}$

Having estimated $t = \dfrac{E(\sigma^2(\Theta))}{Var(\mu(\Theta)) + Var(U_0)}$

we can compare the precisions:

$$mse(R_{BF}) = E(\sigma^2(\Theta))\ (q_k + q_k^2 / t)$$

$$mse(R_{CL}) = E(\sigma^2(\Theta))\ q_k / p_k$$

$$mse(R_c) = c^2\ mse(R_{CL}) + (1-c)^2\ mse(R_{BF}) +$$

$$+ 2c(1-c)q_k\ E(\sigma^2(\Theta))$$

and obtain the following <u>results</u>:

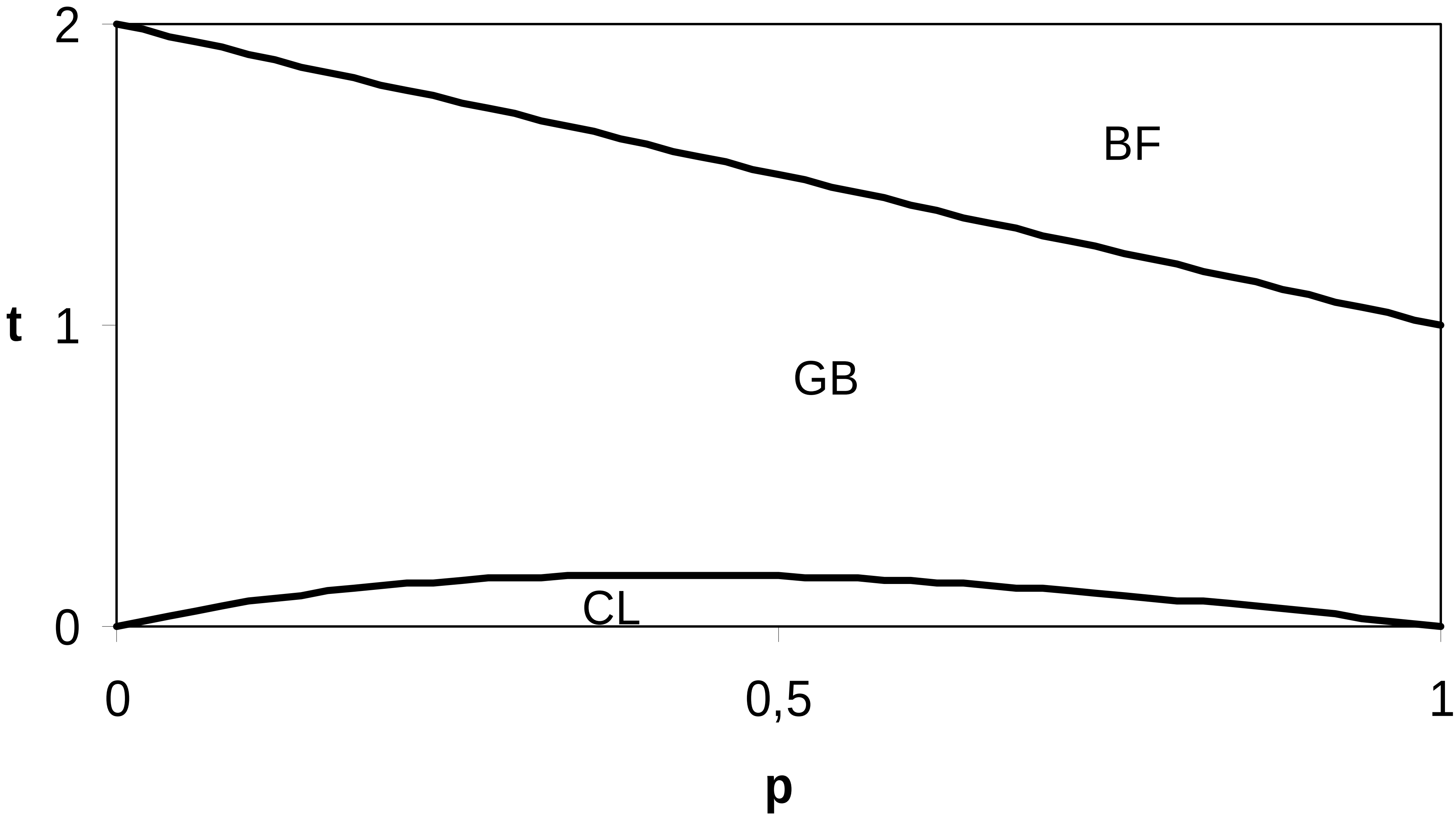
$\mathrm{mse}(R_{BF}) < \mathrm{mse}(R_{CL}) \iff p_k < t$

i.e. use BF for green years

use CL for rather mature years

$\mathrm{mse}(R_{GB}) < \mathrm{mse}(R_{BF}) \iff t < 2-p_k$

$\mathrm{mse}(R_{GB}) < \mathrm{mse}(R_{CL}) \iff t > p_k q_k/(1+p_k)$


2
1
0
t
BF
GB
CL
0
0,5
1
p

<u>Example:</u> $U_0 = 90\%$, $k = 3$

$\{p_j\} = 10\%, 30\%, 50\%, (70, 85, 95, 100\ \%)$

$\{C_j\} = 15\%, 27\%, 55\%$ (of the premium)

=>

$R_{BF} = 45\%$, $U_{CL} = 110\%$, $R_{CL} = 55\%$

$\{m_j\} = 10\%, 20\%, 20\%, (20\%, 15\%, 10\%, 5\%)$

$\{S_j\} = 15\%, 12\%, 28\%$

Inner variability

$$S_1/m_1 = 1.5, \quad S_2/m_2 = 0.6, \quad S_3/m_3 = 1.4$$

$E(\sigma^2\ (\Theta)) =$

$$\frac{0.50}{3-1}\left(\frac{10}{50}(1.5-1.1)^2+\frac{20}{50}(0.6-1.1)^2+\frac{20}{50}(1.4-1.1)^2\right)$$

$= 0.042 = (20.5\%)^2$

Actuary's estimates:

$$\mathrm{Var}(U) = (35\%)^2$$

(e.g. lognormal with 5% above 150%)

$$\mathrm{Var}(U_0) = (15\%)^2$$

=>

$$\mathrm{Var}(\mu(\Theta)) = (35\%)^2 – (20.5\%)^2 = (28.4)^2$$

$$t = (20.5\%)^2 \Big/ \Big((28.4\%)^2 + (15\%)^2 \Big) = 0.408$$

Results:

$$R_{BF} = 45.0\% \pm 21.6\%$$

$$R_{CL} = 55.0\% \pm 20.5\%$$

$$R_{GB} = 50.0\% \pm 18.1\%$$

$$R_{c^*} = 50.5\% \pm 18.0\% \qquad \text{with } c^* = 0.55$$

Note the high standard errors!

Check by <u>distributional assumptions</u>:

$U \sim \text{Lognormal}(\mu, \sigma^2)$ with

$$E(U) = 90\%,\ Var(U) = (35\%)^2$$

$\Rightarrow$ $\mu = -0.176$, $\sigma^2 = 0.141$

$C_k|U \sim \text{Lognormal}(\nu, \tau^2)$ with

$$E(C_k|U) = p_k U,\ Var(C_k|U) = p_k q_k \alpha^2 U^2$$

where α^2 is such that $Var(C_k)$ is as before

$\Rightarrow$ $\alpha^2 = 0.045$, $\tau^2 = 0.044$

Then, according to Bayes' theorem:

$U|C_k \sim \text{Lognormal}(\mu_1, \sigma_1^2)$

with $\mu_1 = z(\tau^2 + \ln(C_k/p_k)) + (1-z)\mu = 0.0643$

$\sigma_1^2 = z\tau^2 = 0.0335$

$z = \sigma^2 / (\sigma^2 + \tau^2) = 0.762$

$\Rightarrow E(U|C_k) = \exp(\mu_1+\sigma_1^2/2) = 108.4\%$

$\Rightarrow E(R|C_k) = \quad 53.4\%$

$\text{Var}(U|C_k) = (20.0\%)^2 = \text{Var}(R|C_k)$

- No estimation error
- standard error still very high

Results:

$$R_{BF} = 45.0\% \pm 21.6\%$$

$$R_{CL} = 55.0\% \pm 20.5\%$$

$$E(R|C_k) = 53.4\% \pm 20.0\%$$

$$R_{GB} = 50.0\% \pm 18.1\%$$

$$R_{c^*} = 50.5\% \pm 18.0\% \qquad \text{with } c^* = 0.55$$

Conclusions:

- Use of a priori knowledge (U_0) may be better than distributional assumptions
- A way is shown how to assess the variability of the Bornhuetter/Ferguson reserve, too.

Conclusions (ctd.):

- Benktander's credibility mixture of BF and CL is simple to apply and gives almost always a more precise estimate.
- The volatility measure t is not too difficult to estimate and improves the precision even more or helps to decide on BF, CL, GB.