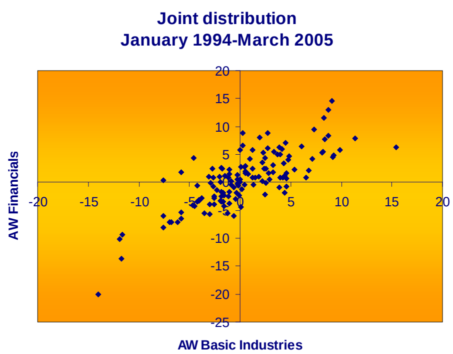
Copulation – are actuaries up to it?

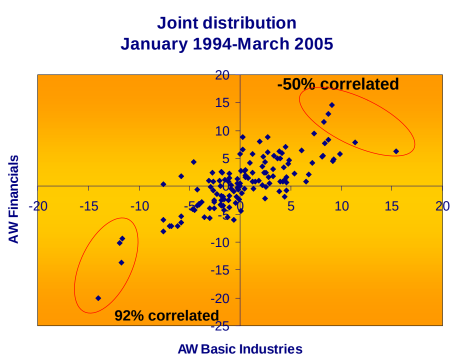
Younger members convention
5 December 2005

Martyn Dorey

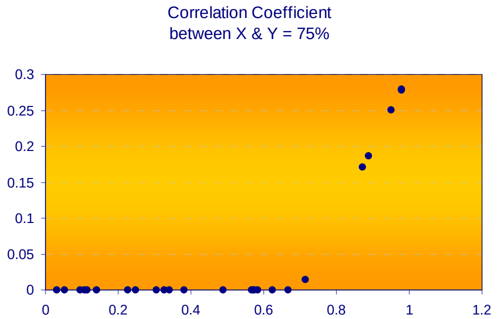
Murphy's law



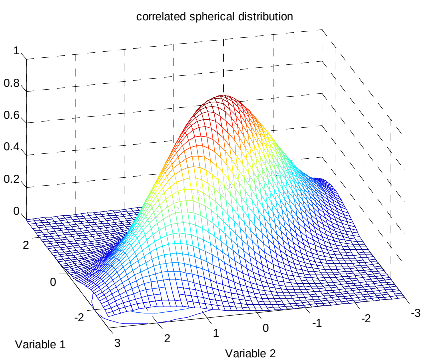
Murphy's law



Motivating Question 1



Motivating Question 2



Why is correlation so big?

- A linear combination of Normal rv's is Normal
- Regression:

$$Y = mX + \varepsilon$$

- m is the correlation coefficient

Desirable features of a measure of dependence

- Range [-1,1]?
- Scale invariance?
- 0 for independence?
- Extremes meaningful?

Measuring relationships

- Pearson's correlation:

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma(X)\sigma(Y)}$$

- Spearman's rank ρ:

$$\upsilon = 1-\frac{6D^2}{n(n^2-1)} \qquad with\ \ D^2=\sum_1^n\left(r_{y_i}-r_{x_i}\right)^2$$

- Kendall's τ (non linear):

$$\tau = 1-\frac{4Q}{n(n\ \ -1)}$$

Properties of rank correlations

✓ $\rho(X,X^3)$ and $\tau(X, X^3) = 1$

✓ If X and Y are independent then
$\rho(X,Y) = \tau(X,Y) = 0$

✓ $-1\leq \rho(X,Y), \tau(X,Y) \leq 1$

✓ If T:**R**→**R** is an increasing transformation, then
$\rho(X,Y) = \rho(T(X),Y)$ and $\tau(X,Y) = \tau(T(X),Y)$

Multivariate distributions - Copulas

F_X and $F_Y \rightarrow F_{XY}$?

Sklar's Theorem

If F_{XY} is a joint distribution function with marginal distributions F_X and F_Y, then there exists a function $C:[0,1]^2 \rightarrow [0,1]$ st

$$F_{XY} = C(F_X, F_Y)$$

We call C a copula.

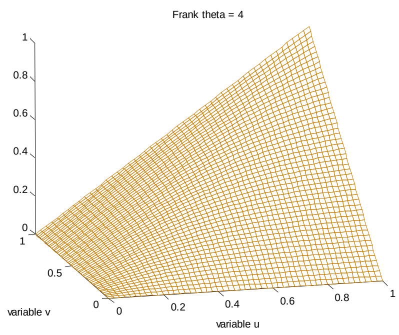
Conversely, if C is a copula, and F_X and F_Y are any marginal distributions, then F_{XY} defined above is a joint distribution with those marginals.

Copula – a definition

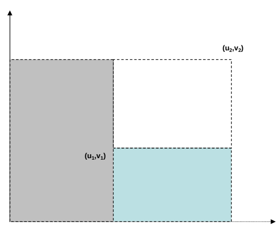
A function $C:[0,1]^2 \rightarrow [0,1]$ is a copula if:

1. $C(0,v) = C(u,0) = 0$
2. $C(1,v) = v$ and $C(u,1) = u$ all $(u,v) \varepsilon [0,1]$
3. $C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$
   all $(u_1,v_1),(u_2,v_2) \varepsilon [0,1]^2$ s.t. $u_1 \leq u_2$ and $v_1 \leq v_2$

What one looks like

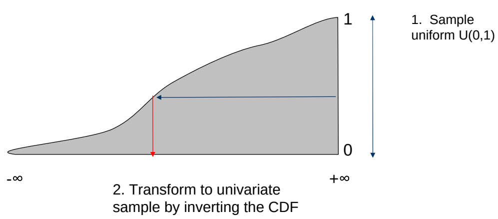


“2-increasing”

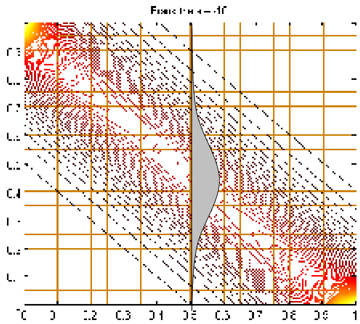


How does it work?

Start with a univariate distribution..



Copula is the 2-d version



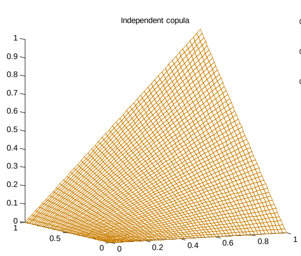
Some examples of Copulas

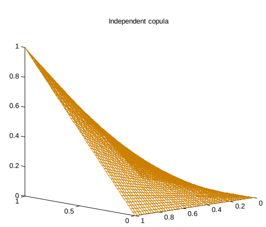
The independent copula:

$$C(u,v) = uv$$

1. $C(0,v) = C(v,0) = 0$
2. $C(1,v) = v$ and $C(u,1) = u$
3. $C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1)$
   $= u_2v_2 - u_1v_2 - u_2v_1 + u_1v_1$
   $= (u_2 - u_1)(v_2 - v_1) \geq 0$

Independent copula





The Gaussian copula

Copulas can be generated from known multivariate distributions using Sklar's theorem:

$C(u,v) = F_{XY}(F_X^{-1}(u), F_Y^{-1}(v))$

$C(F_X, F_Y) = F_{XY}(F_X^{-1}(F_X), F_Y^{-1}(F_Y)) = F_{XY}$

e.g.

$C(u,v) = \Phi^2\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$

The Gaussian copula

Say we look at two variables:

1. Generate 100x2 normally distributed samples
   [x1 x2] = randn(100,2)
2. Apply a transformation to make them correlated
   y1 = x1+x2*rho
   y2 = x2*sqrt(1-rho^2)
3. Take each column in turn and apply norm inv.
   U = Norminv(y1,0,1)
4. Treat these as samples from a cumulative dist with a Gaussian joining function.

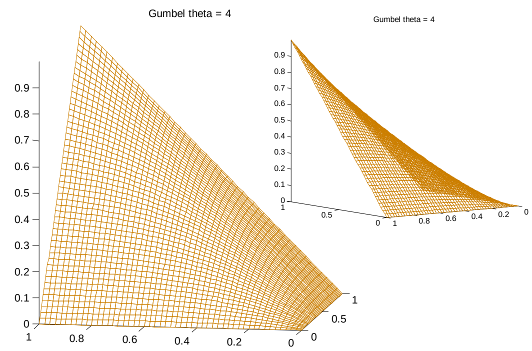
Archimedean Copulas

- Copulas can also be generated directly by considering a formula.
- A large class of such copulas are called archimedean copulas:
- Let $\varphi:[0,1] \to [0, \infty]$ be strictly increasing and convex with $\varphi(0) = \infty$ and $\varphi(1) = 0$. Then

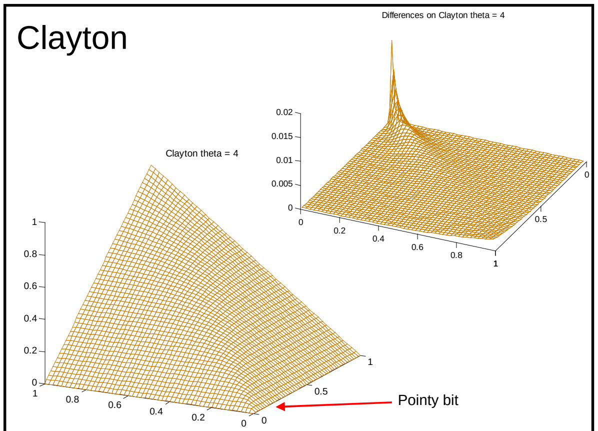
  $C(u,v) = \varphi^{-1}(\varphi(u)+\varphi(v))$

  is a copula

Gumbel



Clayton

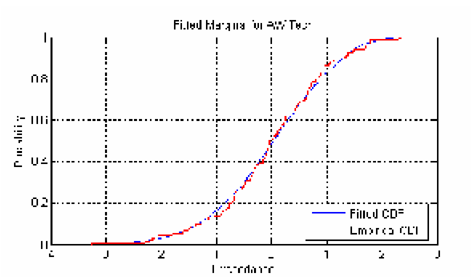


Skew elliptical distributions

- Brand new.
- Normal (σ)
  - Y = z* σ
- Skew Normal (α)
  - SkewN = max(Y1,Y2)
- T distribution (v)
  - TDist= z* σ *stretch
- Skew T
  - Y = SkewN*stretch

Conditioning

Marginal distributions

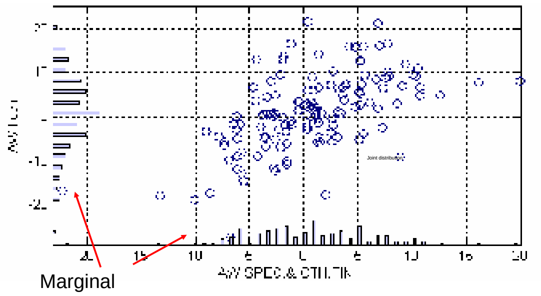


What copula to join marginals

- There are a large number of known copulas – how do we choose?
- Many copulas are impractical for large variables or conditioning
- Often the Gaussian is used – for convenience and familiarity
- Personally prefer the T.

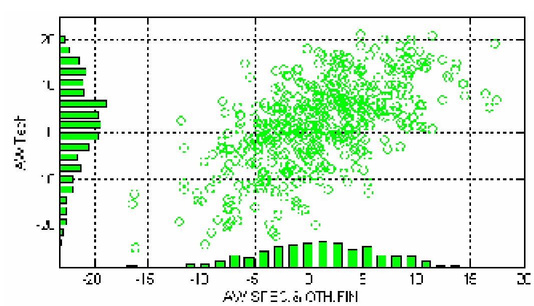
Historic returns

Historic Joint distribution between AW Tech AW Specialist Finance

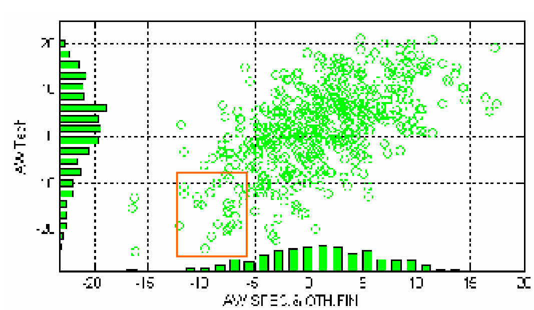


Marginal distributions

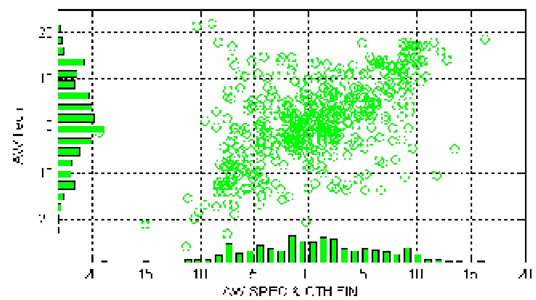
Gaussian simulations



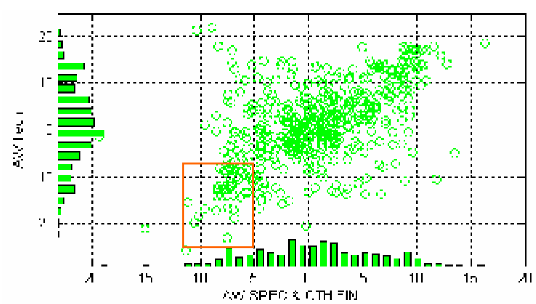
Gaussian simulations



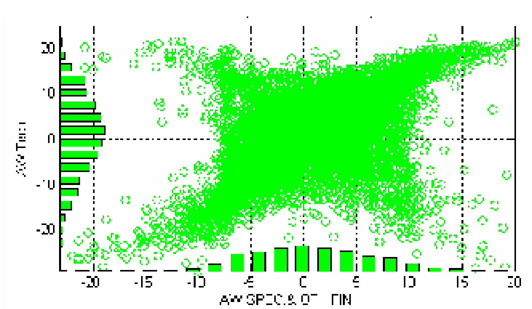
T simulations



T simulations



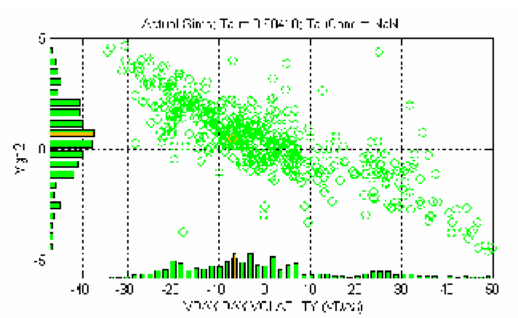
Overcooked T.



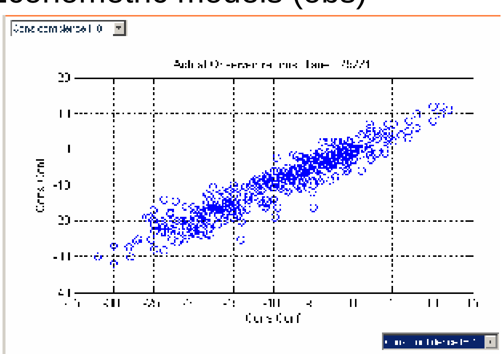
Uses of copulas in practice

- V@R analysis
- Creating multi-business line loss distributions
- Asset pricing, e.g. CDO’s
- Often used to generate simulations for Monte-Carlo implementations

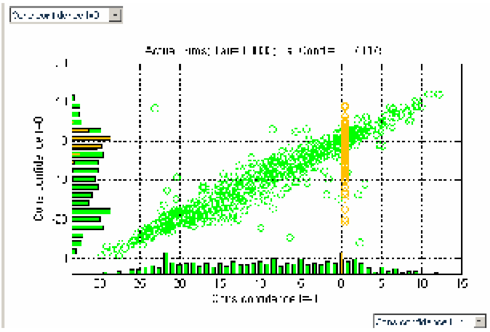
Manager stress testing



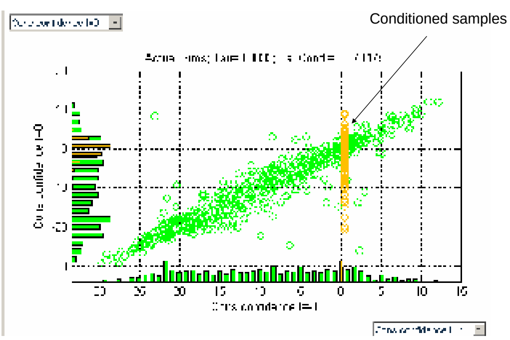
Econometric models (obs)



Econometric models (sims)



Regression (conditioning)



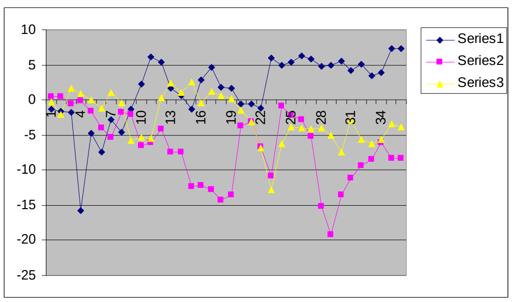
Comparison

- GLM (& neural networks)
  - Does it have to be a straight line?
  - Transformation loses transparency
  - Complex transformations lead to instability
  - Generalisation ability can be poor
  - Learning process can be slow

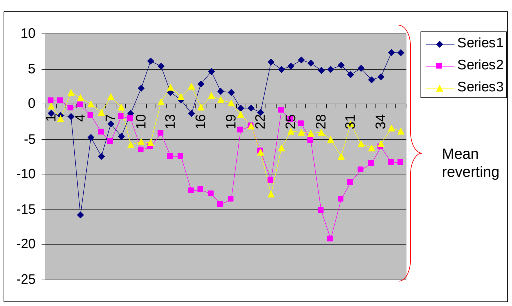
Time series modelling

- Feedback outputs into the input.
- Roll forward
- Condition on economic environment.
- Like a transition matrix.

Autoregression copulas – cons conf.



Autoregression copulas – cons conf.



What next

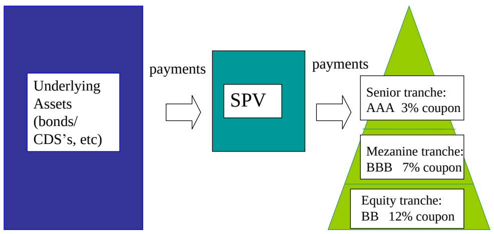
- Asymmetric copulas (clayton & adj. t)
- Bolting together different designs
  - Segment input space
  - Different copulas on different relationships
- Discontinuous copulas
- More work on feedback
- Artificial intelligence

Parameterisation

- Gaussian & T copula
  - Rho = sin(pi/2*Tau)
- Clayton
  - Rho = 2*Tau/(1-Tau)
- Gumbel
  - Rho = 1/(1-Tau)

CDOs and Copulas

CDO structure



What is a CDS?

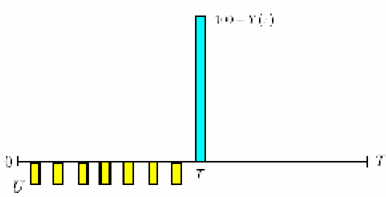
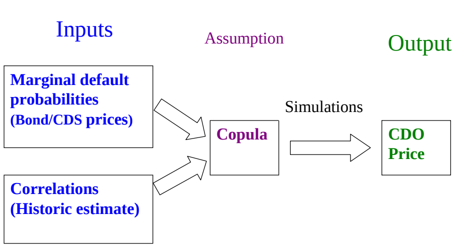


Figure 3: Default swap: buyer of protection pays the CDS rate U quarterly, and at the default time τ delivers bond worth $Y(\tau)$ in exchange for notional (100).

Source: Duffie, Ixis CIB presentation

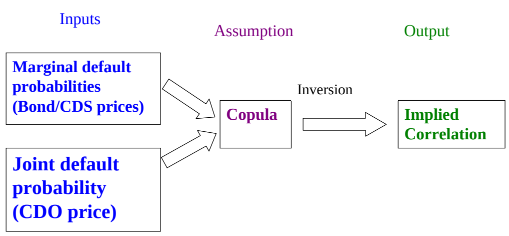
Pricing using a Copula



Pricing using a Copula

- Generate correlated uniform random variables using Gaussian copula
- Use these to simulate random default times
- Use these to generate Monte Carlo estimate of payoffs

Marking to market using a Copula



Base correlation

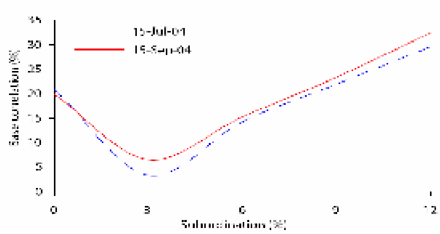


Figure 19: Base Correlation Smiles. iTraxx Europe. Source: Deutsche Bank, September, 2004.

Source: Duffie, Ixis CIB presentation

Portfolio loss distribution