TFA 40 (1985-1987) 98-134

THE CHANGING SHAPE OF ENGLISH LIFE TABLES

by

D. O. FORFAR, B.A., F.F.A. and D. M. SMITH, M.A., F.F.A.

1. In a paper to the Institute of Actuaries (*JIA* 107, p. 49) Heligman and Pollard fitted certain mathematical curves to Australian national mortality. They showed that a good fit could be obtained throughout the whole of life with the following curve:

$$q_x = \frac{f(x)}{1+f(x)} \text{ where}$$

$$f(x) = A^{(x+B)^C} + De^{-E\left(\log_e \frac{x}{F}\right)^2} + GH^x$$

2. The above curve has been fitted to English Life Tables ELT1 to ELT13 for both males and females.

The parameters were obtained by minimising S^2 where,

$$S^2 = \sum_{x=0}^{85} \left(\frac{\hat{q}_x}{q_x} - 1\right)^2$$

$\hat{q}_x$ = mortality rate estimated from the formula

q_x = mortality rate from the appropriate ELT table.

3. The work was undertaken to see if there was a good fit over all the twenty-six life tables and whether there was a regular progression in the eight parameters from one life table to the next.

Tables 1 and 2 show the values of the parameters A to H for each English Life Table, ELT1 to ELT13, for males and females.

It will be seen from the value of S^2 that all twenty-six life tables gave a reasonably good fit and that there was a reasonable progression of values from one life table to the next apart from ELT12 (Females) which is discussed later.

TABLE 1

English Life Tables (Males)

ELT	Year	$A \times 10^3$	$B \times 10^3$	$C \times 10^2$	$D \times 10^4$	E	F	$G \times 10^5$	H	S^2
1	1841	87·279	337·03	34·209	85·171	1·3736	32·615	12·379	1·0938	0·2500
2	1841	92·247	369·58	35·587	76·233	1·7474	30·928	19·044	1·0878	0·1353
3	1846	84·419	293·40	33·493	75·633	2·0190	31·260	23·169	1·0852	0·1096
4	1876	92·064	401·72	38·215	80·881	1·3751	39·946	19·391	1·0880	0·0890
5	1886	89·251	439·62	42·740	67·345	1·6940	38·154	19·480	1·0889	0·2835
6	1896	65·958	244·46	38·508	49·213	1·0794	39·843	23·886	1·0858	0·2223
7	1906	48·994	227·52	35·446	29·641	1·6416	30·844	26·036	1·0838	0·1422
8	1911	34·472	163·80	29·399	22·806	2·2570	27·939	23·370	1·0851	0·1739
9	1921	24·175	147·04	24·931	28·181	2·1728	28·592	11·580	1·0946	0·1602
10	1931	14·468	85·848	20·565	20·262	4·2398	23·402	12·875	1·0930	0·3185
11	1951	2·2665	5·4778	11·249	7·0870	8·8602	21·772	5·4925	1·1052	0·1680
12	1961	1·5119	3·7199	10·123	7·4030	14·788	19·929	4·0468	1·1090	0·4525
13	1971	1·2150	3·4898	9·5794	7·0351	17·252	19·355	3·7853	1·1093	0·5194

TABLE 2

English Life Tables (Females)

ELT	Year	$A \times 10^3$	$B \times 10^3$	$C \times 10^2$	$D \times 10^4$	E	F	$G \times 10^5$	H	S^2
1	1841	86.589	455·82	33·423	88·490	1·4390	31·768	8·6592	1·0976	0·2159
2	1841	119·96	792·77	40·511	86·193	1·4476	30·606	13·243	1·0913	0·1327
3	1846	86·892	438·06	33·378	88·005	1·9161	31·659	13·401	1·0915	0·0598
4	1876	100·59	625·93	39·858	74·323	1·3738	35·953	11·310	1·0939	0·0975
5	1886	113·21	814·46	47·352	62·350	2·0146	32·825	14·179	1·0915	0·2931
6	1896	64·892	330·84	37·251	53·544	0·84719	43·670	10·978	1·0947	0·1523
7	1906	47·257	292·74	33·736	41·079	0·72420	47·199	8·8769	1·0963	0·1435
8	1911	36·024	256·07	29·906	33·336	0·71357	45·389	7·5496	1·0981	0·1525
9	1921	23·906	226·51	24·655	28·989	1·4672	30·844	5·1298	1·1030	0·0927
10	1931	13·807	137·94	20·546	22·847	2·0037	27·640	4·8421	1·1036	0·1944
11	1951	2·1387	17·670	12·846	7·2126	2·2223	30·673	2·1927	1·1125	0·1545
12	1961	1·3333	9·8127	11·181	35·306	0·32698	408·50	1·2486	1·1184	0·2274
		(1·3175)	(11·325)	(11·574)	(1·2804)	(9·7113)	(19·880)	(3·1437)	(1·1044)	(0·3639)
13	1971	1·0996	11·779	11·044	1·8566	19·993	19.010	2·8313	1·1047	0·2486

4. The thirteen ELT tables were based on deaths during the periods shown in the table below.

ELT	Period
1	1841
2	1838-44
3	1838-54
4	1871-80
5	1881-90
6	1891-1900
7	1901-10
8	1910-12
9	1920-22
10	1930-32
11	1950-52
12	1960-62
13	1970-72

In Table 1 the mid-year of the appropriate period has been shown (or the later of the two mid-years where the period contained an even number of years).

5. Heligman and Pollard gave an interpretation of each term in the formula.

The first term, a rapidly declining exponential, reflects the fall in mortality during the early childhood years. The value of the parameter A is close to q_1, the rate of mortality at age one year. The parameters B and C reflect the progression of mortality in the first few years, in particular the relationship between q_0, q_1 and q_2. For example in ELT1 (Males) the rate of mortality is halved, broadly speaking, in each of the first three years of life, whereas in ELT13 (Males) there is a very sharp decline from q_0 to q_1 and a much more gradual decline from q_1 to q_2 *to* q_3. This type of pattern accounts for the lower values of B and C in the later life tables.

The second term represents a distinct 'hump' in the mortality curve. The parameter D represents the magnitude of the hump, E is proportional to the severity of the hump and F represents the location of the hump. This distinct hump is evident in all the tables. In the early tables the hump was a gradual one centred around age 32. In the recent tables the hump has become the well-known accident hump centred around age 19.

The third term is the Gompertz exponential and represents the near geometric progression of mortality rates with age. The parameter G reflects the level of mortality and H the rate of increase of mortality.

6. Graphs 1 to 8 show the progression of the parameters A to H over time for both males and females. The parametric values plotted for 1841 were taken from ELT2 rather than ELT1 because of the longer period over which deaths were measured. The graphs can be interpreted in relation to the physical meaning of the parameters. For example the fall in the value of A represents the decline in q_1, the rate of mortality at age one year. The increase in E represents the increasing severity of the hump in the curve, etc. It is noticeable that H, representing the near geometric progression of mortality with age, has remained relatively constant.

The graphs of the parametric values become very much less steep over more recent years showing, as is well known, that the rate of improvement in mortality has slowed down. Improvements in mortality will be more difficult to achieve in the future.

The only mortality table for which the parameters differed substantially from a reasonable progression of parametric values was ELT12 (Females). This table has only a very modest hump around age 20 but the beginning of a more pronounced hump starting at a much higher age, around age 70. The best fit was obtained by fitting the second term of the mathematical formula to the second hump rather than the first. The figures in brackets show the values where the curve is “forced” to fit the first hump rather than the second and these are much more consistent with the other figures in the table.

7. Tables 1 and 2 showing the parametric values of male and female mortality rates, show many similarities. The general progression of the parameters is similar. The parameter H, reflecting the near geometric progression of mortality rates, is similar but the lower value of G for females shows that the mortality rates increase from a lower base. A comparison of parameters D, E and F shows that the mortality hump in the female table has been less pronounced (lower D) and less severe (lower E) but centred around a similar age (similar F). The value of A has in more recent times been lower for females reflecting a lower value of q_1 for females. The higher values of B and C for females reflect broadly speaking the fact that the mortality rate q_0 for females is less than that for males and the progression from q_0 to q_1 is not so steep.

8. Graphs 9, 10, 11 and 12 show the formula mortality rates together with the actual ELT (Males) mortality rates for ELT3, 6, 11 and 13 corresponding to the years 1846, 1896, 1951 and 1971. The closeness of fit of the mathematical curves can be judged from these

graphs as well as from the value of S^2. The graphs plot $\log_e (10^5 \times q_x)$ against age.

9. Graph 13 plots $\log_e (10^5 \times q_x)$ for ELT (Males) 3, 6, 11 and 13 on the same graph. From these it can be clearly seen how the hump has turned from a gradual hump into the steep accident hump which we associate with recent mortality tables. Also evident is the lack of improvement in mortality at ages above about 70.

10. Graph 14 shows a 3-dimensional plot of $\log_e (10^5 \times q_x)$ for ELT 2-13 (Males) and graph 15 shows a contour plot of this surface. The contour plot, which connects ages with equal mortality rates, brings out very clearly the improvements in mortality with age at the younger ages, and the lack of improvement at the older ages. Graphs 16 and 17 show the same for ELT 2-13 (Females).

11. The expectations of life are shown in Table 3 below where the expectations have been derived from the ELT tables (and from the formula rates for comparison).

TABLE 3

Complete Expectations of Life

	ELT2(M)		ELT6(M)		ELT13(M)	
Age	Actual	Formula	Actual	Formula	Actual	Formula
0	40·4	40·4	44·1	44·3	69·0	69·3
20	40·0	40·0	41·0	41·0	51·1	51·4
40	26·5	26·5	25·6	25·7	32·0	32·4
60	13·6	13·6	12·9	12·8	15·4	15·6
80	5·0	5·0	4·6	4·7	5·5	4·9

TABLE 4

Complete Expectations of Life

	ELT2(F)		ELT6(F)		ELT13(F)	
Age	Actual	Formula	Actual	Formula	Actual	Formula
0	42·0	42·1	47·8	47·9	75·3	75·3
20	40·7	40·6	43·4	43·4	56·9	57·0
40	27·5	27·5	27·8	27·8	37·5	37·6
60	14·5	14·4	14·1	14·1	20·0	20·0
80	5·3	5·3	5·1	5·0	7·0	7·4

The effect of the accident hump can be quantified by calculating the expectation from the formula but omitting the middle term. This shows the following results for ELT13(M) and (F).

Age	Male	Female
0	69·6	75·4
20	51·6	57·0

It can be seen that the improvement at age zero is just over half a year of life for males but only 0·1 of a year for females.

12. From the parameters calculated, and the progression of these parameters over time, possible values were estimated for future life tables. The values were taken as follows:

	$A\times10^3$	$B\times10^3$	$C\times10^2$	$D\times10^4$	E	F	$G\times10^5$	H
Males								
Estimated 1981 parameters:	0·8	3·4	9·4	7·2	18·2	19·3	3·0	1·114
Estimated 1991 parameters:	0·7	3·2	9·2	7·2	19·0	19·2	2·6	1·116
Females								
Estimated 1981 parameters:	0·7	11·2	10·7	1·6	20·0	18·8	2·2	1·107
Estimated 1991 parameters:	0·6	10·9	10·4	1·6	20·0	18·7	1·9	1·108

Graphs 18 and 19 show the values of $\log_e (10^5 \times q_x)$ for the above parameters (males and females) and the parameters for 1971, 1961 and 1951. These graphs are described as ELT (FORMULA) 11, 12, 13, (1981), (1991).

The expectations of life are shown in Table 5.

TABLE 5

Estimated Complete Expectations of Life

	1981		1991	
Age	Males	Females	Males	Females
0	69.6	76·9	70·0	77·9
20	51·2	58·1	51·5	58·9
40	32·0	38·6	32·3	39·4
60	15·1	20·8	15·2	21·4
80	4·5	7·7	4·5	8·1

13. Graphs 20 and 21 plot the expected age at death at each age ($x + \mathring{e}_x$) for ELT 2 to 13 inclusive for both males and females. The values are also shown for the estimated 1981 and 1991 mortality tables. They show, as pointed out in paragraph 6, how the slowing down in the rate of improvement of mortality has affected the progression over time of the expected age at death. These graphs show that while the expected age at death in 1841 was very similar for males and females, the expected age at death for females has subsequetly shown the greater improvement.

14. The authors are grateful to the Edinburgh Regional Computing Centre for permission to use their NAG algorithms to find the values of the parameters and to their own office for computing facilities to draw the graphs.

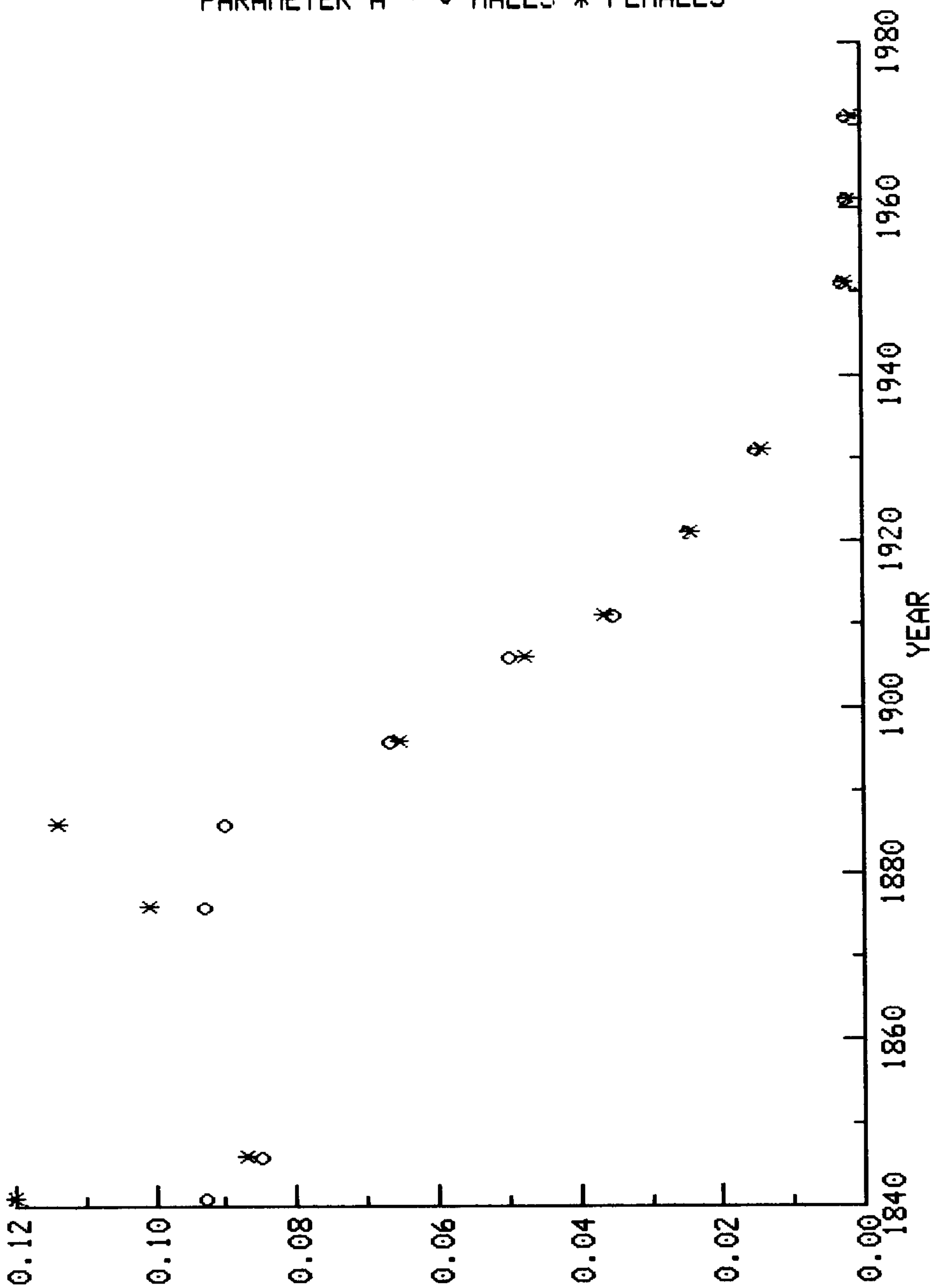
REFERENCES

HELIGMAN, L. and POLLARD, J. H. The age pattern of mortality. *J.I.A.*, 107, p. 49.

BENJAMIN, B. and POLLARD, J. H. The analysis of mortality and other actuarial statistics. Heinemann, London.

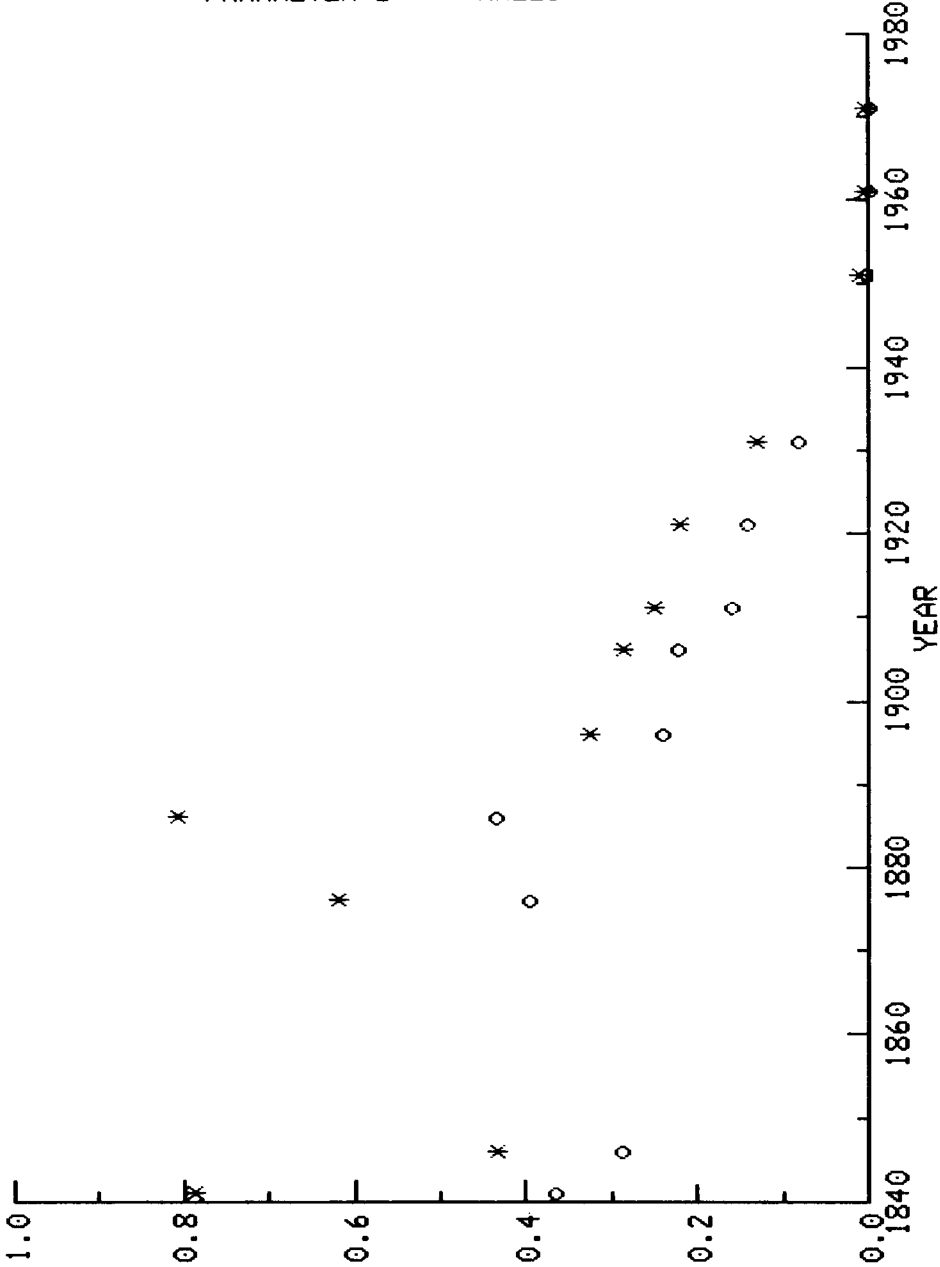
GRAPH 1

PARAMETER A - o MALES * FEMALES



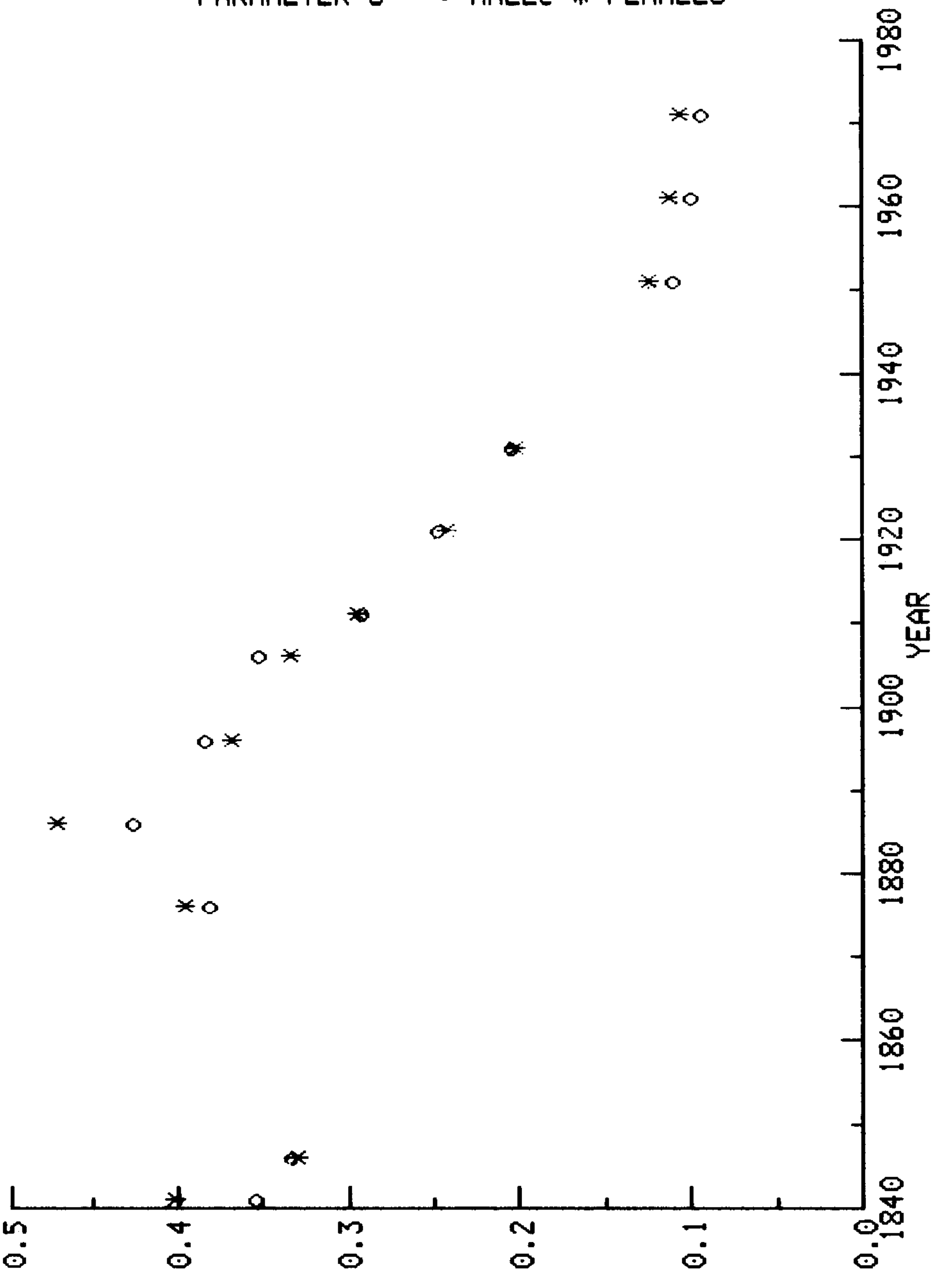
GRAPH 2

PARAMETER B - o MALES * FEMALES



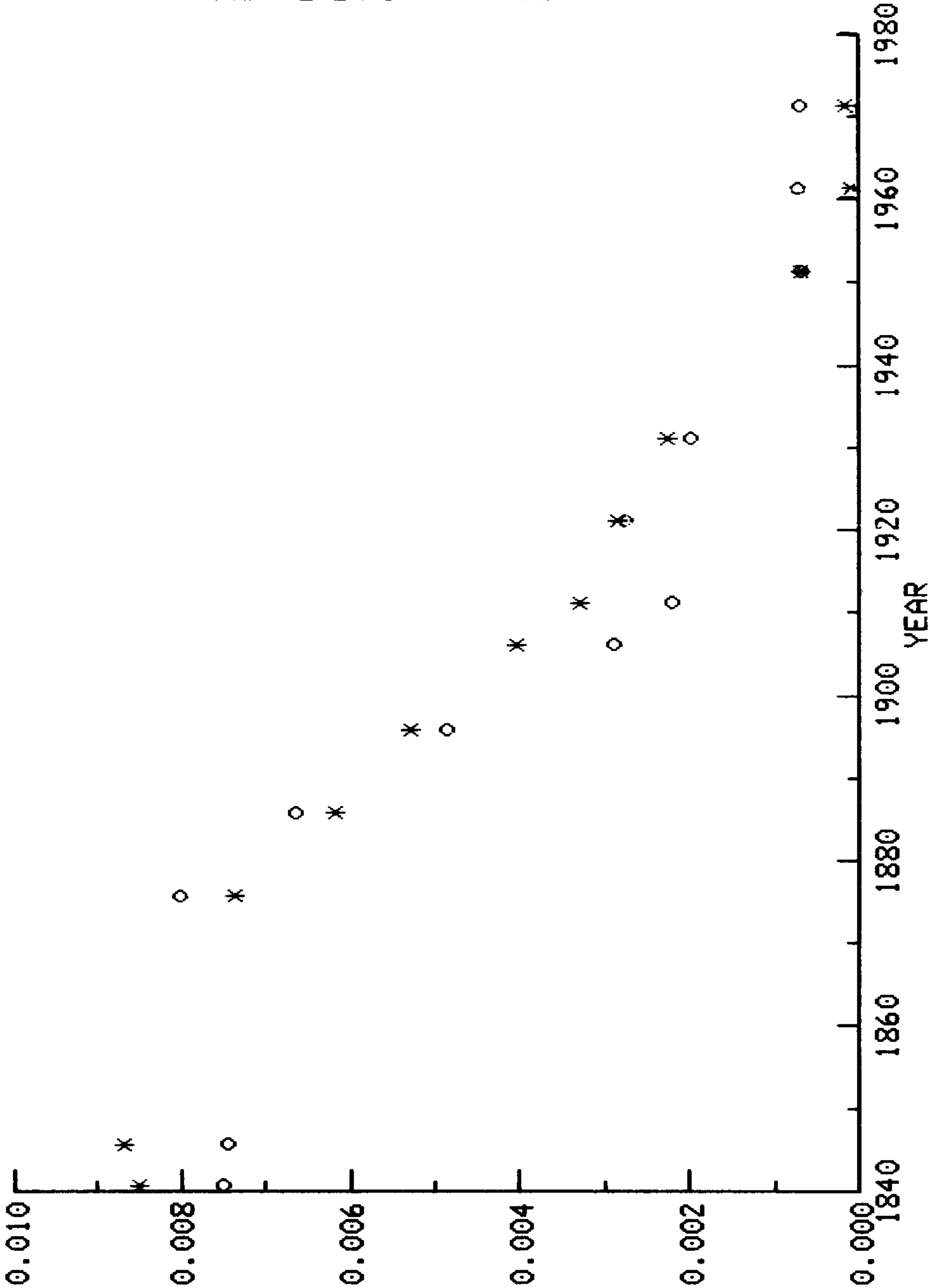
GRAPH 3

PARAMETER C - o MALES * FEMALES



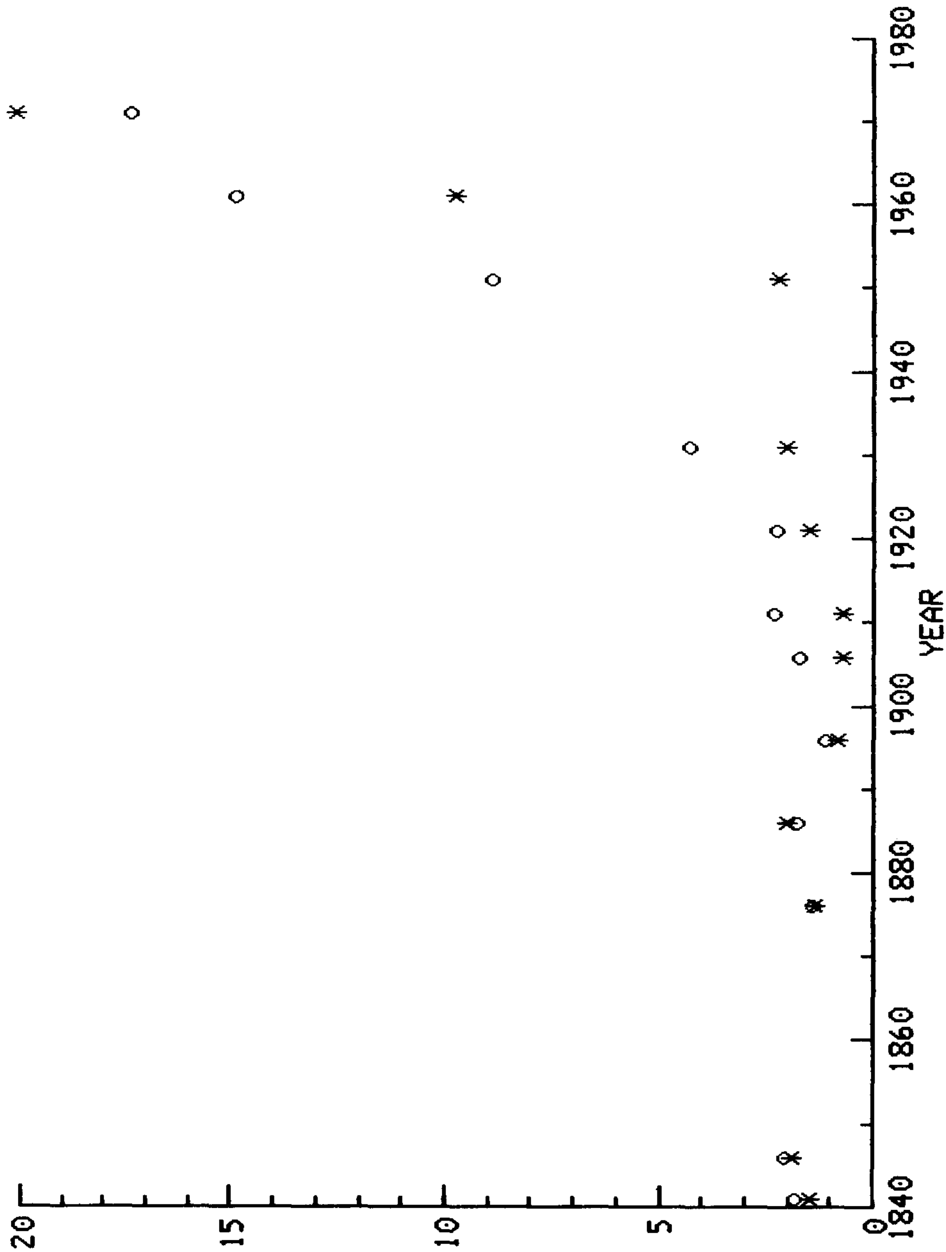
GRAPH 4

PARAMETER D - o MALES * FEMALES

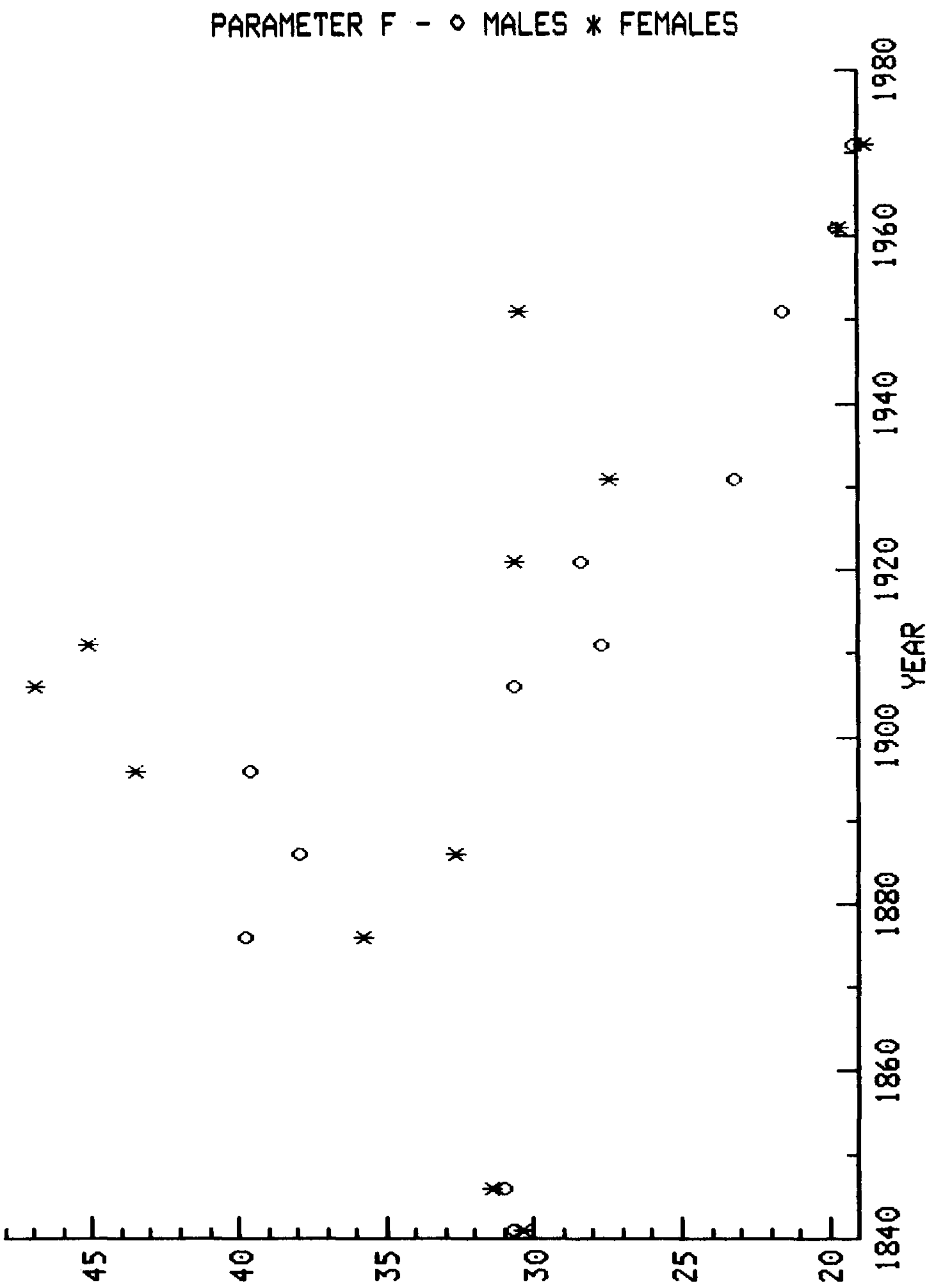


GRAPH 5

PARAMETER E - o MALES * FEMALES

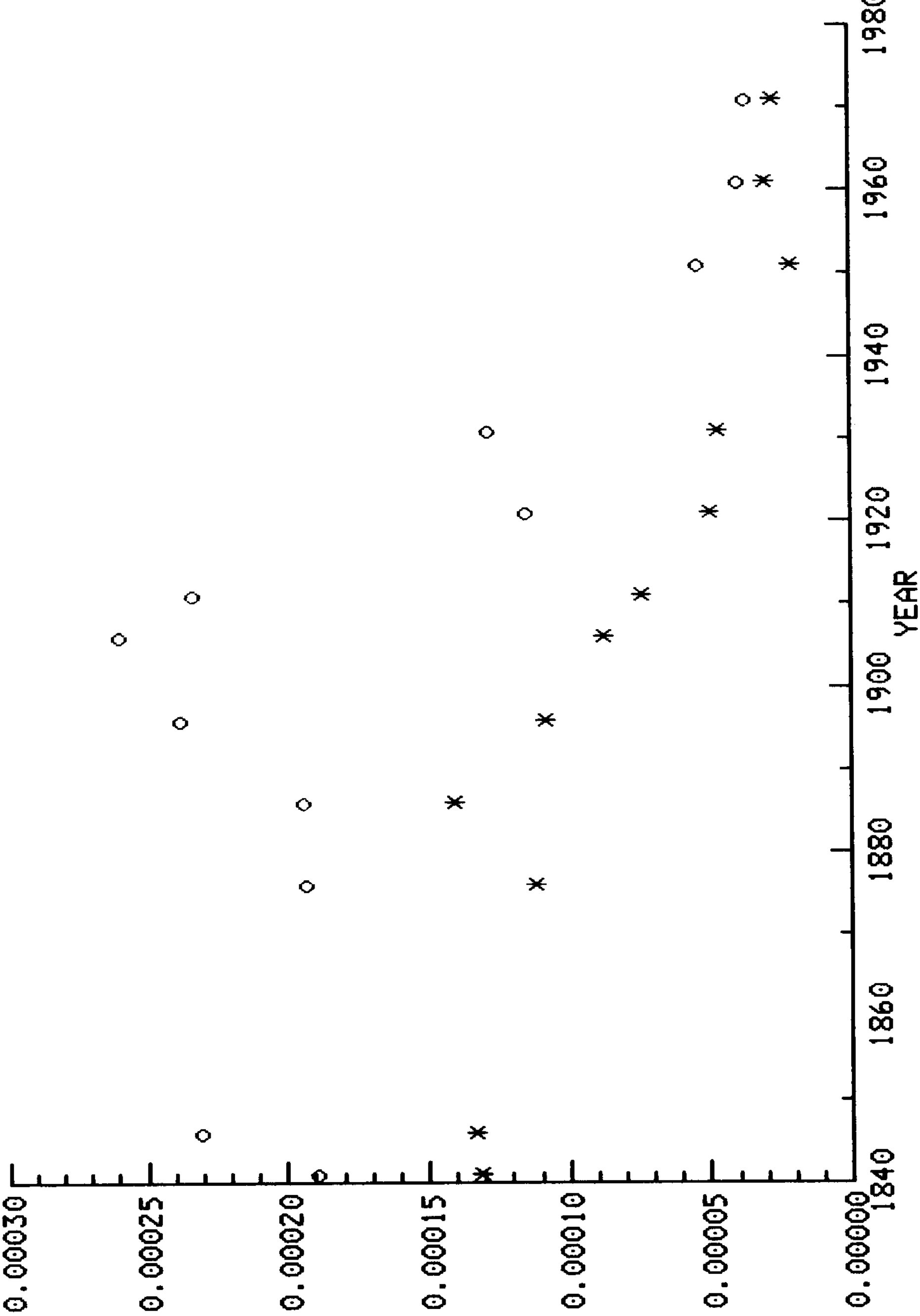


GRAPH 6


PARAMETER F - o MALES * FEMALES
YEAR
1840
1860
1880
1900
1920
1940
1960
1980
20
25
30
35
40
45

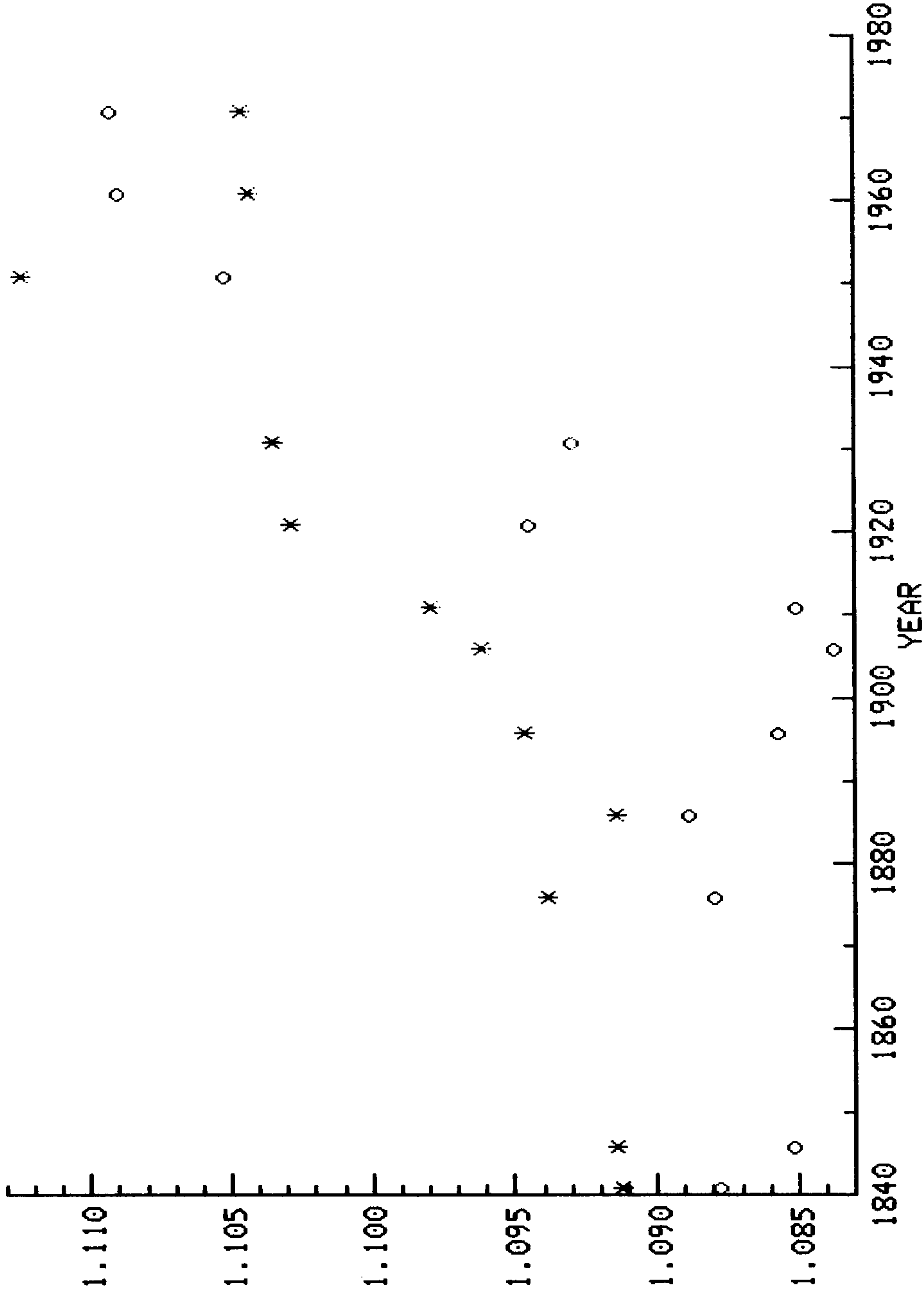
GRAPH 7

PARAMETER G - o MALES * FEMALES



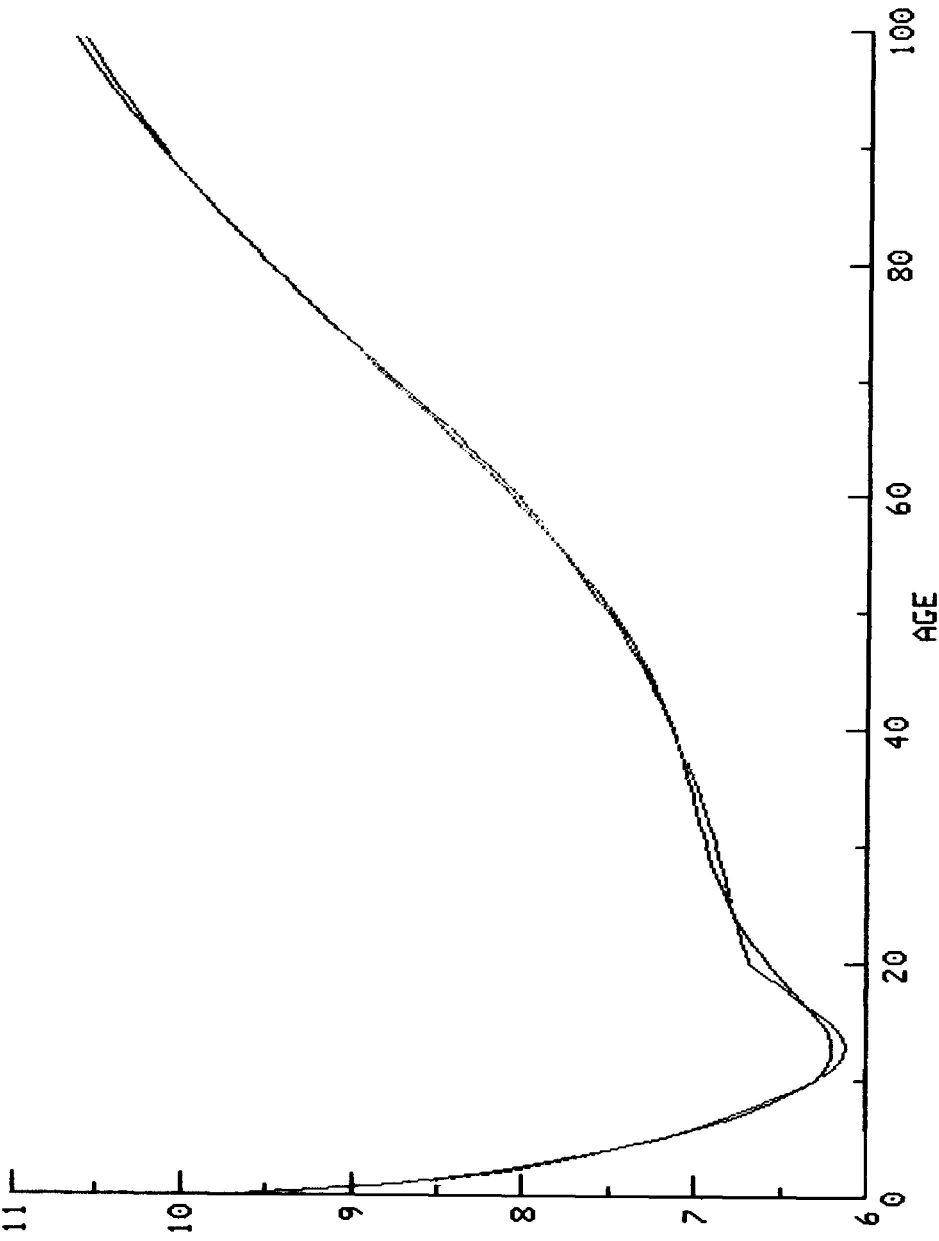
GRAPH 8

PARAMETER H – o MALES * FEMALES



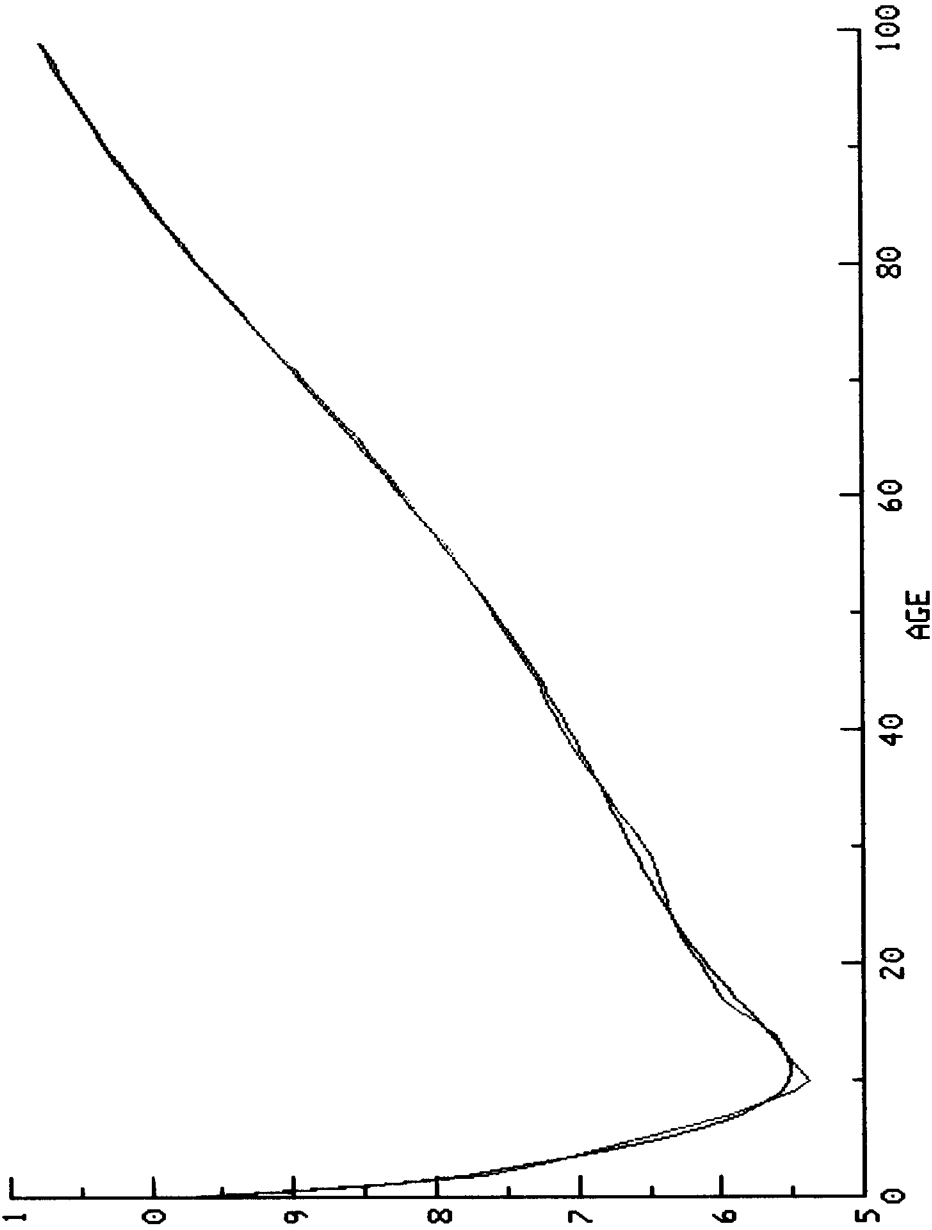
GRAPH 9

ELT 3 ACTUAL VS FORMULA - MALES - LN(100,000 x q)



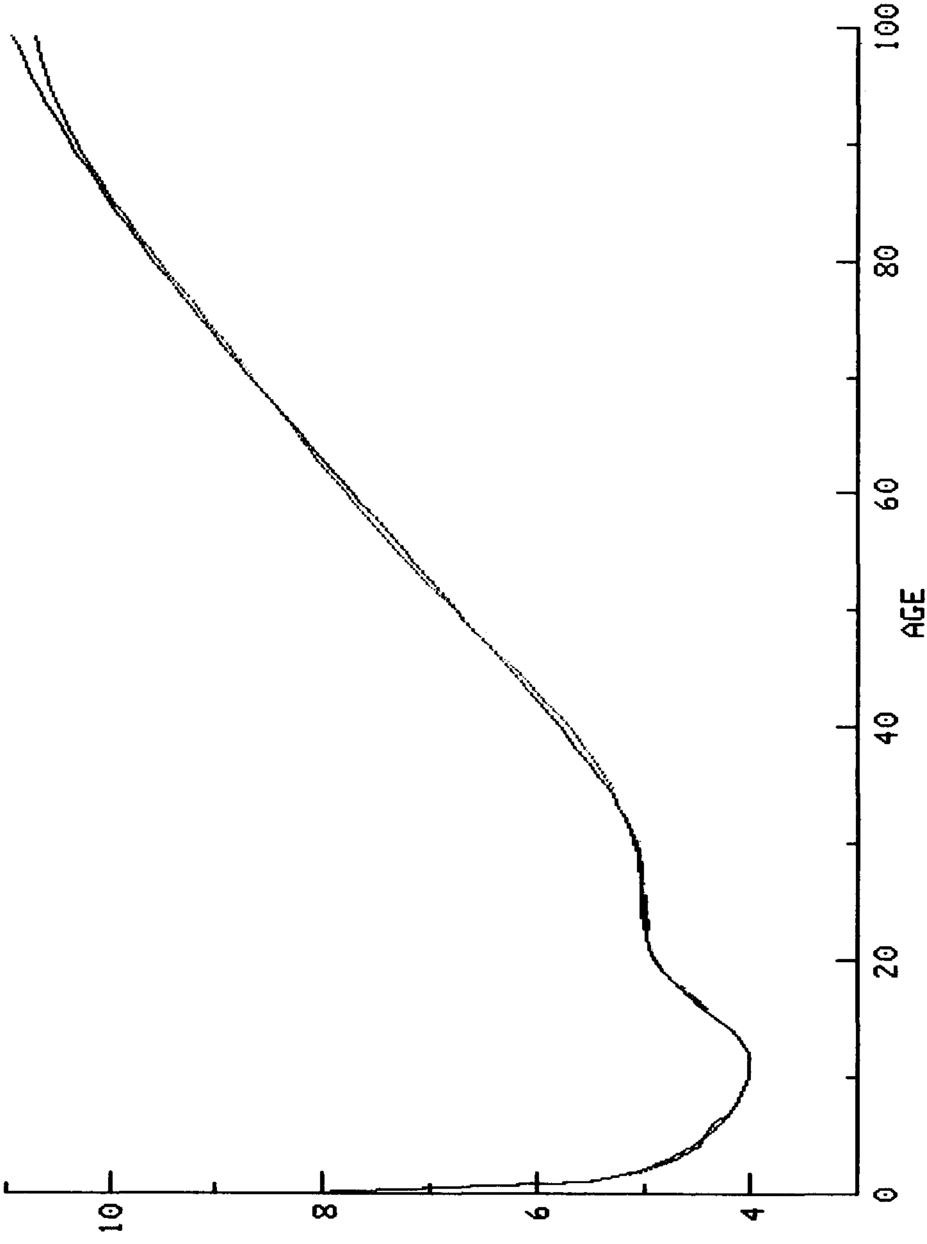
GRAPH 10

ELT 6 ACTUAL VS FORMULA - MALES - LN(100,000 x q)



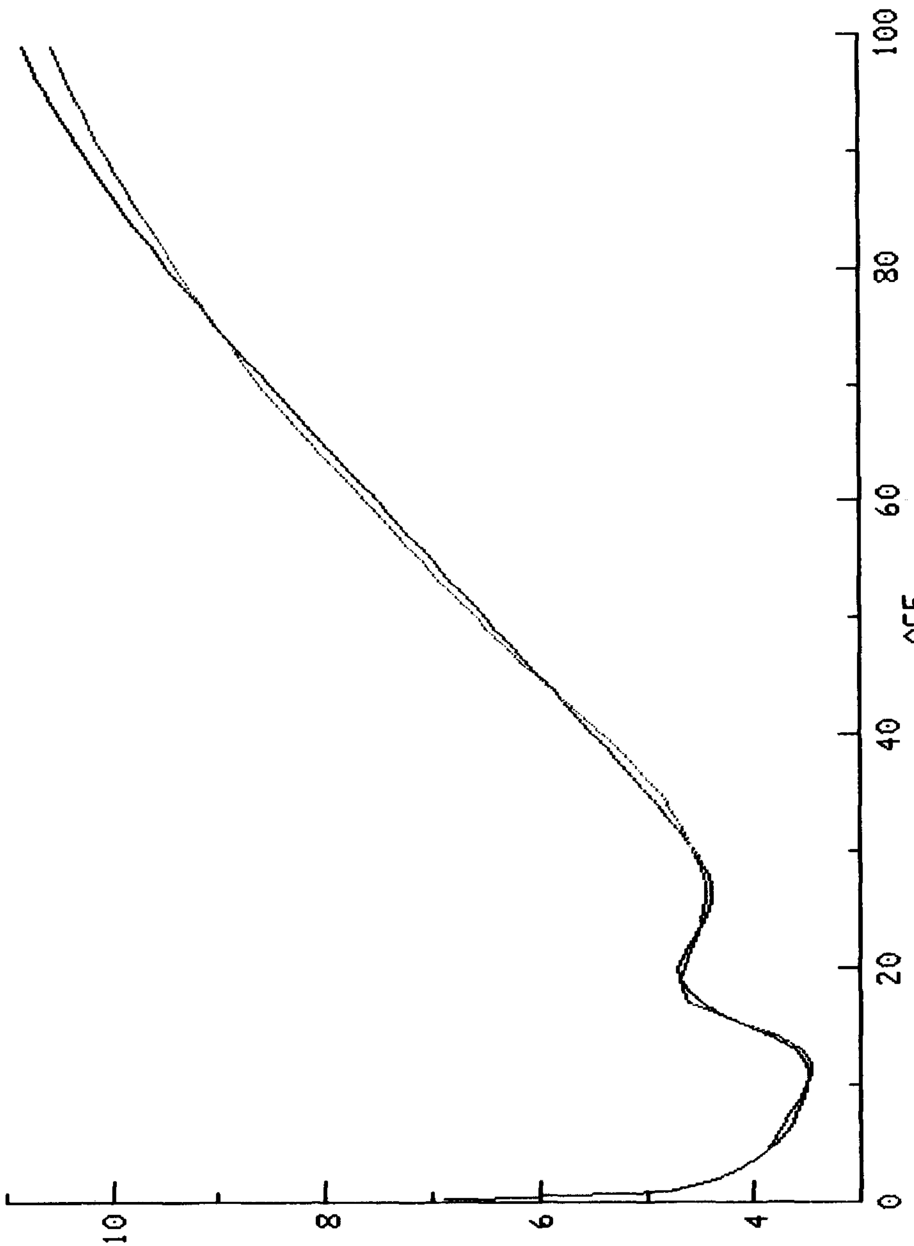
GRAPH 11

ELT 11 ACTUAL VS FORMULA - MALES - LN(100,000 × q)

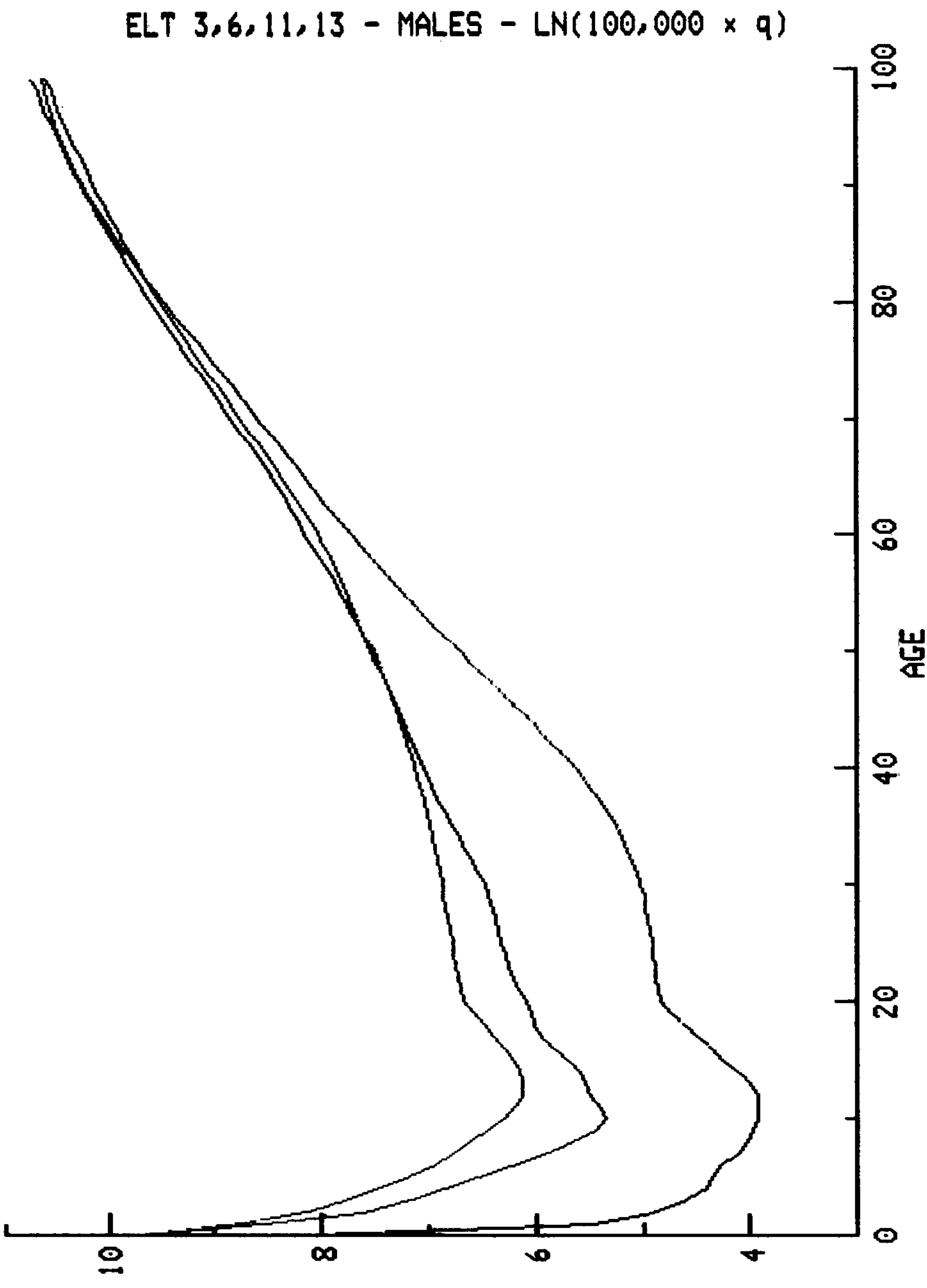


GRAPH 12

ELT 13 ACTUAL VS FORMULA - MALES - LN(100,000 x q)

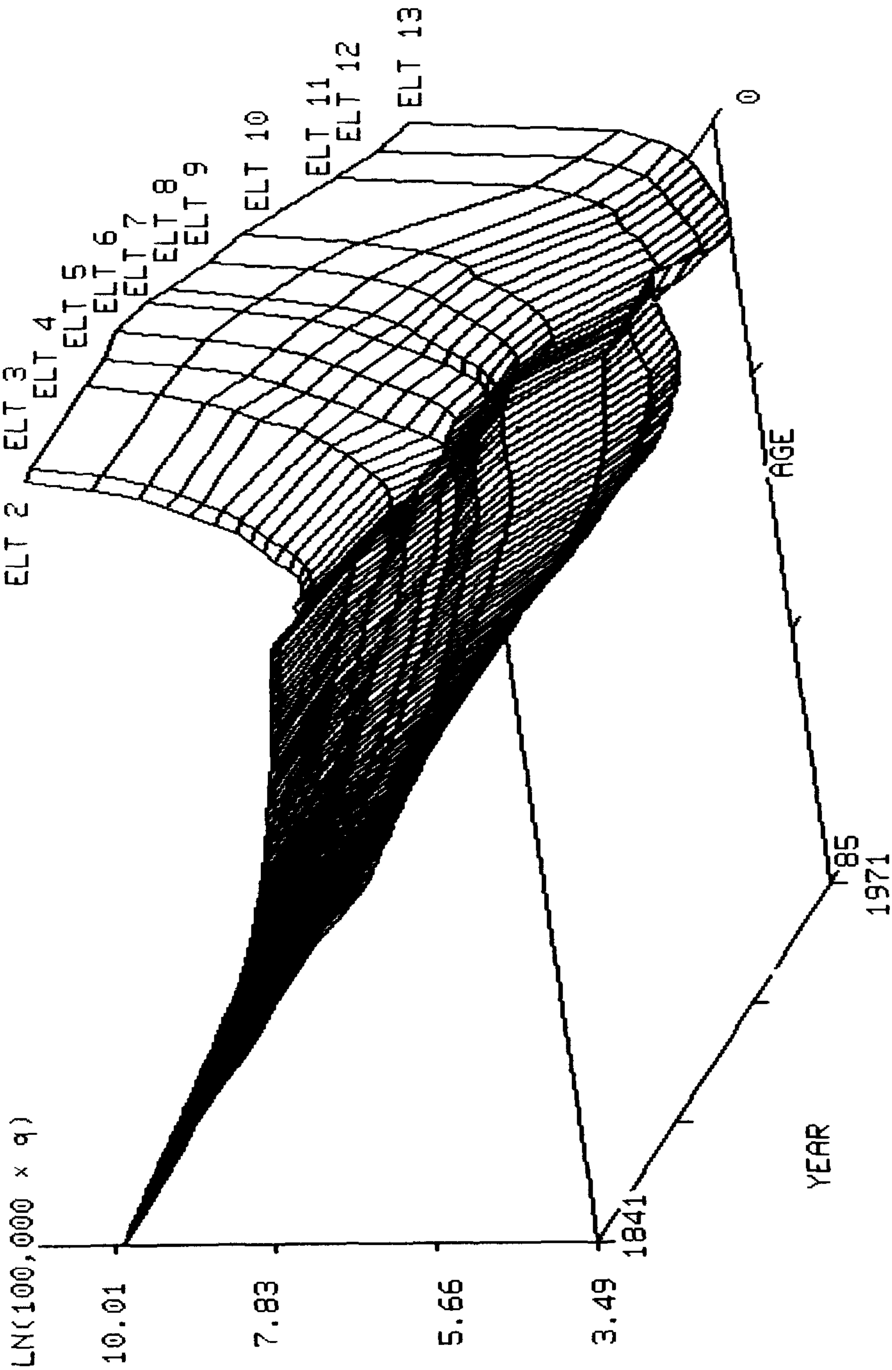


GRAPH 13



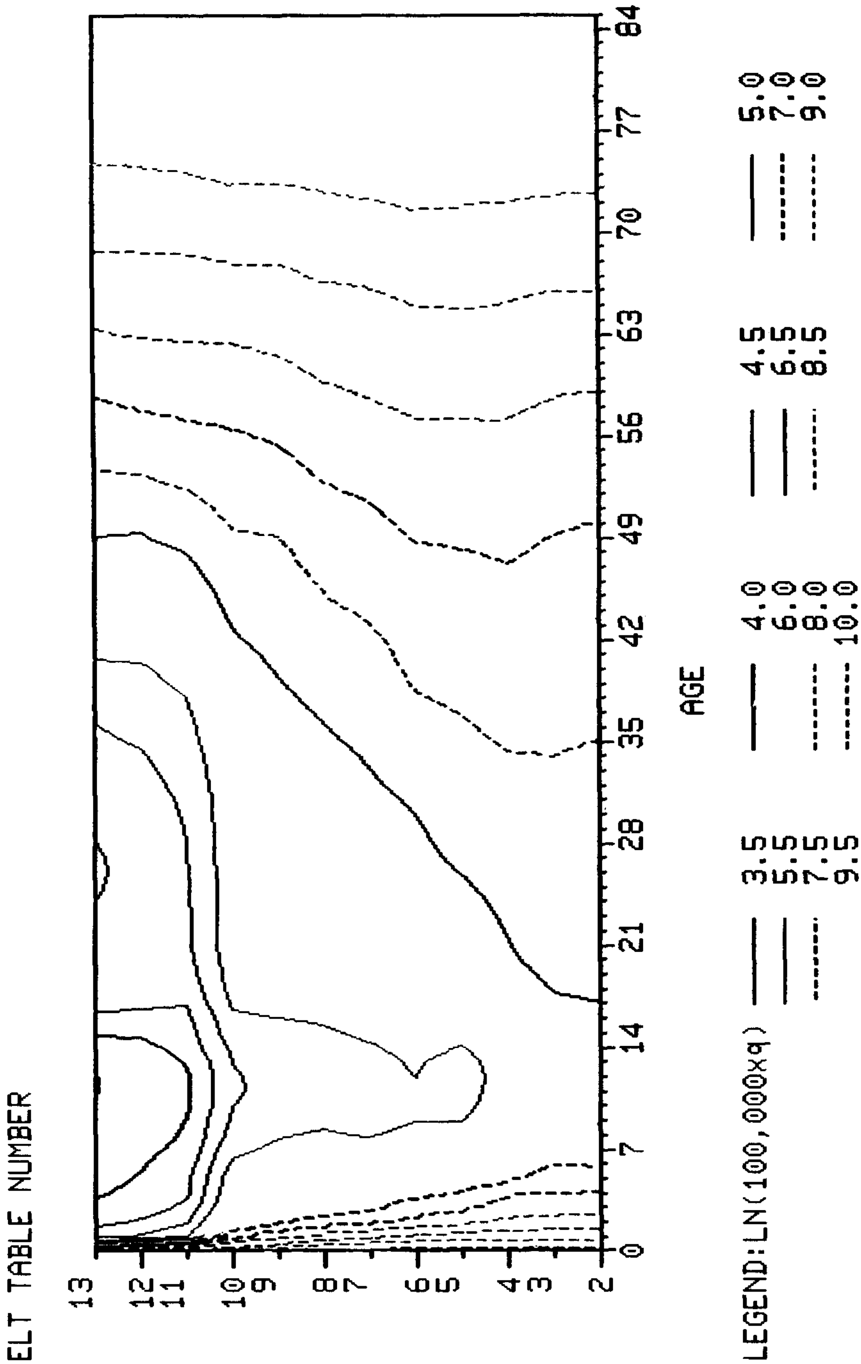
GRAPH 14

MALE MORTALITY RATES 1841 - 1971



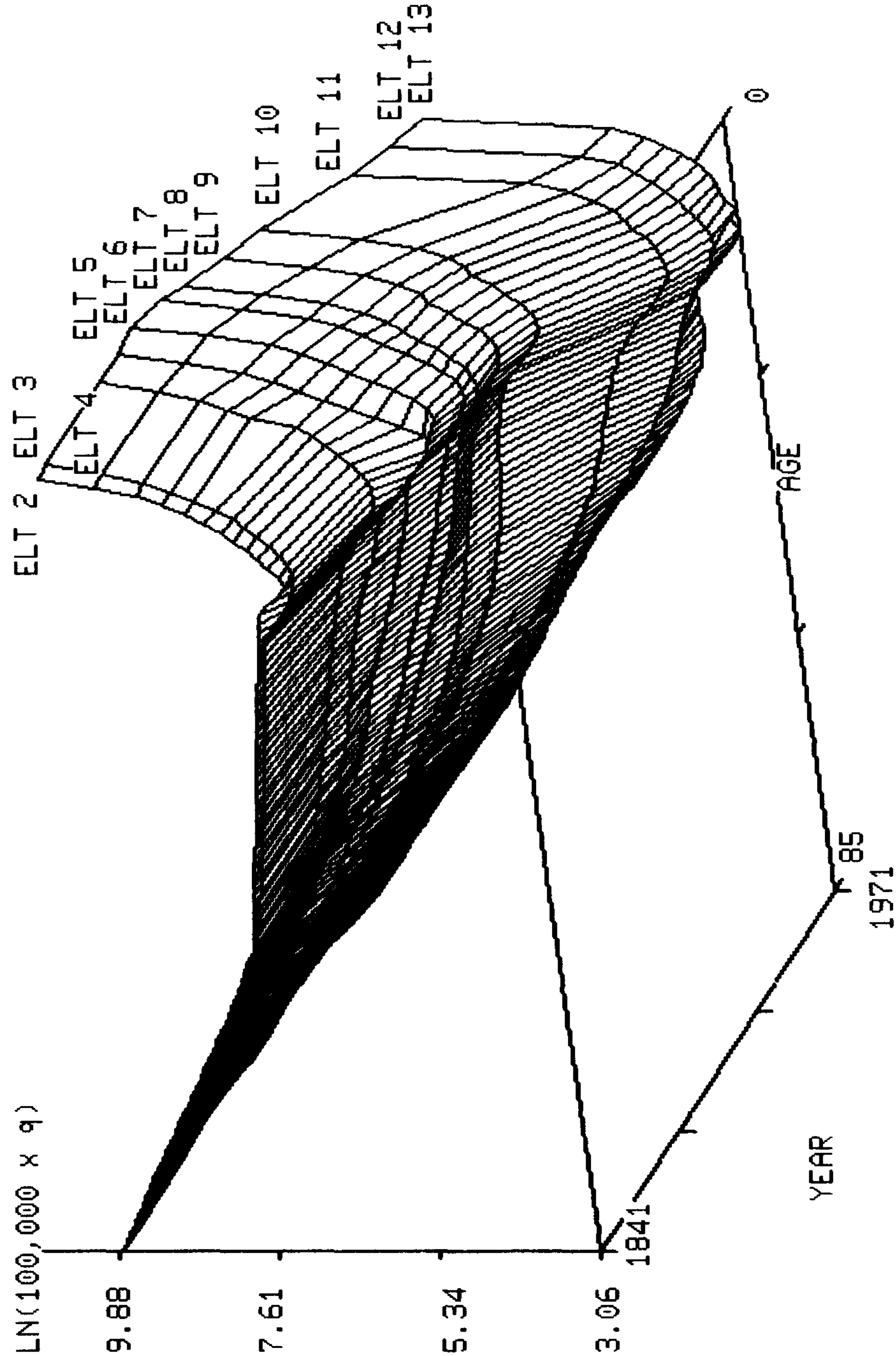
GRAPH 15

MALE MORTALITY RATES 1841 - 1971



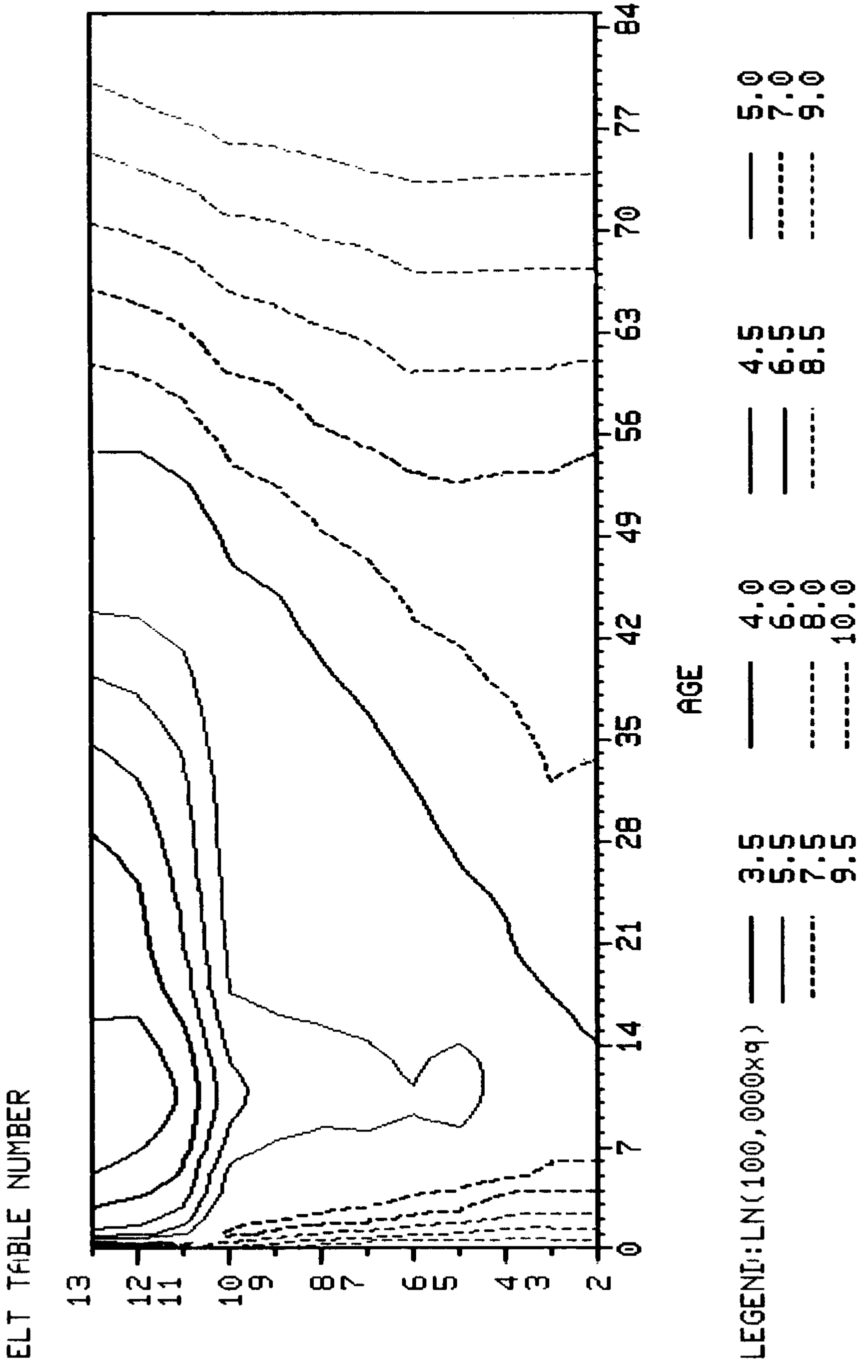
GRAPH 16

FEMALE MORTALITY RATES 1841 - 1971



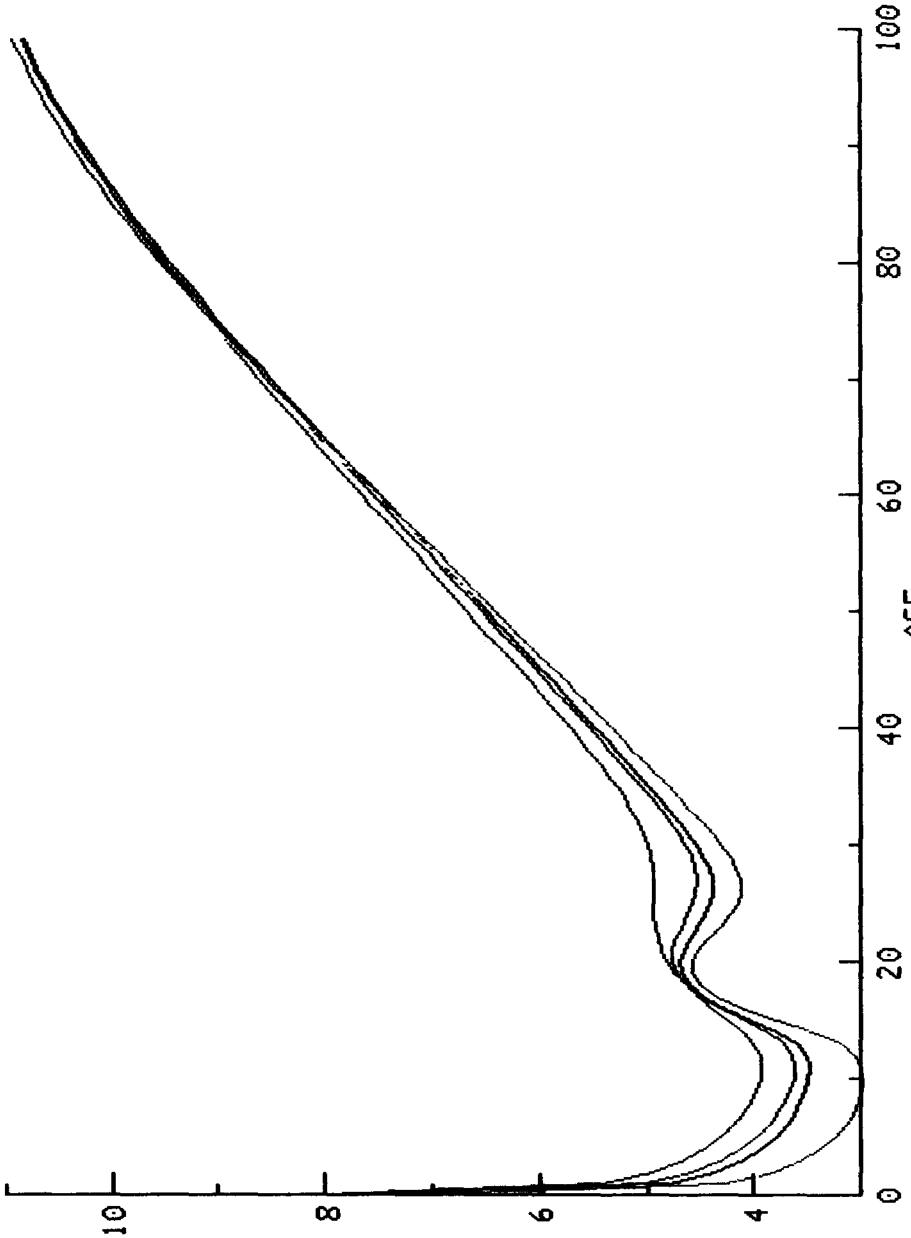
GRAPH 17

FEMALE MORTALITY RATES 1841 - 1971



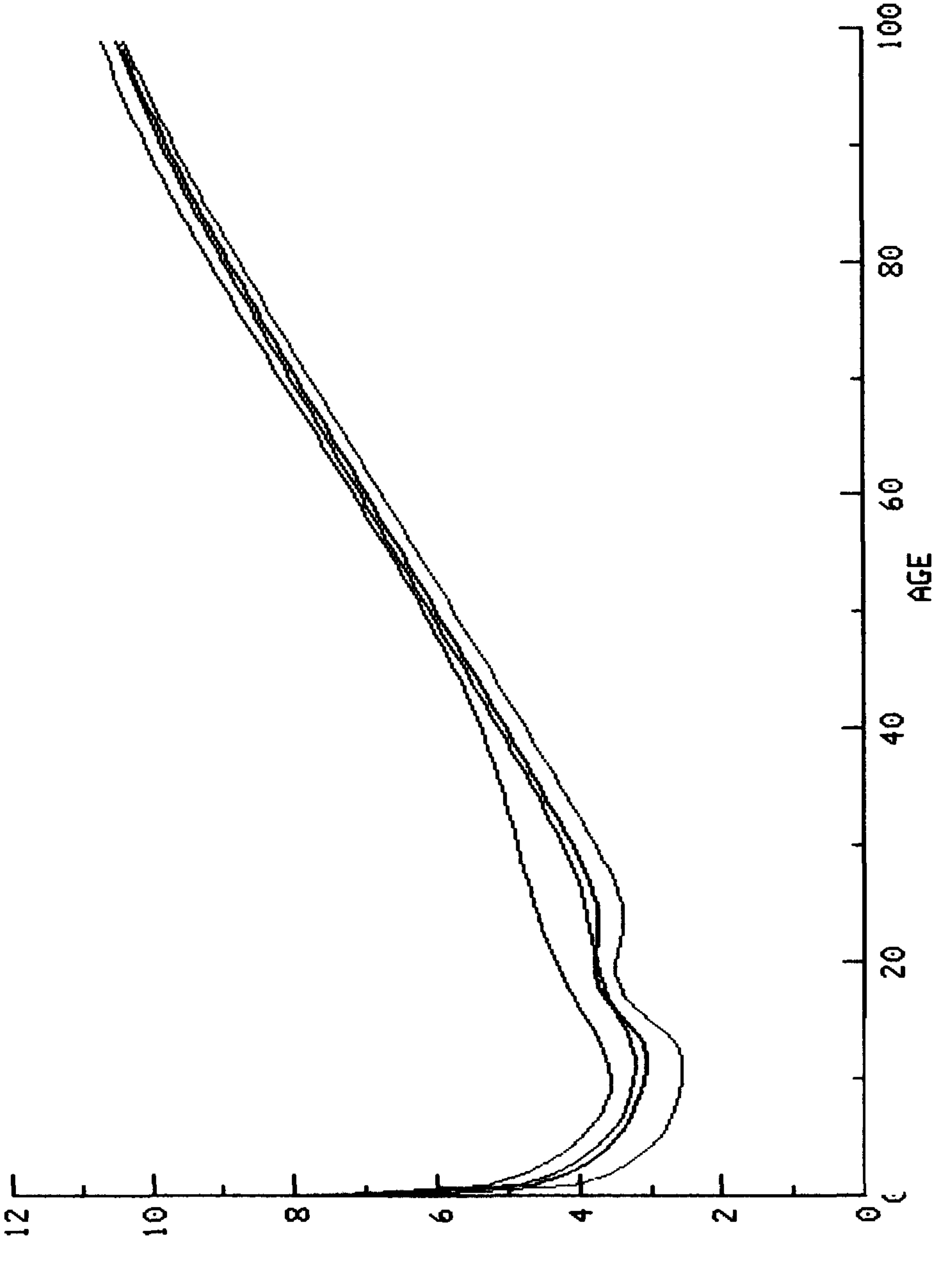
GRAPH 18

ELT(FORMULA) 11, 12, 13, (1981), (1991)
MALES - LN(100,000 × q)



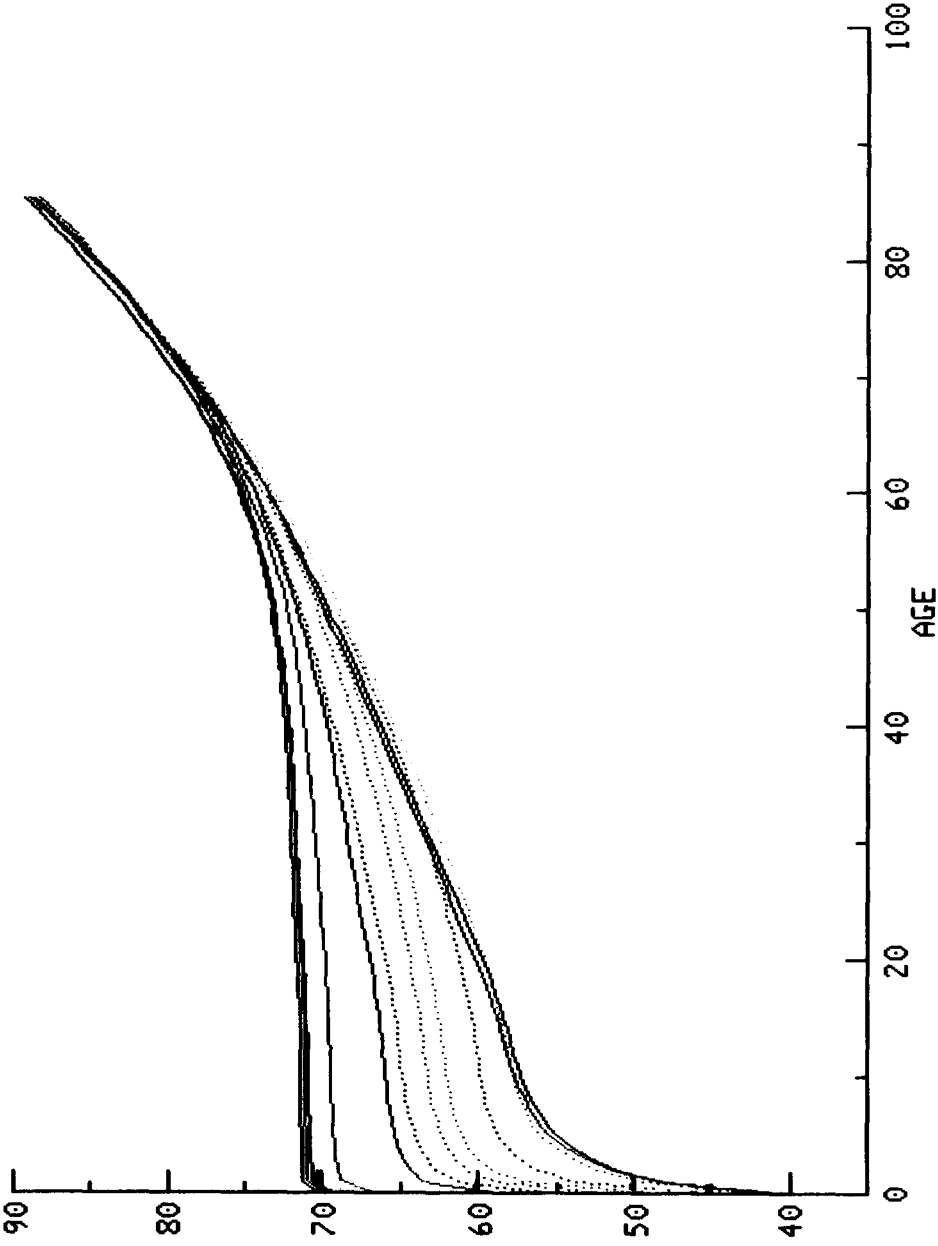
GRAPH 19

ELT(FORMULA) 11,12,13,(1981),(1991)
FEMALES - LN(100,000 x q)



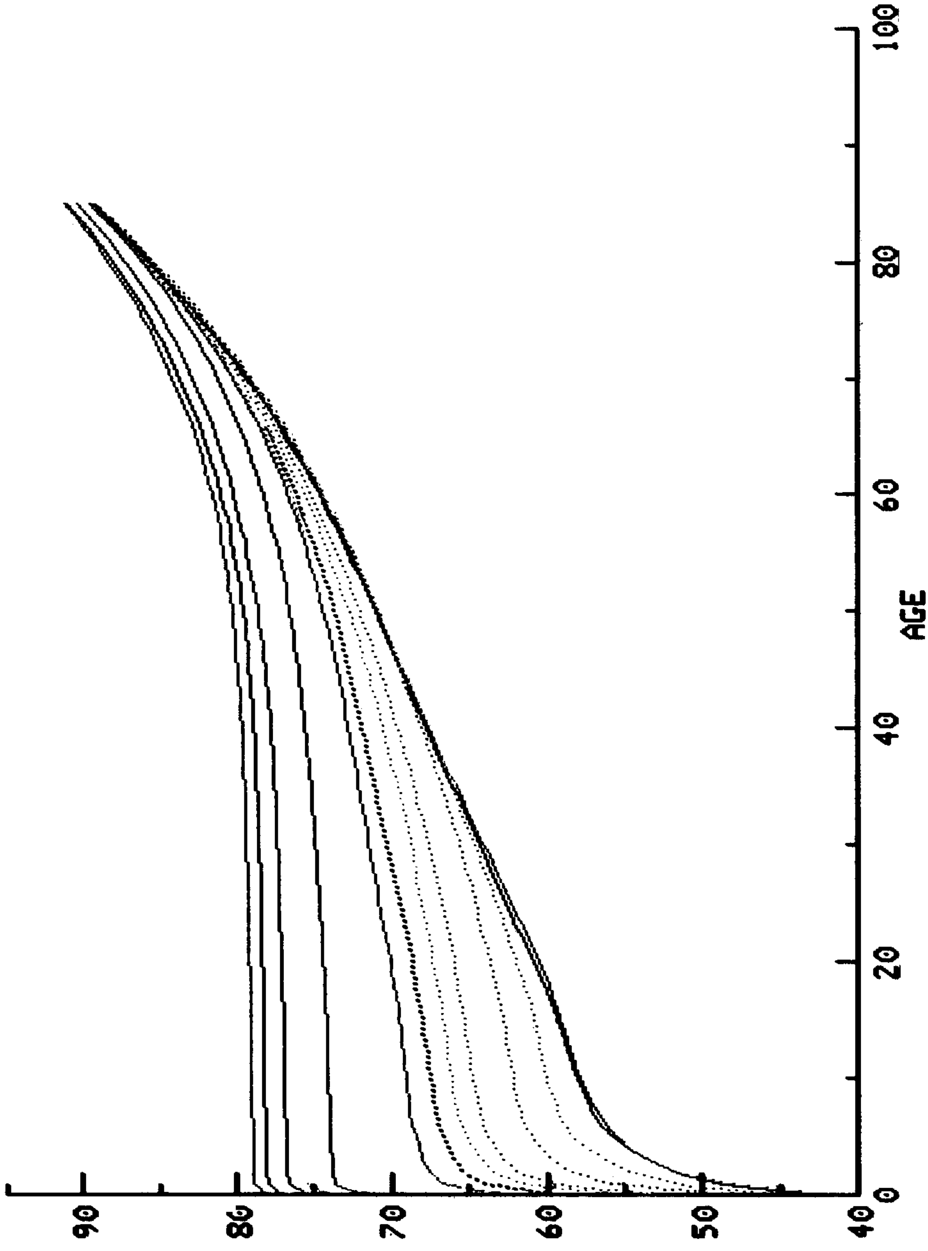
GRAPH 20

X + E[X] - ELT 2 - 13,(1981),(1991) - MALES



GRAPH 21

X + E[X] - ELT 2 - 13,(1981),(1991) - FEMALES



Addendum

1. In the paper parameters were found for one formula type for all of the English Life Tables numbers 1 to 13 for both males and females. Using these parameters, estimates were made of possible parameters for the years 1981 and 1991.

Since the paper was prepared, the mortality tables for the year 1981 (ELT 14) have been published and it is felt that it may be of interest to compare the estimated parameters for the year 1981 with those which could be calculated from the published tables. The results are as follows:

	$A \times 10^3$	$B \times 10^3$	$C \times 10^2$	$D \times 10^4$	E	F	$G \times 10^5$	H
Males								
Est. 1981	0·8	3·4	9·4	7·2	18·2	19·3	3·0	1·114
ELT 14	0·7760	3·5780	8·8473	6·4195	14·647	19·749	2·955	1·1113
Females								
Est. 1981	0·7	11·2	10·7	1·6	20·0	18·8	2·2	1·107
ELT 14	0·6750	9·3212	9·8176	1·4415	11·085	18·771	2·243	1·1065

As before, the "best fit" to the published table was found by minimising S^2 where

$$S^2 = \sum_{x=0}^{85} ((\hat{q}/q) - 1)^2$$

$\hat{q}$ = mortality rate estimated from the formula
q = mortality rate from the appropriate ELT table

The value of S^2 comparing the estimated tables ($\hat{q}$) with the actual published tables (q) is 2·1455 for males and 0·4095 for females confirming the reasonableness of the estimates for the female parameters. The "best fit" for male lives gives a sum of squares (S^2) of 0·6505 for male lives and 0·1486 for female lives.

It is clear from the values of S^2 that the estimates for 1981 were very good for female lives but much poorer for male lives.

Graph I shows the formula mortality rates together with the actual ELT (Males) mortality rates for ELT 14. Graph II plots the same curves for female mortality. The graphs plot $\log_e(10^5 \times q_x)$ against age.

2. It appears from the values of S^2 for males for ELT 12, 13 and 14 that it is becoming increasingly difficult to get a good fit using the formula. The differences are greatest at higher ages where the Heligman and Pollard formula produces values which are too high.

After some experimentation, it was decided to fit the male tables to the following formula:

$$q_x = f(x) \,/\, (1 + f(x))$$

where

$$f(x) = A^{(x+B)^C} + D\,e^{-E(\log_e(x/F))^2} + \frac{GH^x}{(1 + GH^x)}$$

Table 1 below shows the revised parameters and values of S^2 for ELT 1 to ELT 14 inclusive. It can be seen that the fit for the more recent tables is now much better than before. Graphs III and IV show the formula mortality rates together with the actual ELT (Males) mortality rates for ELT 13 and 14 corresponding to the years 1971 and 1981. As before the graphs plot $\log_e(10^5 \times q_x)$ against age.

TABLE 1

English Life Tables (Males)

ELT	$A \times 10^3$	$B \times 10^3$	$C \times 10^2$	$D \times 10^4$	E	F	$G \times 10^5$	H	S^2
1	88·023	342·80	34·394	93·354	1·1760	35·377	6·267	1·1055	0·2376
2	93·239	376·85	35·768	84·548	1·4545	33·400	10·164	1·0986	0·1466
3	84·870	295·41	33·505	85·276	1·6460	33·880	12·172	1·0962	0·1305
4	92·743	406·07	38·251	101·86	1·0567	47·577	8·353	1·1016	0·1048
5	86·739	416·02	41·869	81·060	1·4242	42·190	10·069	1·1003	0·3021
6	66·037	242·90	38·337	77·023	0·6872	59·035	10·205	1·0996	0·2523
7	48·509	219·81	34·943	40·137	1·1402	37·598	14·444	1·0940	0·1727
8	34·371	159·59	29·071	30·433	1·4592	33·473	13·340	1·0950	0·2195
9	24·341	147·80	24·167	32·675	1·6765	31·724	6·845	1·1040	0·1950
10	14·512	84·621	20·438	21·762	3·2433	24·743	9·052	1·0999	0·4352
11	2·2408	4·4789	10·874	7·3962	7·9098	22·113	4·497	1·1099	0·1166
12	1·4846	2·8092	9·6955	7·4803	13.501	20·125	3·364	1·1135	0·2910
13	1·2060	2·8356	9·2678	7·1292	15·904	19·483	3·196	1·1136	0·2835
14	0·7781	3·1621	8·6434	6·3725	12·719	20·105	2·479	1·1155	0·4889

3. For completeness, we have recalculated our best estimates for the year 1991 and provided best estimates for the year 2001. The projections for female lives use the original formula while those for male lives use the alternative formula specified above. The projections are based on historical data and cannot make allowance for future, unknown trends (e.g. the effect of AIDS).

One possible use that could perhaps be made of these projections is to use these tables as a "standard" against which to measure future mortality which may well be significantly different because of the effects of AIDS. The figures are given below.

TABLE 2

	A × 10^3	B × 10^3	C × 10^2	D × 10^4	E	F	G × 10^5	H
Males								
Est. 1991	0·6	2·7	8·0	6·0	12·7	20·0	2·0	1·117
Est. 2001	0·45	2·7	7·5	5·6	12·7	20·0	1·5	1·120
Females								
Est. 1991	0·6	8·0	9·0	1·4	20·0	18·7	1·9	1·108
Est. 2001	0·5	6·5	8·2	1·4	20·0	18·6	1·6	1·1095

NB—The parameters for males are for the adjusted Heligman and Pollard formula described in paragraph 2 of this Addendum. The parameters for females are for the original Heligman and Pollard formula.

Tables 3 and 4 show values of q_x for ELT 14 and estimated values using the parameters in Table 2 for the years 1991 and 2001. Table 3 shows male rates and Table 4 female rates. Rates are given for decennial ages only.

TABLE 3 (Males)

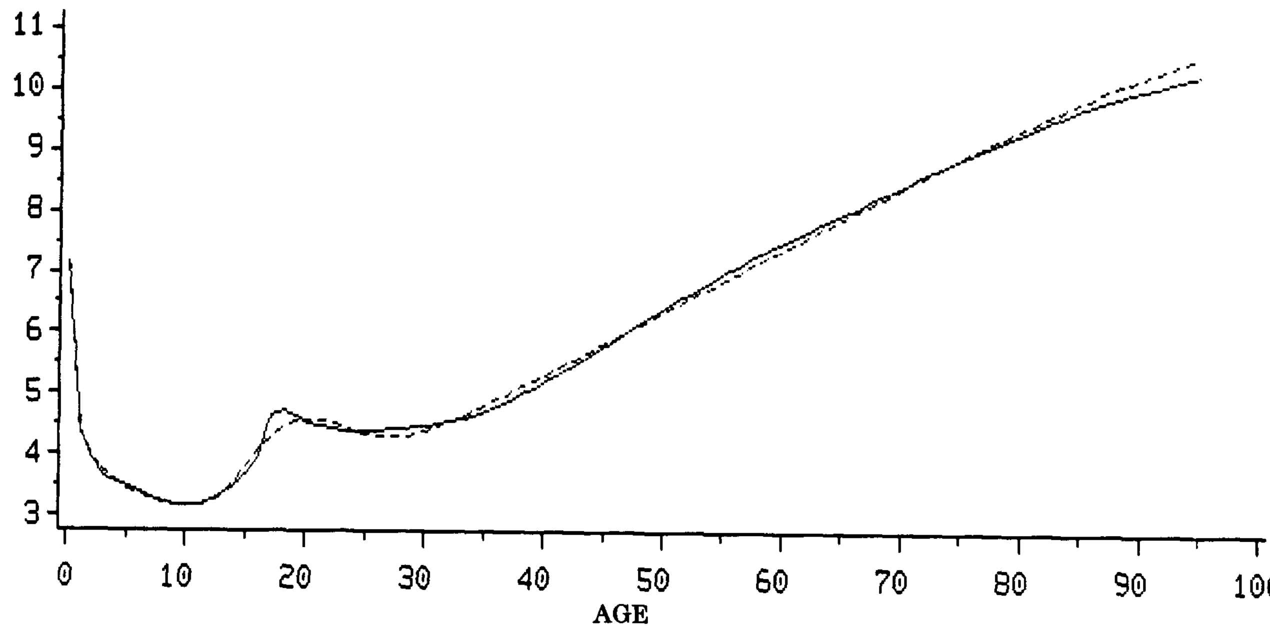
Age	q_x (ELT 14)	q_x (1991)	q_x (2001)
0	·01271	·00974	·00707
10	·00024	·00020	·00015
20	·00093	·00086	·00077
30	·00088	·00069	·00057
40	·00184	·00171	·00143
50	·00615	·00504	·00433
60	·01843	·01486	·01314
70	·04703	·04233	·03861
80	·11334	·10923	·10312

TABLE 4 (Females)

Age	q_x (ELT 14)	q_x (1991)	q_x (2001)
0	·00984	·00813	·00650
10	·00018	·00016	·00015
20	·00035	·00034	·00031
30	·00052	·00046	·00041
40	·00127	·00118	·00105
50	·00378	·00322	·00291
60	·00986	·00888	·00812
70	·02443	·02433	·02257
80	·06982	·06499	·06123

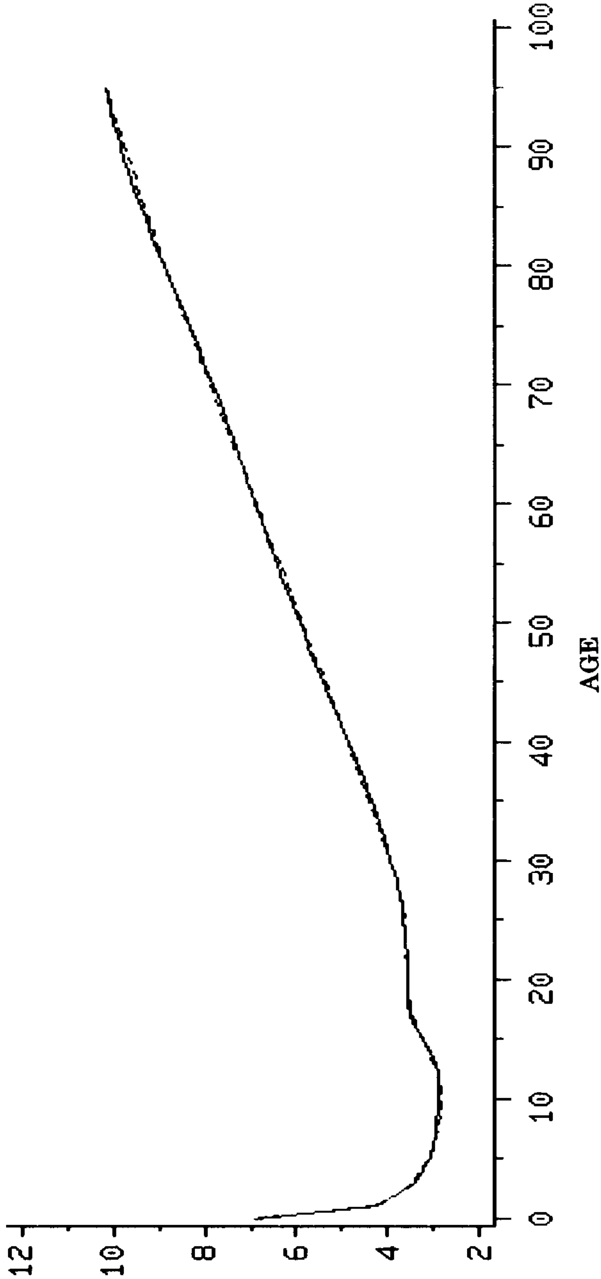
GRAPH I

PLOT OF ELT 14 (MALES) vs ELT 14 (Formula)—POLLARD
Continuous line actual, dotted line formula

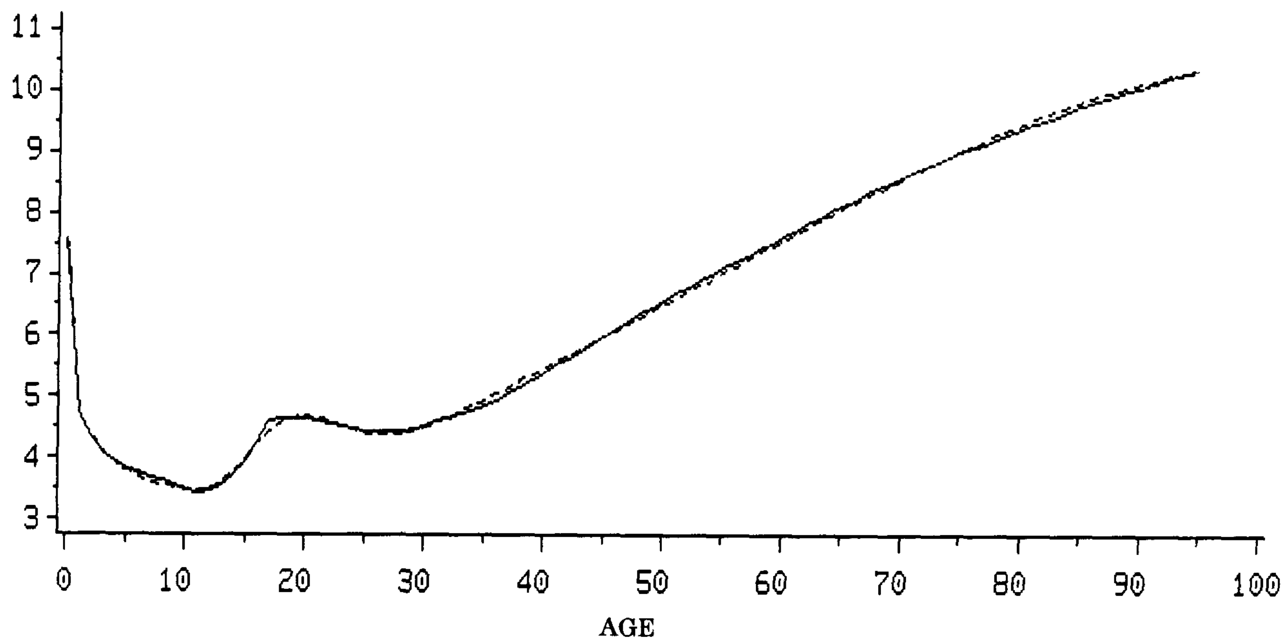


GRAPH II

PLOT OF ELT 14 (FEMALES) vs ELT 14 (Formula)—POLLARD
Continuous line actual, dotted line formula



GRAPH III PLOT OF ELT 13 (MALES) vs ELT 13 (Formula)—POLLARD ADJ.
Continuous line actual, dotted line formula



GRAPH IV

PLOT OF ELT 14 (MALES) vs ELT 14 (Formula)—POLLARD ADJ.
Continous line actual, dotted line formula