

Plan

1. Does studying advanced mathematics develop general reasoning skills?
2. Short break: have a go at the question on your sheet!
3. Using comparative judgement to improve mathematics teaching and learning.
4. A demonstration of the NoMoreMarking system.

Does studying advanced mathematics develop general reasoning skills?

Matthew Inglis

Royal Society Worshipful Company of Actuaries Research Fellow
Mathematics Education Centre, Loughborough University

Plan

- Why should people study mathematics?
- The Plato/Vorderman Hypothesis:
Theory of Formal Discipline.
- Reasons to doubt the value of mathematics.
- Do mathematicians reason differently to non-mathematicians?
- Is this developmental?

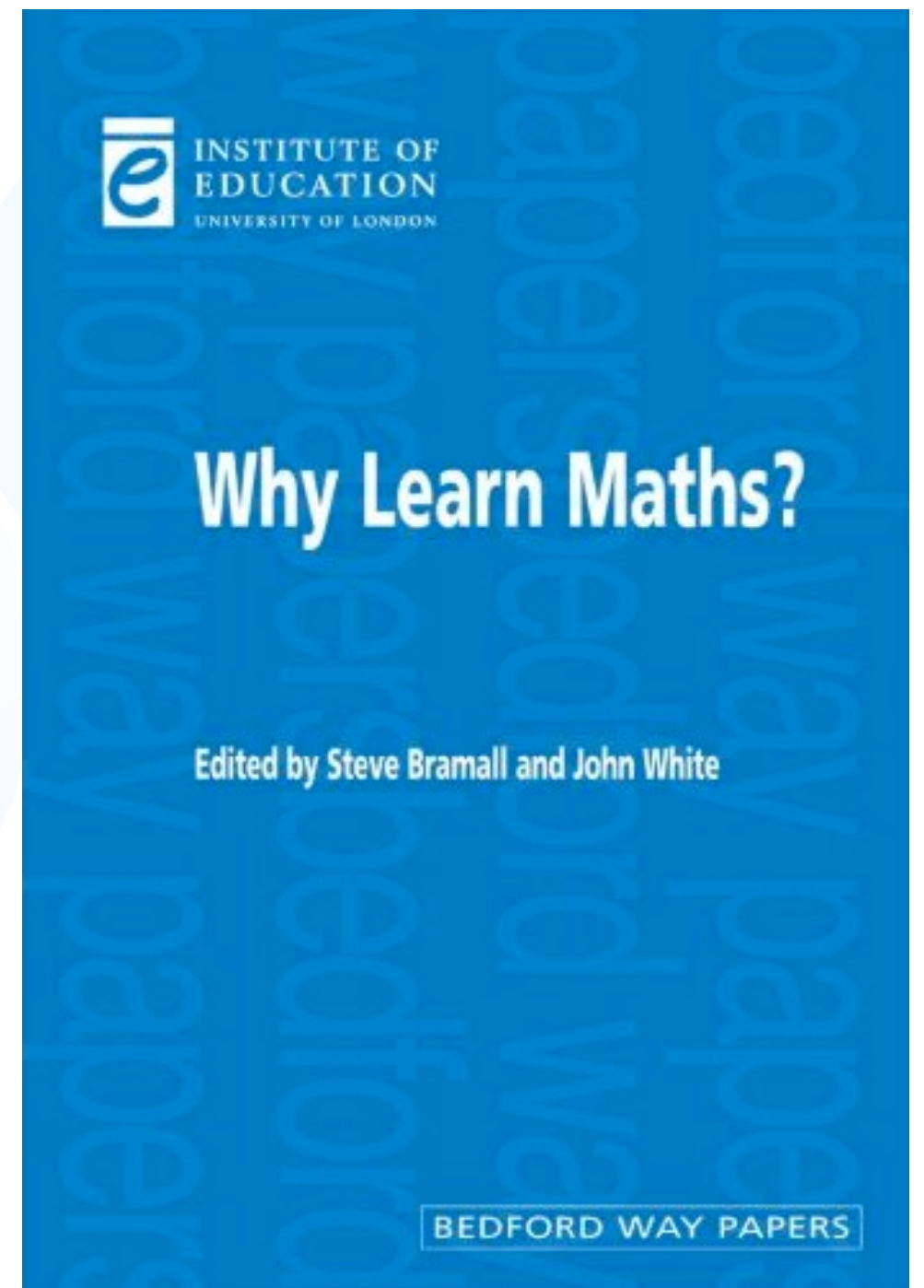
Why Study Mathematics?

Mathematics has a privileged place on the school curriculum. Why? Two traditional reasons:

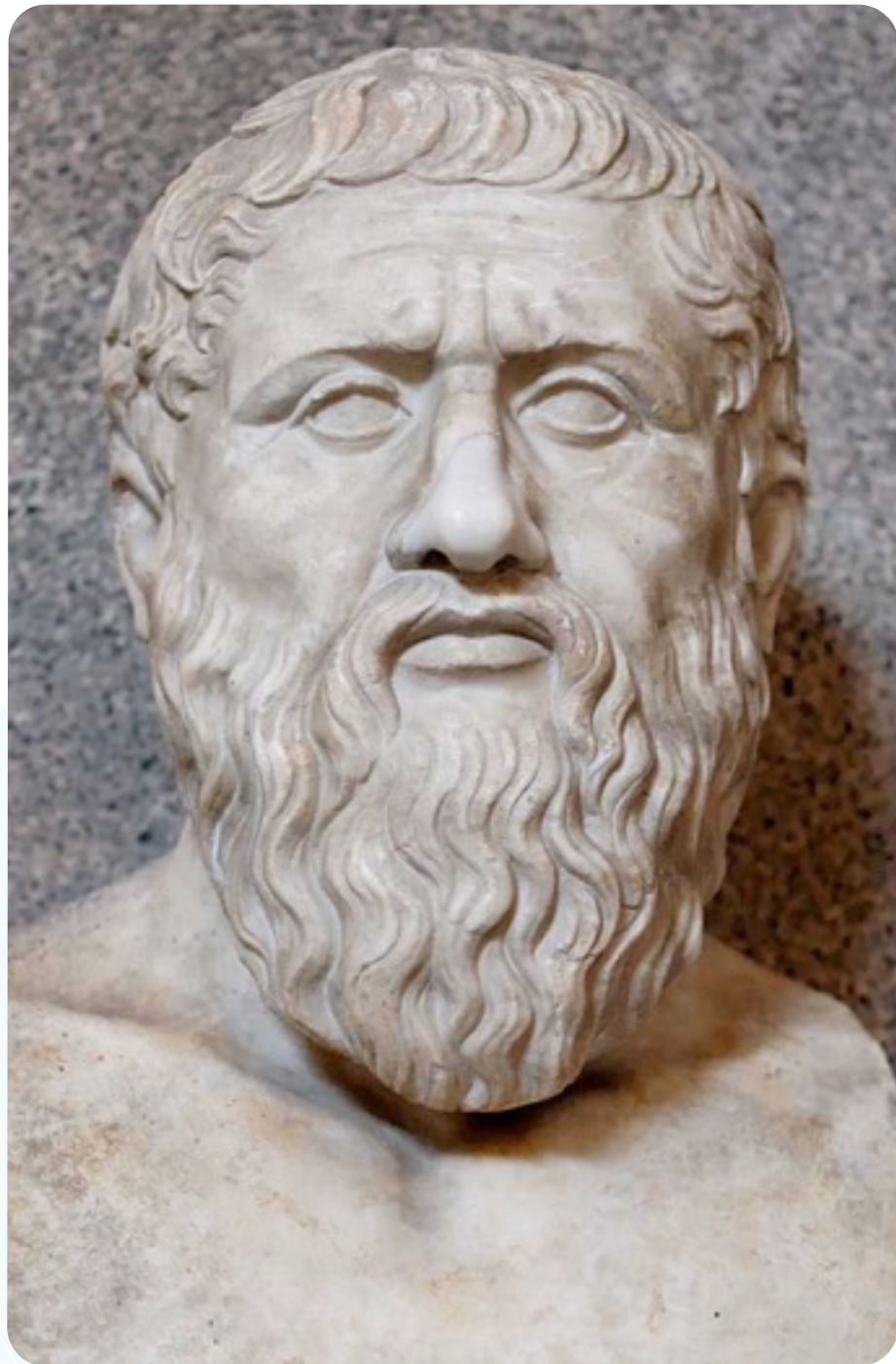
1. It's useful in real life
2. It teaches you to think

Focus of talk:

The Theory of Formal Discipline



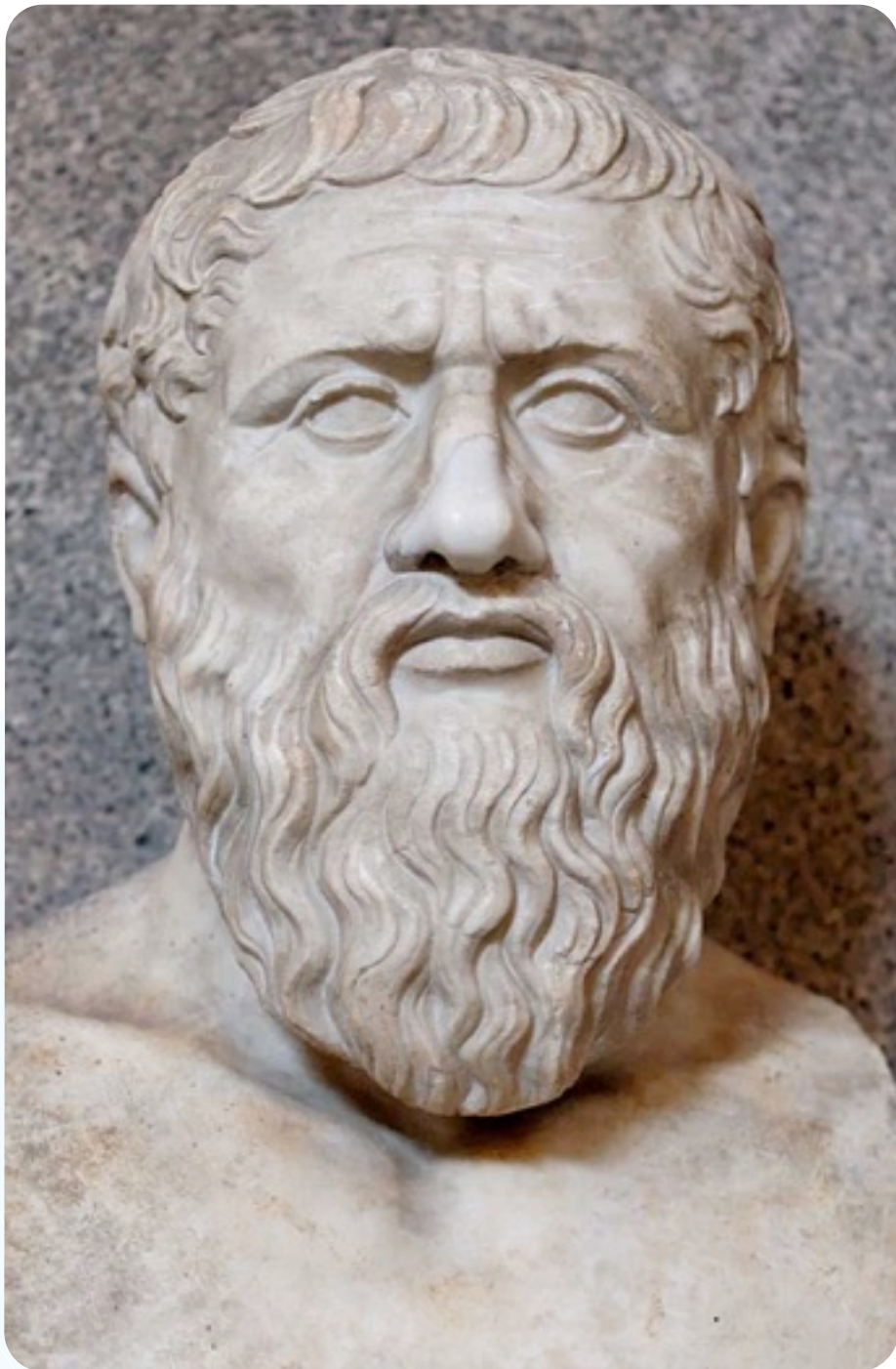
Why Study Mathematics?



Plato (400 BC):

“Those who have a natural talent for calculation are generally quick at every other kind of knowledge; and even the dull, if they have had an arithmetical training... become much quicker than they would otherwise have been.”

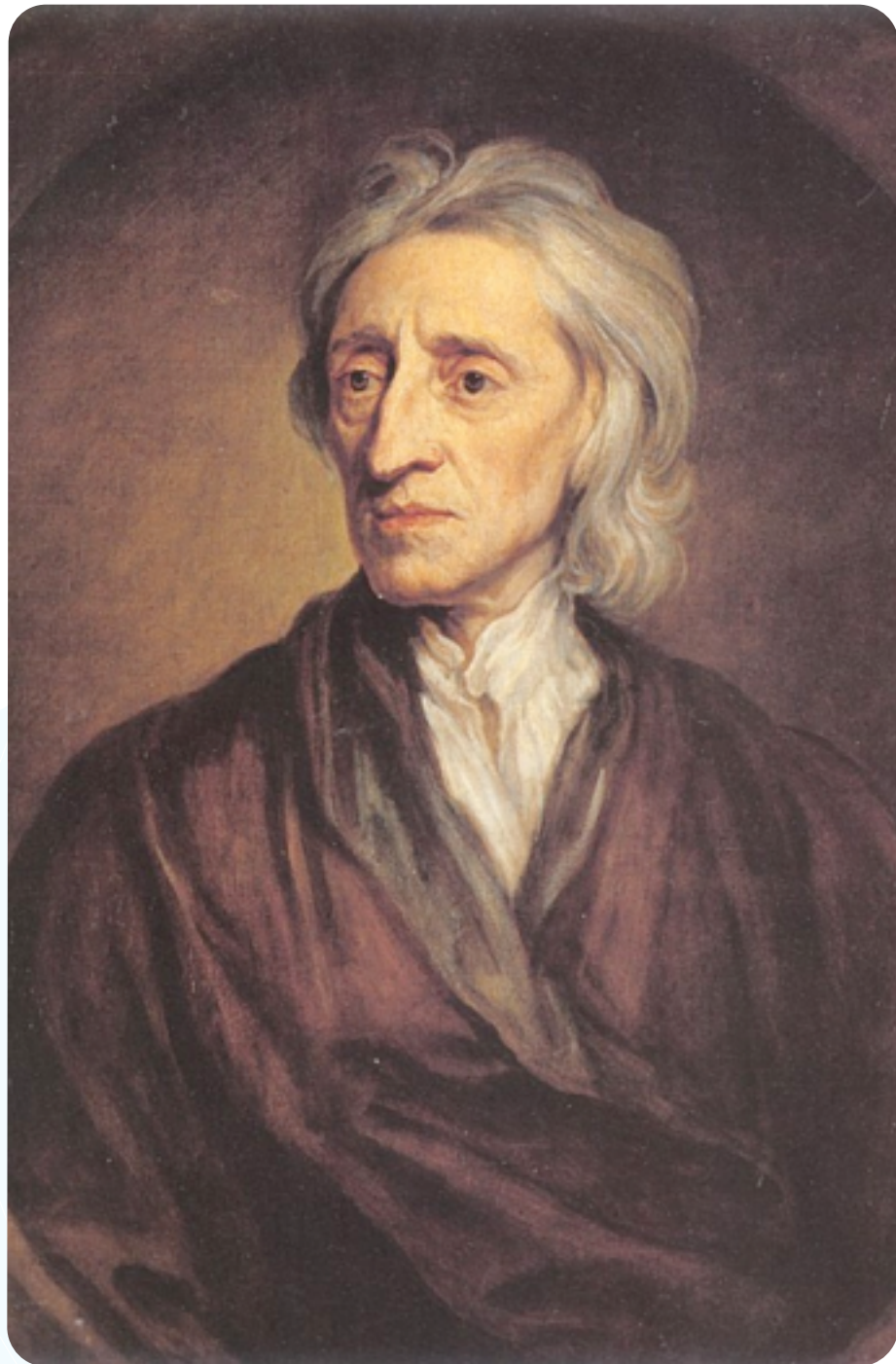
Why Study Mathematics?



Plato (400 BC):

“We must endeavour to persuade those who are to be the principal men of our state to go and learn arithmetic”

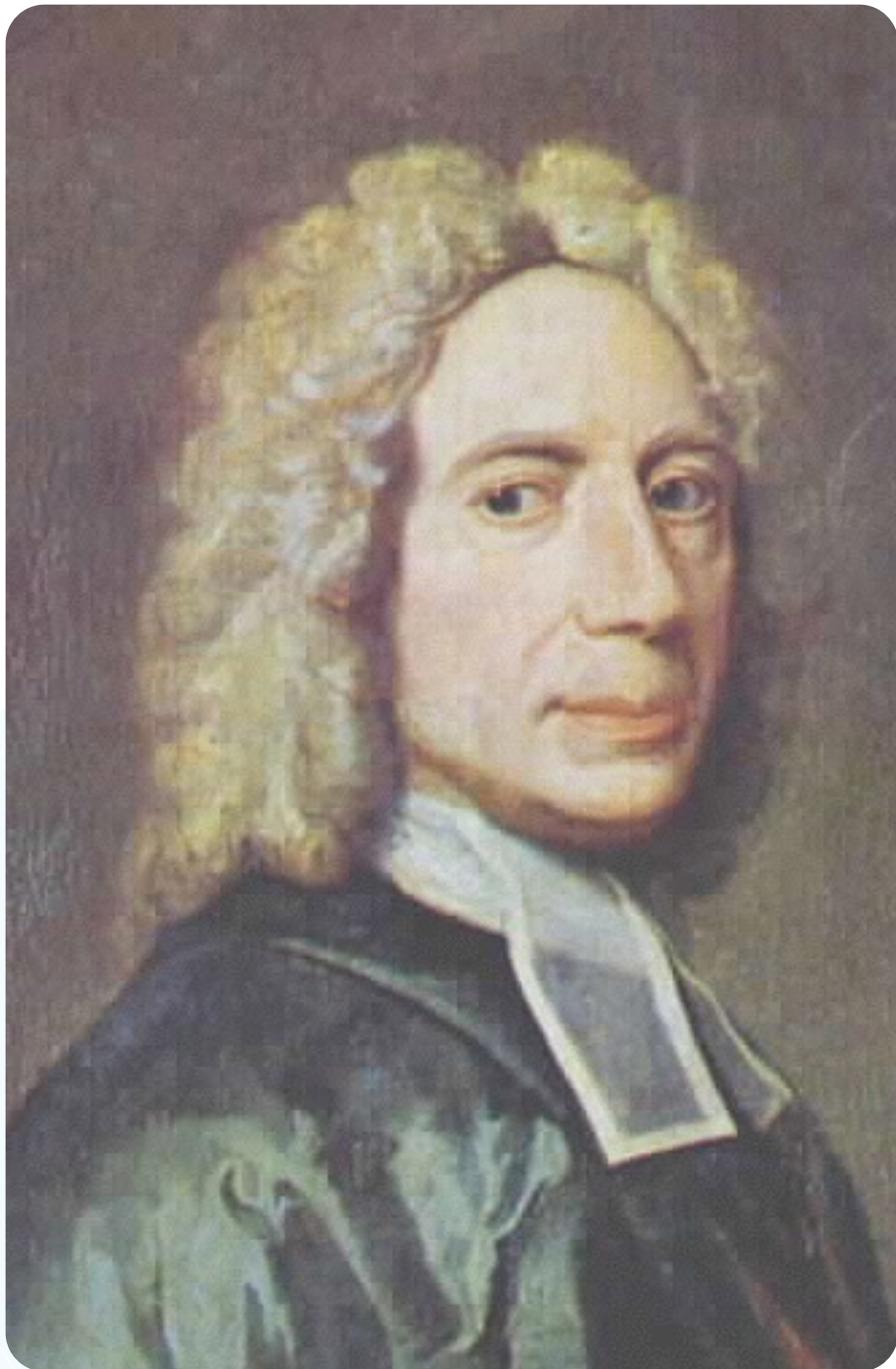
Why Study Mathematics?



John Locke (1706):

Mathematics ought to be taught to “all those who have time and opportunity, not so much to make them mathematicians as to make them reasonable creatures”

Why Study Mathematics?



Isaac Watts (1752)

“If we pursue mathematical Speculations, they will inure us to attend closely to any Subject, to seek and gain clear Ideas, to distinguish Truth from Falsehood, to judge justly, and to argue strongly”

Theory of Formal Discipline

Features of the Theory of Formal Discipline:

- Studying mathematics develops general reasoning abilities, which apply to non-mathematical areas of life;
- This link is causal.

Not just of historical interest.

Why Study Mathematics?



Professor Adrian Smith
(Smith Report, 2004):

“Mathematical training disciplines the mind, develops logical and critical reasoning, and develops analytical and problem-solving skills to a high degree.”

Why Study Mathematics?



The Smith Report recommended tuition fee rebates for mathematics students, and higher salaries for mathematics teachers.

Why Study Mathematics?



Vorderman Report
commissioned by the
Conservative Party:

“Mathematics is not only a language and a subject in itself, but it is also critical in fostering logical and rigorous thinking”

Obvious Question

- Mathematicians are **incredibly** good at arguing for the importance of their subject.
[Compare to psychology: “Psychology, law and media studies: the ‘scandalous’ routes to A-grade success”, *The Independent*, August 2003].
- But notice that none of these advocates offered **any** scientific evidence at all.
- So is the Theory of Formal Discipline correct?
- It could be that those who choose to study mathematics are already better at reasoning: the *filtering hypothesis*.

Obvious Question

- Does studying mathematics cause the development of general reasoning skills?
- In fact (limited) empirical evidence does exist.

Thorndike & Woodworth



Edward Thorndike
(1874 - 1949)

THE INFLUENCE OF IMPROVEMENT IN ONE
MENTAL FUNCTION UPON THE
EFFICIENCY OF OTHER
FUNCTIONS. (I.)

BY DR. E. L. THORNDIKE,
Teachers College, New York,

AND DR. R. S. WOODWORTH,
New York University Medical School.

This is the first of a number of articles reporting an inductive study of the facts suggested by the title. It will comprise a general statement of the results and of the methods of obtaining them, and a detailed account of one type of experiment.

Thorndike & Woodworth



Edward Thorndike
(1874 - 1949)

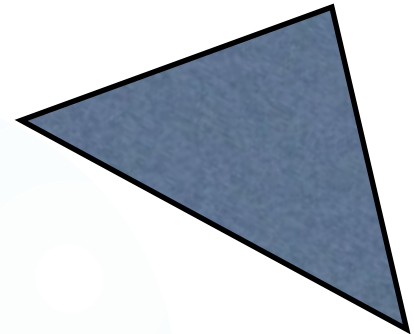
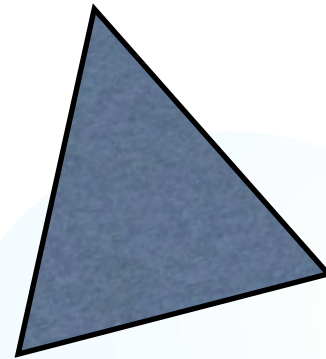
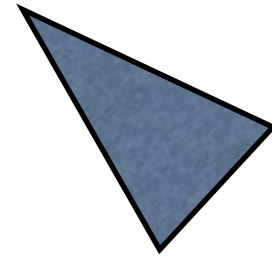
Edward Thorndike investigated the extent to which training on mental function X improves the closely related mental function Y.

Thorndike & Woodworth

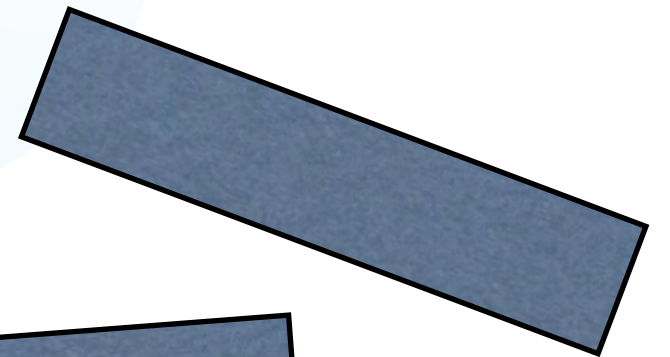
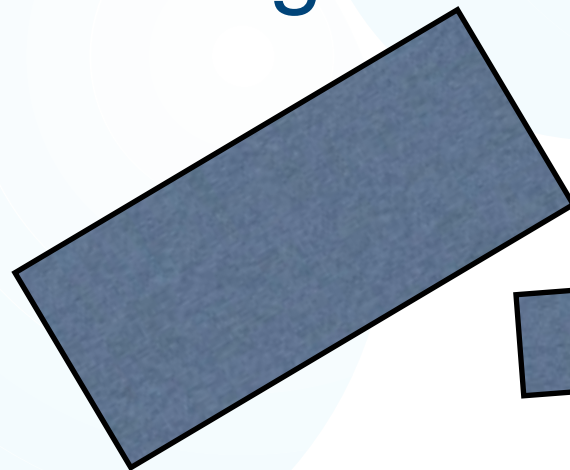


Edward Thorndike
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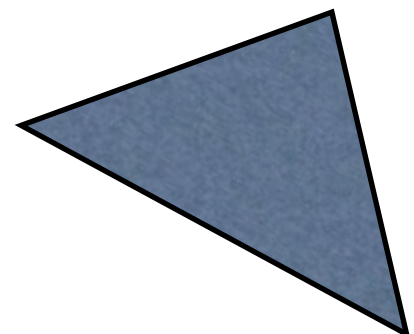
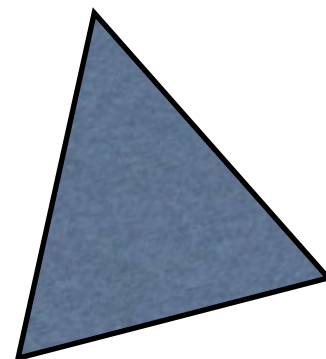
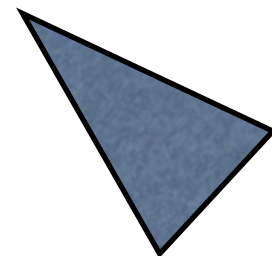
Test:



Training:



Test:



Thorndike & Woodworth



Edward Thorndike
(1874 - 1949)

“Improvement in any single mental function rarely brings about equal improvement in any other function, no matter how similar, for the working of every mental function-group is conditioned by the nature of the data in each particular case.”

Thorndike

What about formal schooling?



Edward Thorndike
(1874 - 1949)

THE JOURNAL OF EDUCATIONAL PSYCHOLOGY

Volume XV

January, 1924

Number 1

MENTAL DISCIPLINE IN HIGH SCHOOL STUDIES¹

E. L. THORNDIKE

With the aid of the staff of The Institute of Educational Research,
Teachers College, Columbia University

The experiment to be reported consisted of an examination in May, 1922, and a reexamination in May, 1923, of 8564 pupils who, in May, 1922, were in grades IX, X and XI. The two examinations were alternative forms of a composite of tests of "general intelligence" that are in common use, plus certain ones added in order to have measures with spatial as well as verbal and numerical content. This composite examination is that described in Vol. V, No. 4 of the *Journal of Educational Research*, April, 1922. Each pupil who took both examinations recorded the subjects which he studied during the school year Sept. 22, 1922 to June 23, 1923; and the gains made in the test were put into relation with the subjects studied. For example, we compare the gains for the pupils who studied English, history, geometry and Latin during the year with the gains for the pupils who studied English, history, geometry and shop-work. If other factors

Thorndike

Selected Findings:

Subject	“Regression Coefficient”
French	+ 0.48
Bookkeeping	+ 0.25
Arithmetic	+ 0.13
Geometry	+ 0.13
Algebra	+ 0.12
Drawing	– 0.01
Economics	– 0.50
Sewing	– 0.66



Edward Thorndike
(1874 - 1949)

Critique of Thorndike



Lev Vygotsky
(1896 - 1934)

Vygotsky suggested that Thorndike's "general intelligence" measure wasn't sensitive enough to measure developmental changes in reasoning skills.

Piaget

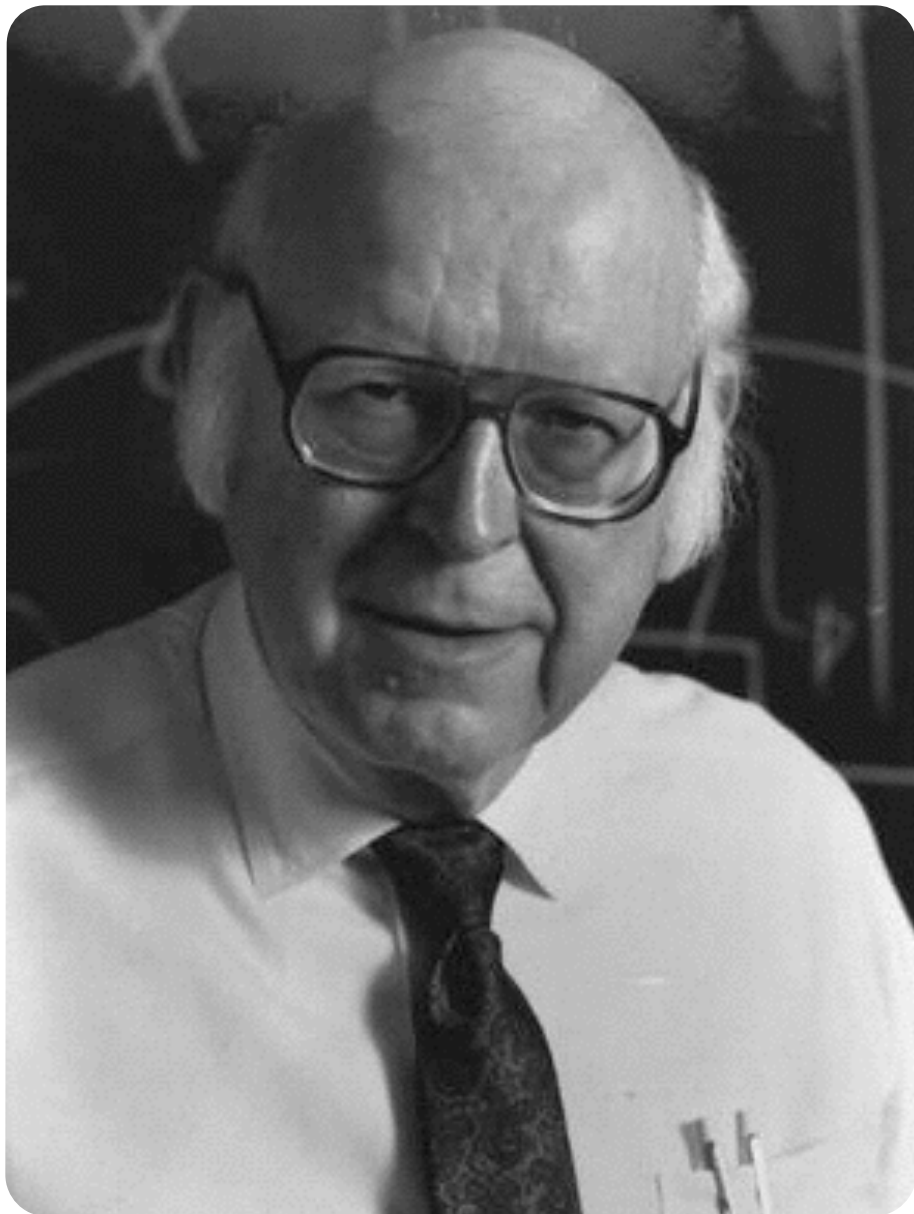


Jean Piaget
(1896 - 1980)

Piaget argued that domain-independent thinking skills did exist, but that they couldn't be taught.

You just have to wait until the child is ready to enter the “stage of formal operations”. You can do nothing at all to help.

The Cognitive Revolution



Alan Newell

Following the cognitive revolution, most cognitive scientists rejected Piaget's claims.

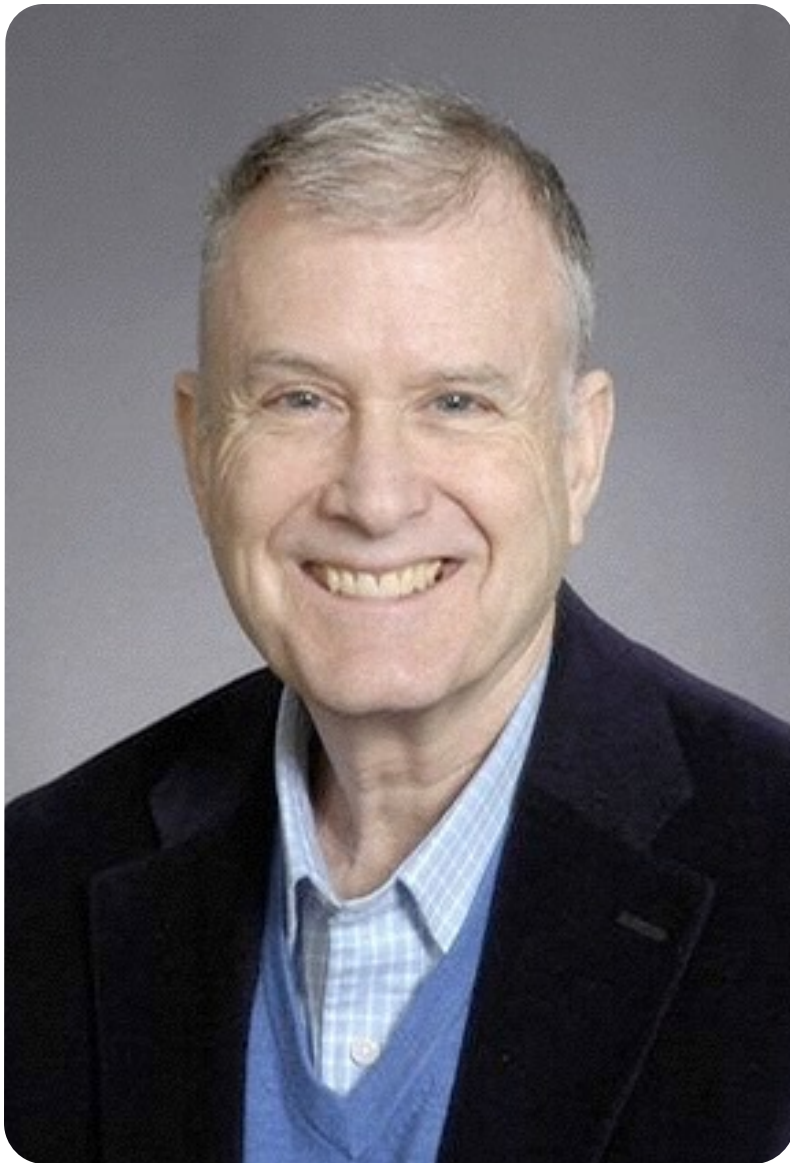
Newell wrote:

“The modern position is that learned problem-solving skills are, in general, idiosyncratic to the task.”

Bad news for Plato/Vorderman:
mathematics cannot develop domain-general skills, as they don't exist!

Newell, A. (1980). One last word. In D. Tuma and F. Reif (Eds.) *Problem Solving and Education*, Hillsdale, NJ: Erlbaum.

Studying Psychology Improves Thinking

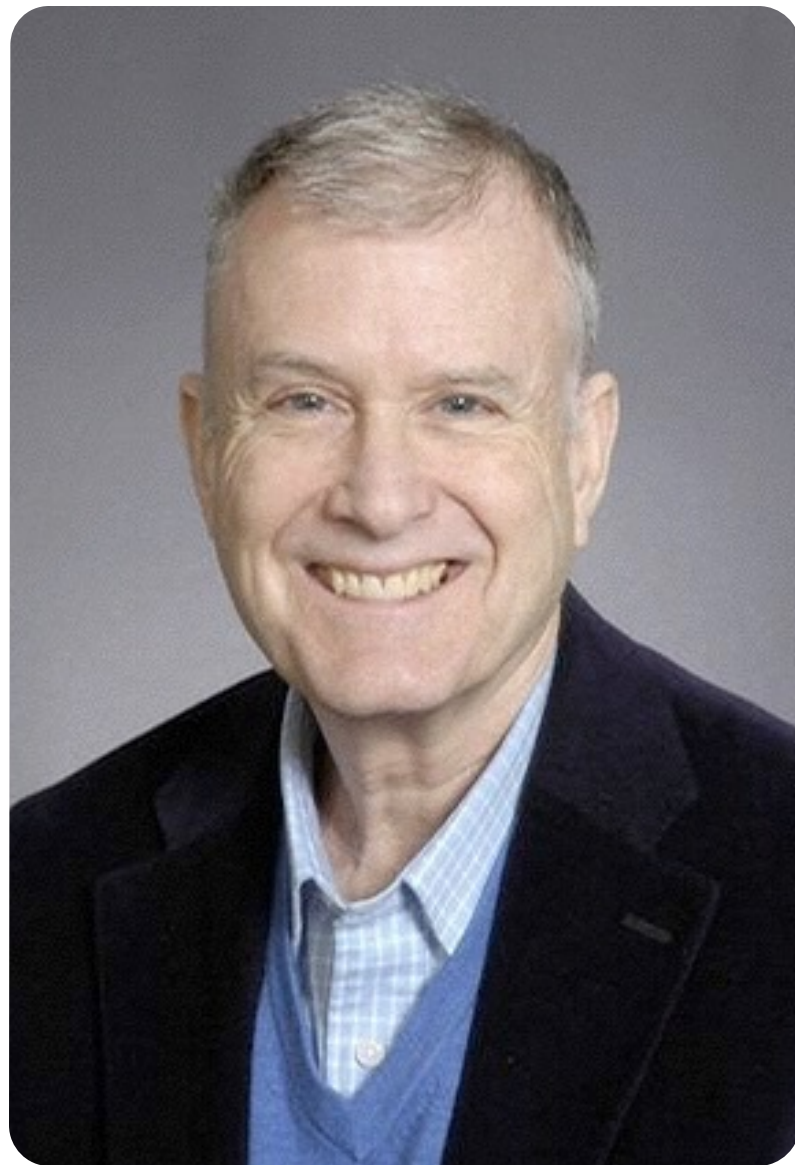


Richard Nisbett

However, more recently Richard Nisbett has found that **some** domain-independent thinking skills **do** exist, and that these can be taught.

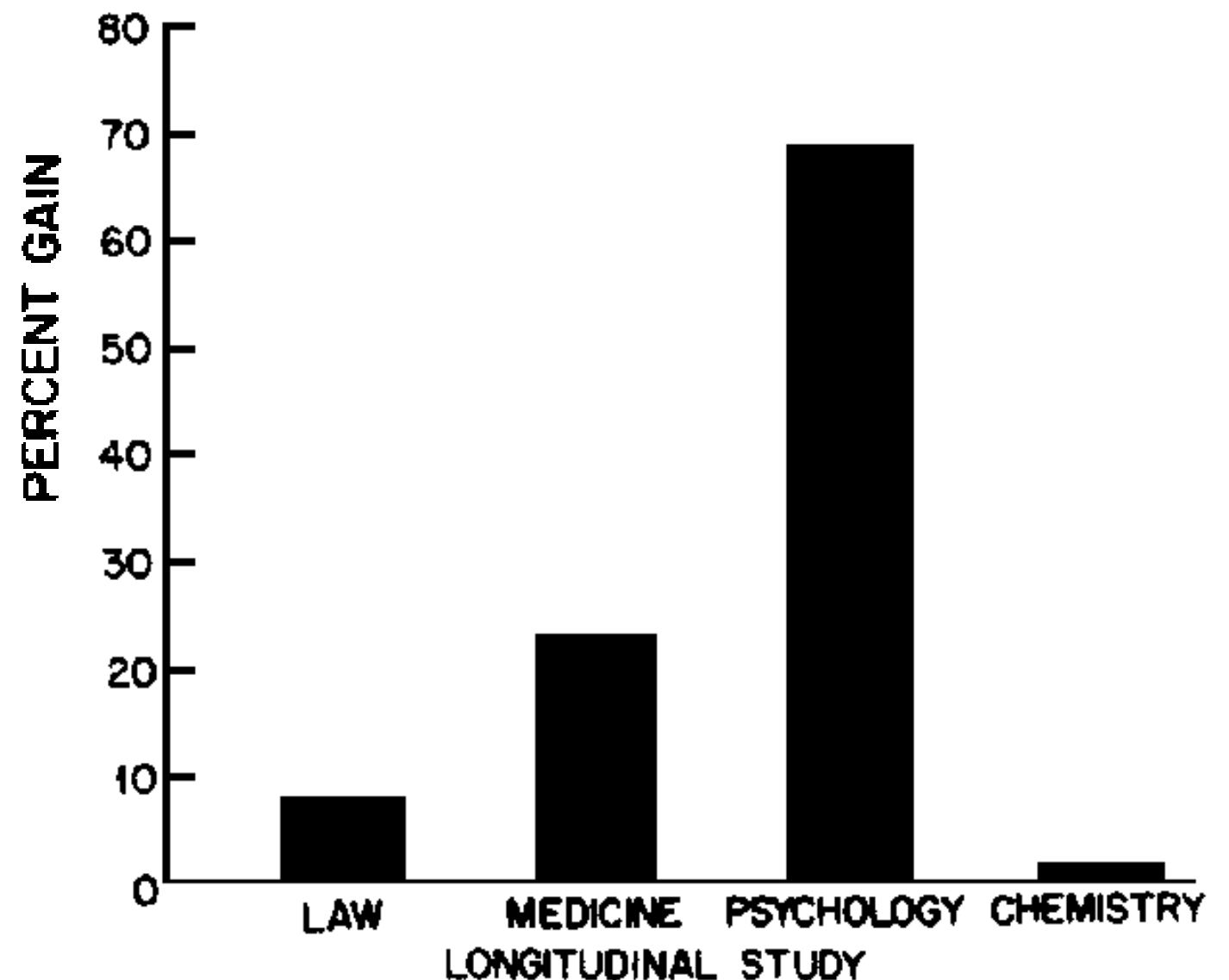
In particular, he has shown that studying psychology makes you better at “statistical and methodological reasoning”. Not so for law or chemistry.

Studying Psychology Improves Thinking

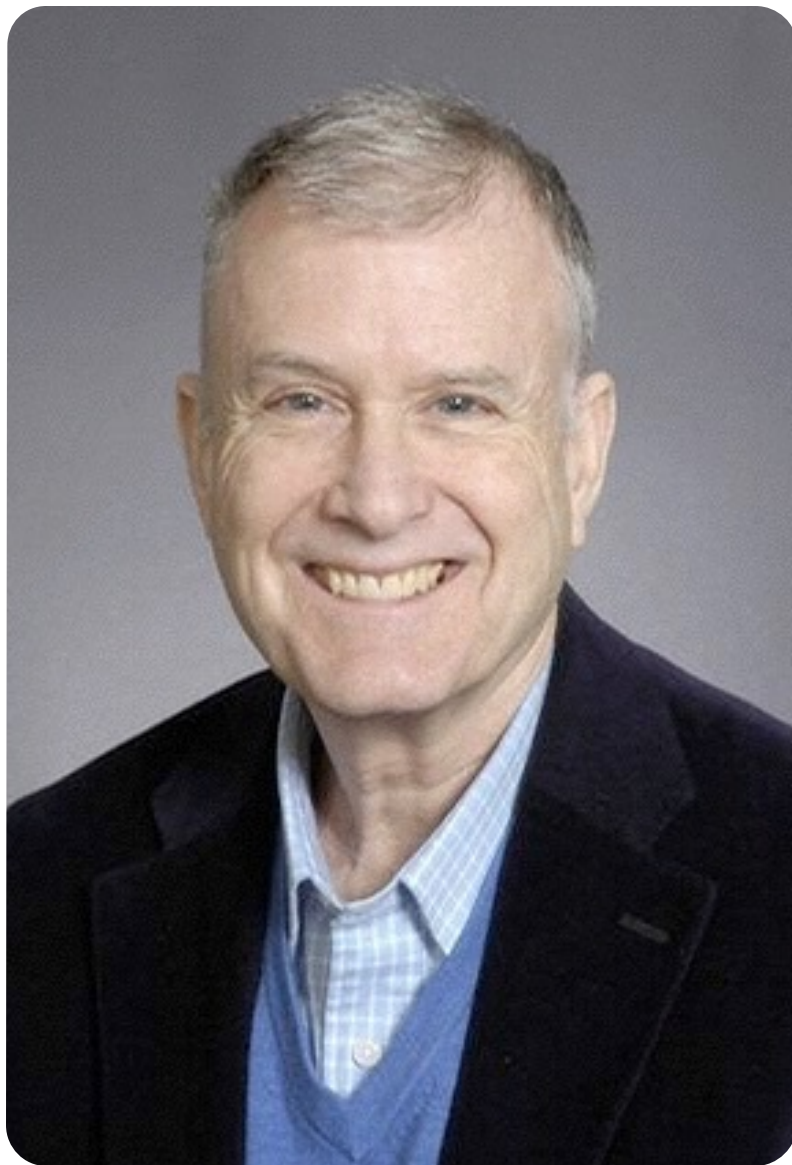


Richard Nisbett

Changes in “Statistical and Methodological Reasoning” across three years of graduate school in Michigan

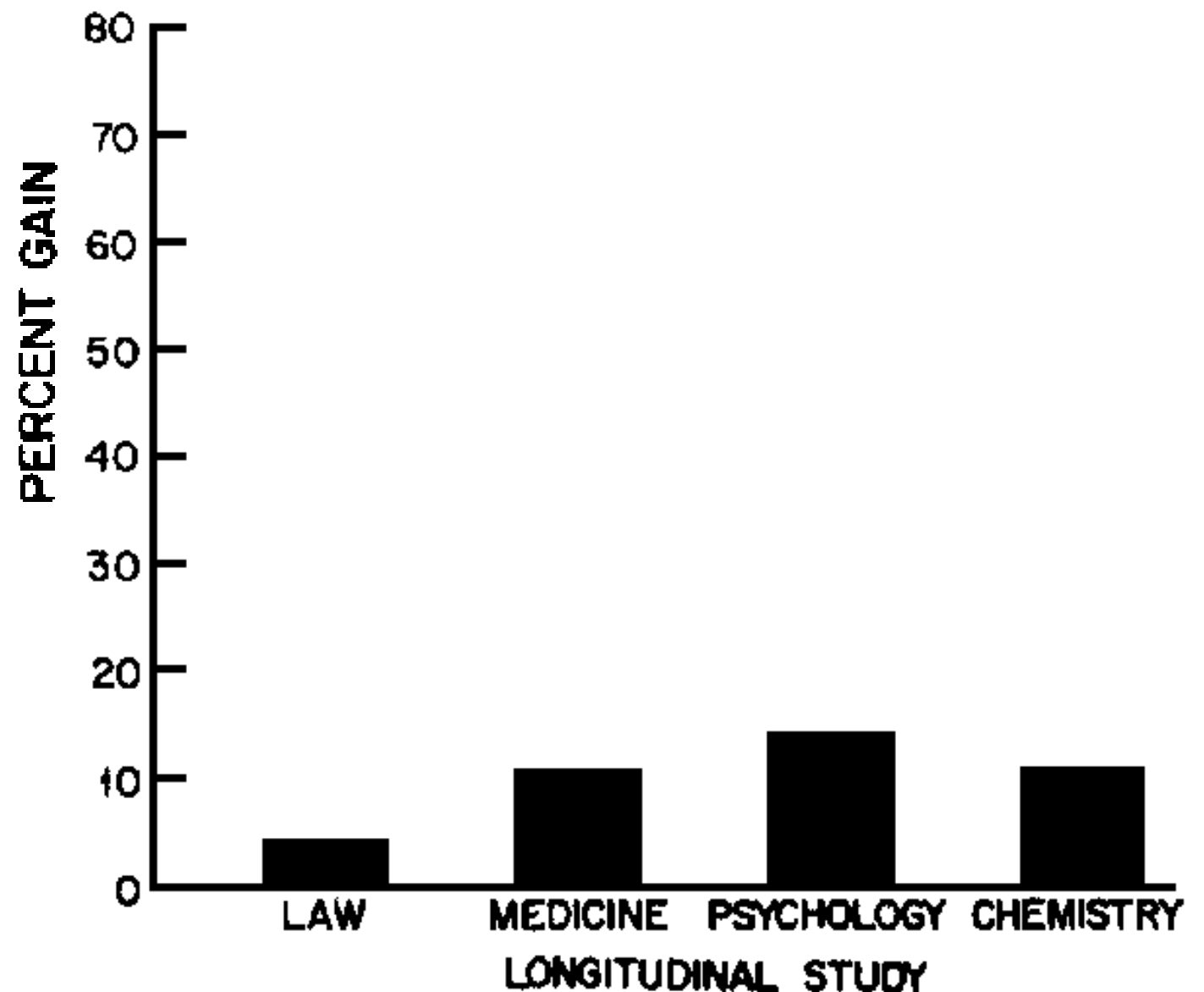


Not the Case for Deductive Logic



Richard Nisbett

Changes in “Verbal Reasoning” across three years of graduate school in Michigan



Not the Case for Deductive Logic

Patricia Cheng even showed that studying a full course in formal logic doesn't improve one's abilities to tackle logic tasks.

(there may be methodological issues with this... see Attridge, Aberdeen & Inglis, in press)

COGNITIVE PSYCHOLOGY 18, 293–328 (1986)

Pragmatic versus Syntactic Approaches to Training Deductive Reasoning

PATRICIA W. CHENG

Carnegie-Mellon University

KEITH J. HOLYOAK

University of Michigan

AND

RICHARD E. NISBETT AND LINDSAY M. OLIVER

University of Michigan

Two views have dominated theories of deductive reasoning. One is the view that people reason using syntactic, domain-independent rules of logic, and the other is the view that people use domain-specific knowledge. In contrast with both of these views, we present evidence that people often reason using a type of

LETTERS

Putting brain training to the test

Adrian M. Owen¹, Adam Hampshire¹, Jessica A. Grahn¹, Robert Stenton², Said Dajani², Alistair S. Burns³, Robert J. Howard² & Clive G. Ballard²

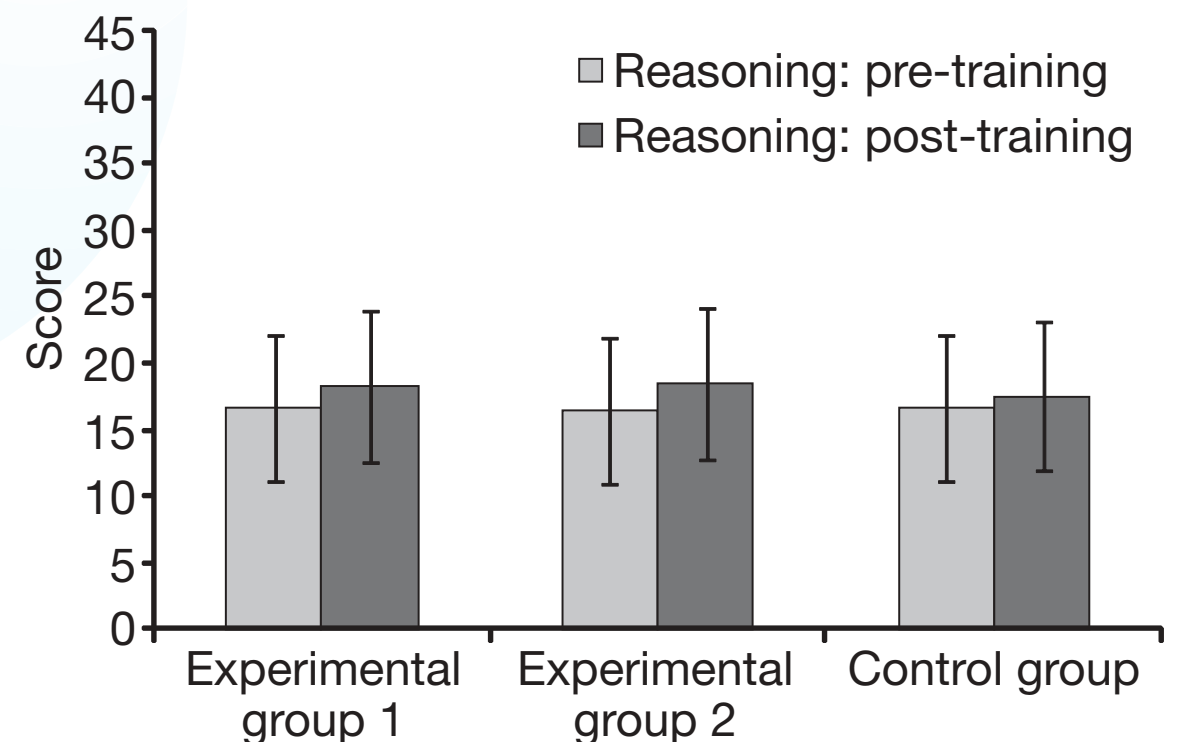
‘Brain training’, or the goal of improved cognitive function through the regular use of computerized tests, is a multimillion-pound industry¹, yet in our view scientific evidence to support its efficacy is lacking. Modest effects have been reported in some studies of older individuals^{2,3} and preschool children⁴, and video-game players outperform non-players on some tests of

broader range of cognitive functions was trained using tests of short-term memory, attention, visuospatial processing and mathematics similar to those commonly found in commercially available brain-training devices. The difficulty of the training tasks increased as the participants improved to continuously challenge their cognitive performance and maximize any benefits of training. The control group

Collaboration with BBC’s
“Bang Goes the Theory”

$N = 11,430$

Used ‘brain training’ for
six weeks.



Background Summary

1. Overwhelming view among mathematicians and policy-makers is that studying mathematics **causally** develops general reasoning skills.
2. Overwhelming view among psychologists is that it does not (or, if you're Nisbett, that it does not develop *logical* reasoning skills, but might develop other non-logical reasoning skills).
3. Very little direct empirical evidence either way.

Background Summary

- This situation is a bit of a mess.
- Clearly unsatisfactory that important educational policy decisions are being made on anecdotal evidence.
- Main goal of the Fellowship, funded by the Worshipful Company of Actuaries via the Royal Society, was to provide some compelling evidence either way.

Research Strategy

1. How can we measure reasoning performance?
2. Do mathematicians “reason differently” to non-mathematicians?
3. Are such differences developmental?
4. Does the curriculum matter?



**How can we measure
reasoning performance?**

Measuring Reasoning

- What reasoning skills do TFD proponents think studying mathematics develops?
- When asked, people say things like “logic, critical thinking, problem solving...”
- But I wanted to pin them down to making specific predictions.
- First I conducted a literature review to identify tasks that seem to be related to the kinds of skills Plato and Vorderman talk about.

Measuring Reasoning

I interviewed a series of “stakeholders” to ask them their views:

- Presidents of learned societies;
- MPs associated with education;
- Mathematicians involved in influencing curriculum development;

I showed them a series of reasoning tasks and asked them to predict the extent to which studying mathematics would help.

I insisted they made specific predictions (1-5 scale).

Measuring Reasoning

Task	Median
Argument Evaluation Task	4
Belief Bias Syllogism Task	5
Cognitive Reflection Task	4
Conditional Inference Task	5
Evaluation of Arguments	3.5
Interpretation of Arguments	4
Recognition of Assumptions	4
Estimation	4.5
Insight Problem Solving	2
Statistical Reasoning	4
Wason THOG Task (disjunctive reasoning)	4
Wason Selection Task (conditional reasoning)	5
Ravens' Matrices (intelligence)	4

Measuring Reasoning

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Wason THOG Task (disjunctive reasoning)	4
Wason Selection Task (conditional reasoning)	5
Ravens' Matrices (intelligence)	4

Conditional Inference Task

This problem concerns an imaginary letter-number pair. Your task is to decide whether or not the conclusion *necessarily* follows from the rule and the premise.

Rule: If the letter is not T then the number is 6.

Premise: The number is not 6.

Conclusion: The letter is T.

☐ YES (it follows) ☐ NO (no, it does not follow)

Denial of the Antecedent,
Affirmation of the Consequent

Modus Ponens,
Modus Tollens

Table 1 The four conditional types and four inference types used in the study

Conditional	MP		DA		AC		MT	
	Pr	Con	Pr	Con	Pr	Con	Pr	Con
if p then q	p	q	$\neg p$	$\neg q$	q	p	$\neg q$	$\neg p$
if p then $\neg q$	p	$\neg q$	$\neg p$	q	$\neg q$	p	q	$\neg p$
if $\neg p$ then q	$\neg p$	q	p	$\neg q$	q	$\neg p$	$\neg q$	p
if $\neg p$ then $\neg q$	$\neg p$	$\neg q$	p	q	$\neg q$	$\neg p$	q	p
Inference-type	Affirmative		Denial		Affirmative		Denial	
Validity	Valid		Invalid		Invalid		Valid	

Conditional Inference Task

If you are a good lecturer
then you will get good
student feedback.

Suppose I get good student
feedback.

Does this mean I am a good
lecturer?

Absolutely not, I might just be
good at telling jokes, or
setting easy examinations.

any letter-number pair. Your task is to decide whether or not the
in the rule and the premise.

the number is 6.

(it does not follow)

Denial of the Antecedent,
Affirmation of the Consequent

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if p then $\neg q$	p	$\neg q$	$\neg p$	q	$\neg q$	p	q	$\neg p$
if $\neg p$ then q	$\neg p$	q	p	$\neg q$	q	$\neg p$	$\neg q$	p
if $\neg p$ then $\neg q$	$\neg p$	$\neg q$	p	q	$\neg q$	$\neg p$	q	p
Inference-type	Affirmative		Denial		Affirmative		Denial	
Validity	Valid		Invalid		Invalid		Valid	

Conditional Inference Task

Normative model, as
taught in logic courses

Four “typical” ways of interpreting an “if p then q ” statement:

1. Material conditional (q or not- p)
2. Defective conditional (irrelevant unless p)
3. Biconditional (p if and only if q)
4. Conjunctive conditional (p and q)

Unfortunate terminology
(from Peter Wason):
not a stupid way of
thinking at all.

Material v Defective

- The difference between the material and defective conditionals is about the MT inference.
- ‘if p then q ’ interpreted materially allows you to conclude not- p from not- q .
- ‘if p then q ’ interpreted defectively does not allow this (as there is no p , the conditional is irrelevant, so the only premise you have is not- q).

(Although: it is possible to draw MT if you have a defective conditional and sufficient Working Memory capacity to construct a mini contradiction proof: evidence suggests few people in this category).

Material v Defective

- The difference between material and defective conditional
- ‘if p then q ’ in a material conditional allows you to conclude not- p if you know q
- ‘if p then q ’ in a defective conditional does not allow you to conclude not- p if you know q (as there is no inference from q to not- p so the only possible inference is from not- q to not- p).

(Although: it is possible to have a material and sufficient Working proof: evidence suggests that the material conditional is irrelevant, not- q).

Defective Conditional:

“If good lecturer then good student feedback” only adds information if we know I’m a good lecturer.

In the case where I’m not, the conditional adds *no extra information*.

Material Conditional:

“Bad feedback” and “if good lecturer then good feedback” allows us to directly conclude “not good lecturer”

material and defective conditional. The material conditional allows you to conclude not- p if you know q .

The defective conditional does not allow you to conclude not- p if you know q .

The material conditional is irrelevant, not- q).

The defective conditional does not allow you to conclude not- p if you know q (as there is no inference from q to not- p so the only possible inference is from not- q to not- p).

Conditional Inference Task

The conditional you adopt influences the validity of the four inferences:

Conditional	MP	DA	AC	MT
Material	Valid	Invalid	Invalid	Valid
Defective	Valid	Invalid	Invalid	Invalid*
Biconditional	Valid	Valid	Valid	Valid
Conjunctive	Valid	Invalid	Valid	Invalid

Conditional Inference Task

The conditional you adopt influences the validity of the four inferences:

Conditional

MP

DA

AC

MT

Material

Valid

Valid

Defective

Valid

Valid*

Biconditional

Valid

Valid

Valid

Valid

Conjunctive

Valid

Invalid

Valid

Invalid

By looking at which inferences are endorsed, you can work out which interpretation the person adopts

Research Strategy

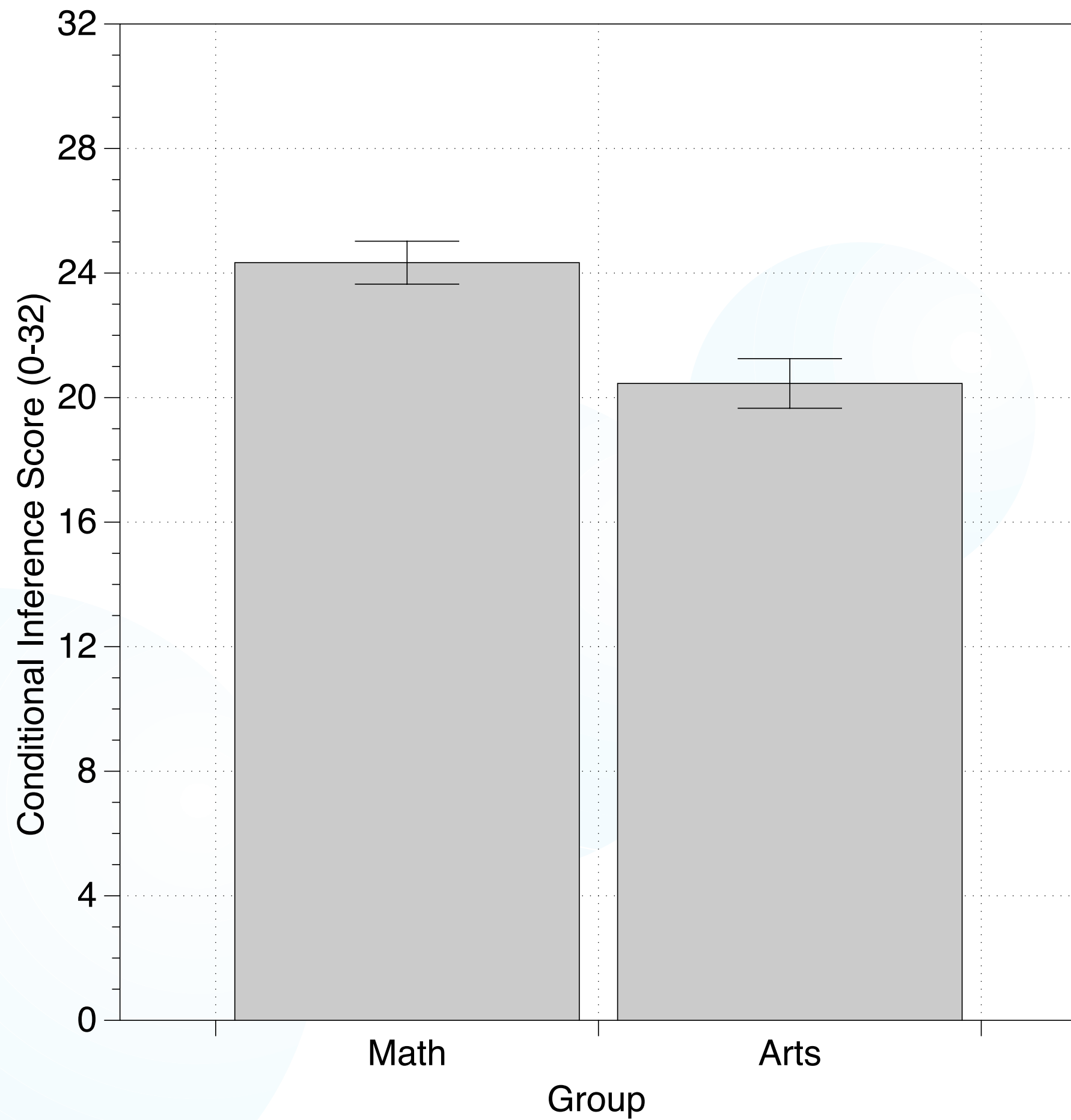
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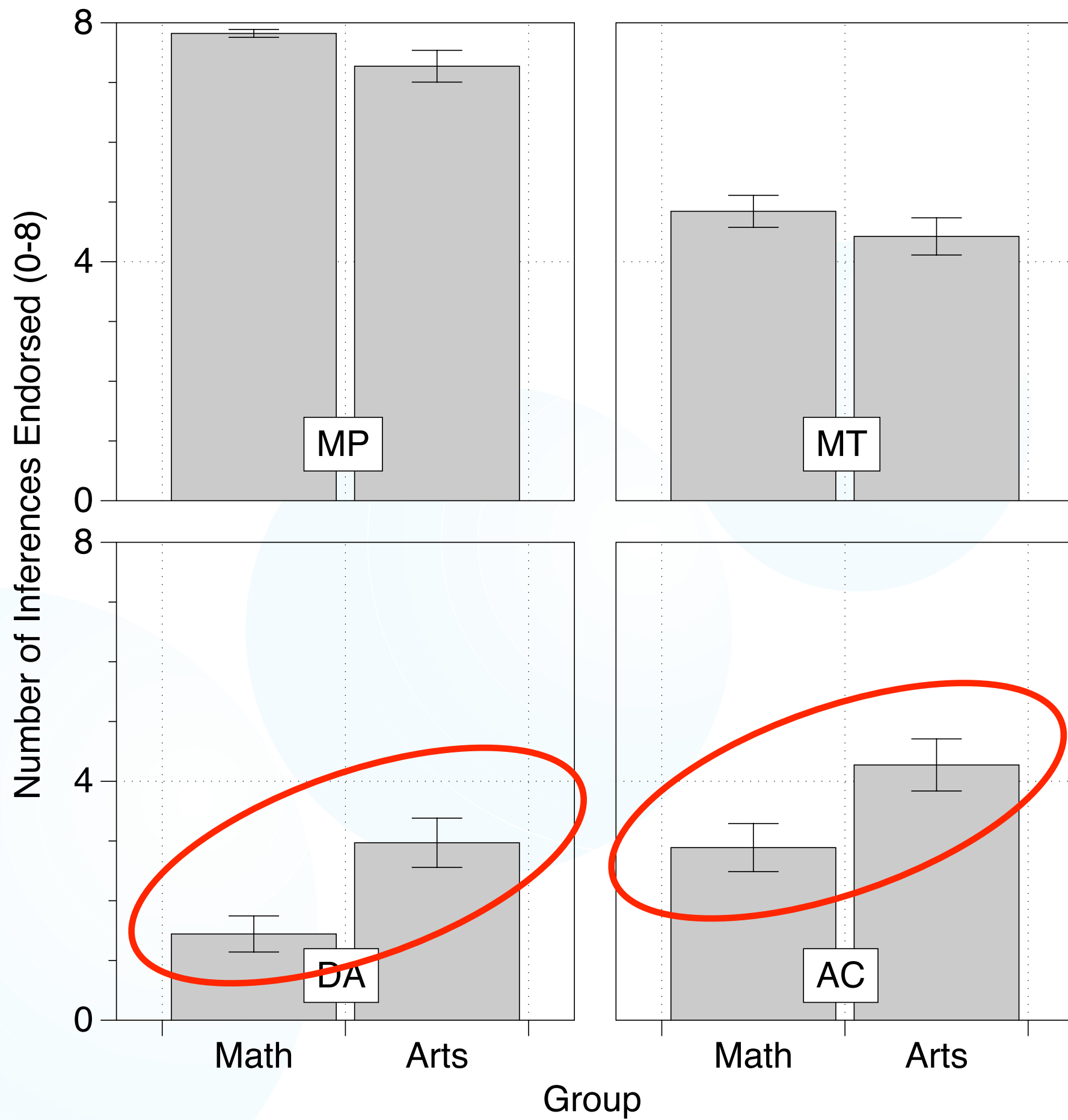
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Study 1

Study 1

- Cross-sectional comparison of first year mathematics undergraduates ($N = 44$) and first year arts undergraduates ($N = 33$) at “highly rated” UK university (high IQ sample);
- Took place in Week 1 of u/g study (no lectures yet);
- Groups matched for IQ (AH5 test);
- Used Evans’s Abstract Conditional Inference Task (Evans et al., 1996);
- Thirty two item test of abstract conditional inference.





Summary

- Maths students show an advantage on the conditional inference task prior to **any** undergraduate study;
- Not the result of differences in intelligence (groups were matched on AH5 scores);
- Advantage was uneven: came from advantage at rejecting DA and AC inferences, not from increased acceptance of MP or MT (move from biconditional to material/defective?).
- (Sort of) Consistent with predictions of Plato/Vorderman. But is it developmental?

Research Strategy

1. ~~How can we measure reasoning performance?~~
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Study 2

Study 2

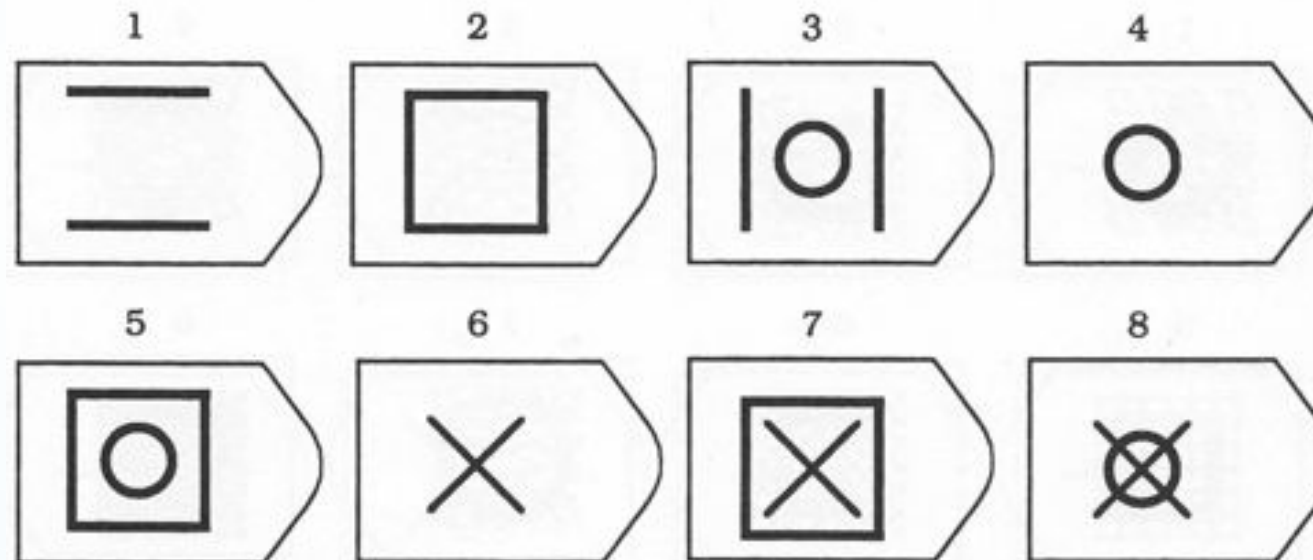
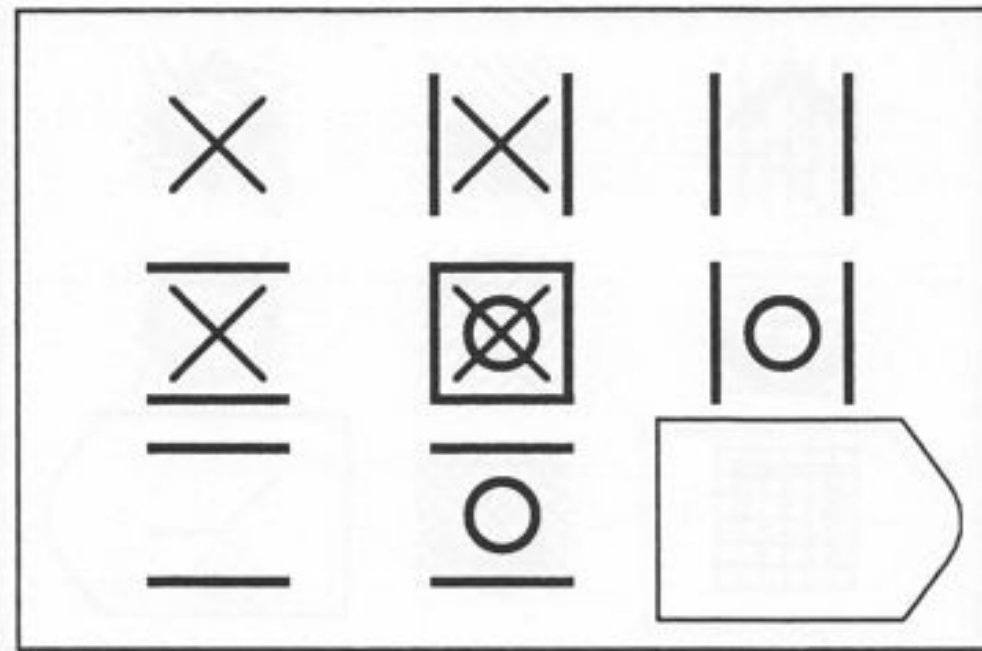
- Were the differences in Study 1 the result of **filtering** or **development**?
- Can't be filtering on intelligence (unless AH5 is a poor measure), so maybe on thinking disposition?
- Longitudinal quasi-experimental design, tracking students across AS level mathematics and AS level English literature.
- Two test points: start and end of year of study.

Study 2

Covariates:

- Raven's Intelligence Test;
- Frederick's Cognitive Reflection Test (measure of thinking disposition).

Raven's IQ Measure



Cognitive Reflection Test

- (1) A bat and a ball cost \$1.10 in total. The bat costs \$1.00 more than the ball.
How much does the ball cost? _____ cents

Study 2

Manipulation Check:

- Maths Test

When expressing $\frac{x}{(x+1)^2(x^2+2)}$ in partial fractions, the appropriate form is

(a) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$

(b) $\frac{A}{x+1} + \frac{B}{x^2+2}$

(c) $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x^2+2}$

(d) $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2}$

Study 2

Dependent Measure:

- Evans's Conditional Inference Task

If the letter is U then the number is not 9.

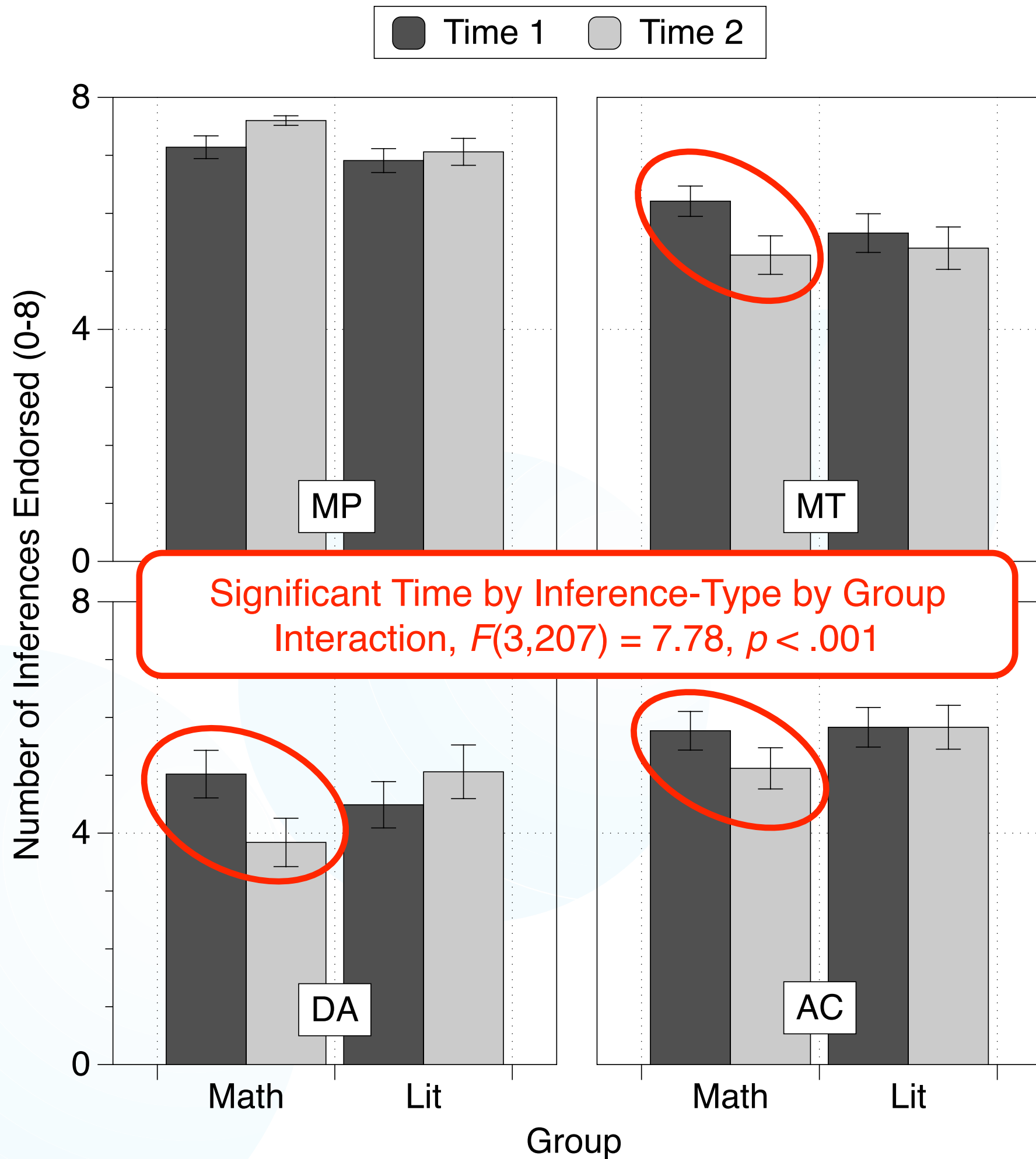
The number is 9.

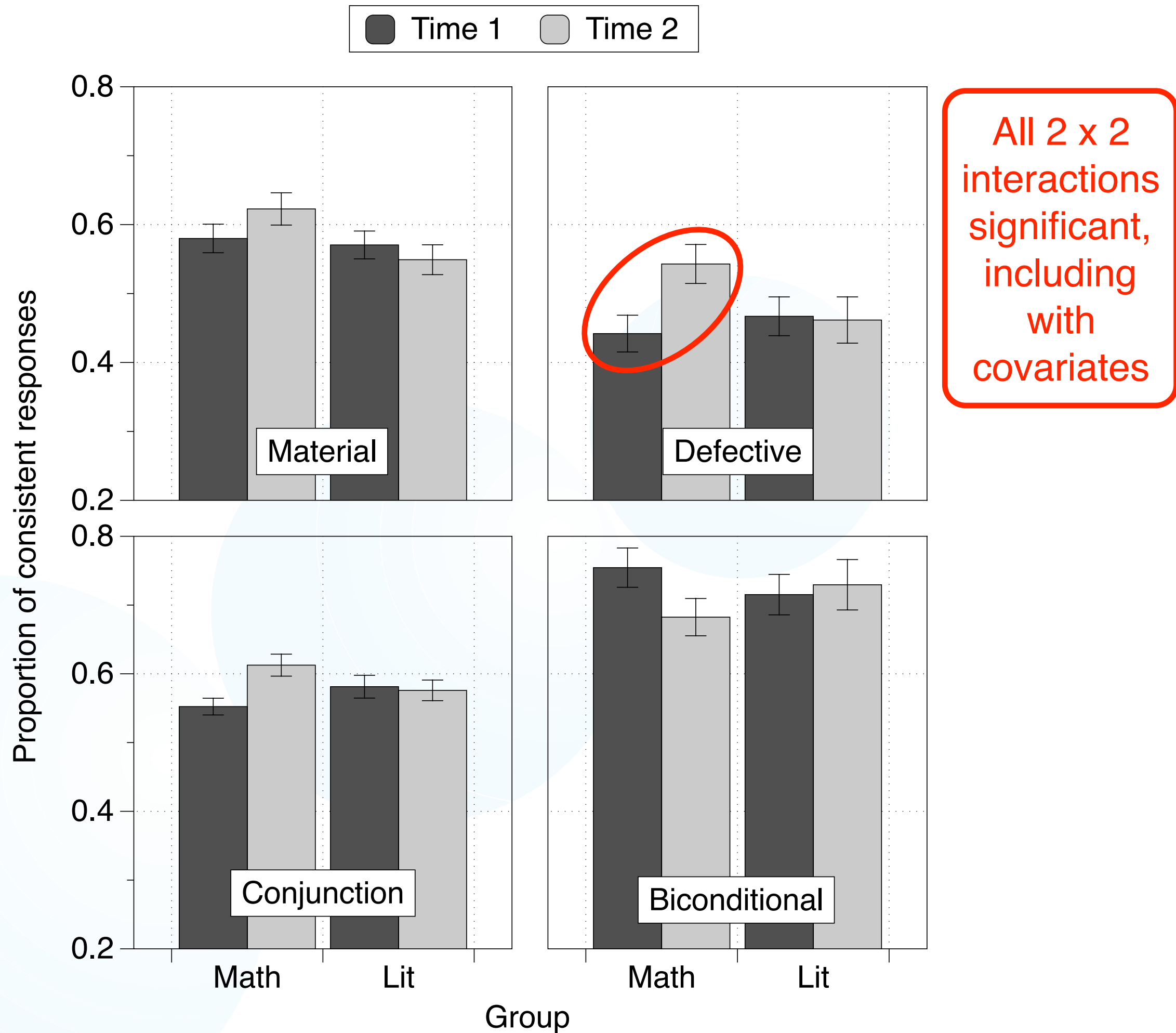
Conclusion: The letter is not U.

☐ YES

☐ NO

Study 2 Results





Causes?

If studying A Level mathematics is associated with a development towards the defective conditional interpretation, is this due to domain general changes (intelligence or thinking disposition), or domain specific experience (mathematical study)?

Ran a regression including change scores.

R^2	Predictors	Beta
.713**	Initial Defective Conditional Index	0.745**
	Initial RAPM (intelligence)	0.065
	Initial CRT (thinking disposition)	0.116
	Prior academic attainment	-0.006
	RAPM (intelligence) change	0.143
	CRT (thinking disposition) change	0.088
	Group (0 = lit, 1 = maths)	0.195*
	RAPM change x Group	0.023
	CRT change x Group	-0.091

Causes?

Apparently not due to general changes in intelligence or thinking disposition, but rather specific to mathematical study.

Obvious question: Were they simply taught how to solve such tasks during their A Level studies?

No. Two sources of evidence:

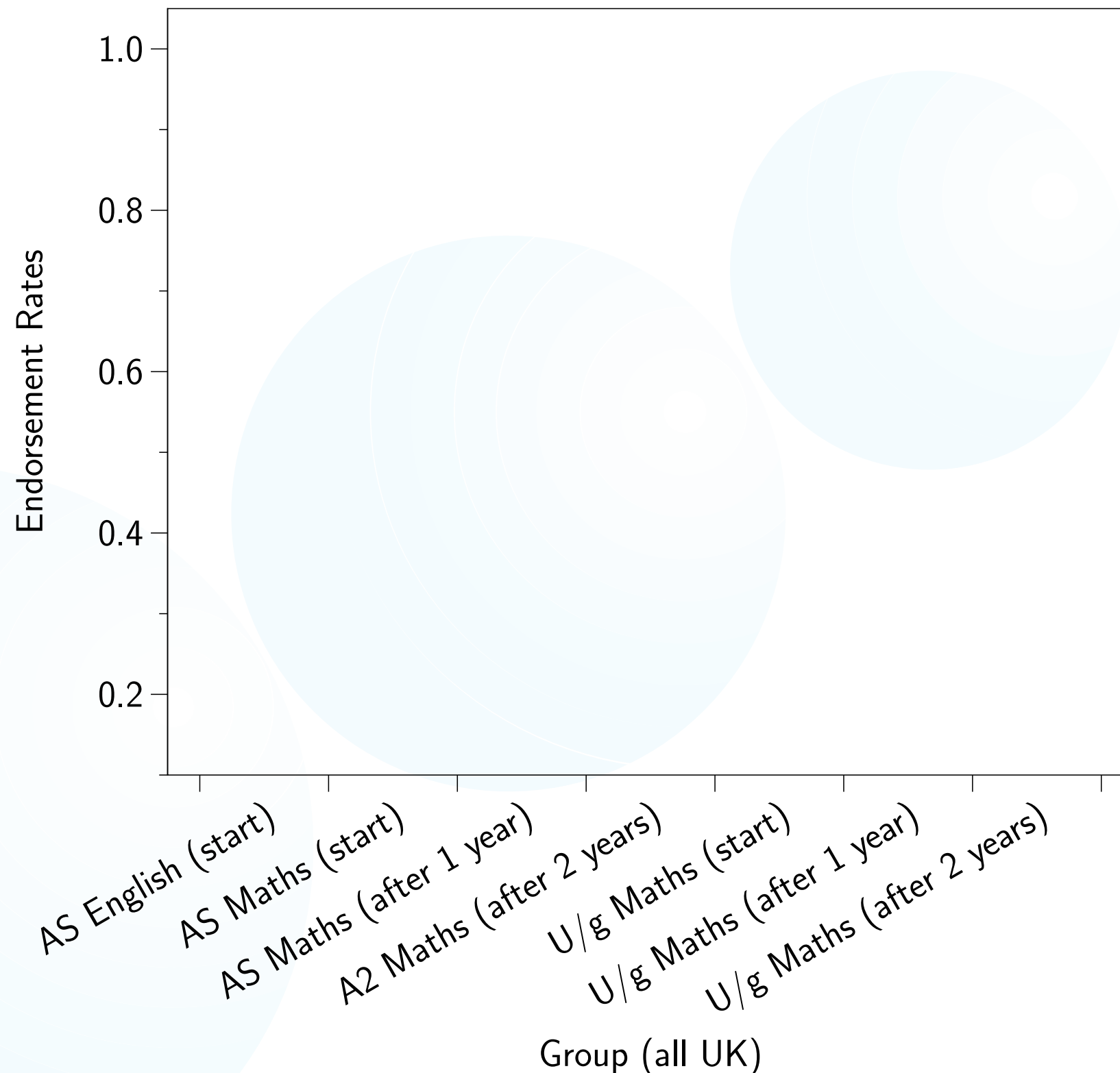
1. Not uniform “improvement” across all inference types.
2. Conditional inference is not on the syllabus, and is not examined: of 929 A Level mathematics examination questions set between 2009 and 2011, only one contained an explicit “if...then” sentence, and there were no mentions of “modus ponens”, “modus tollens” or “conditional”.

Summary

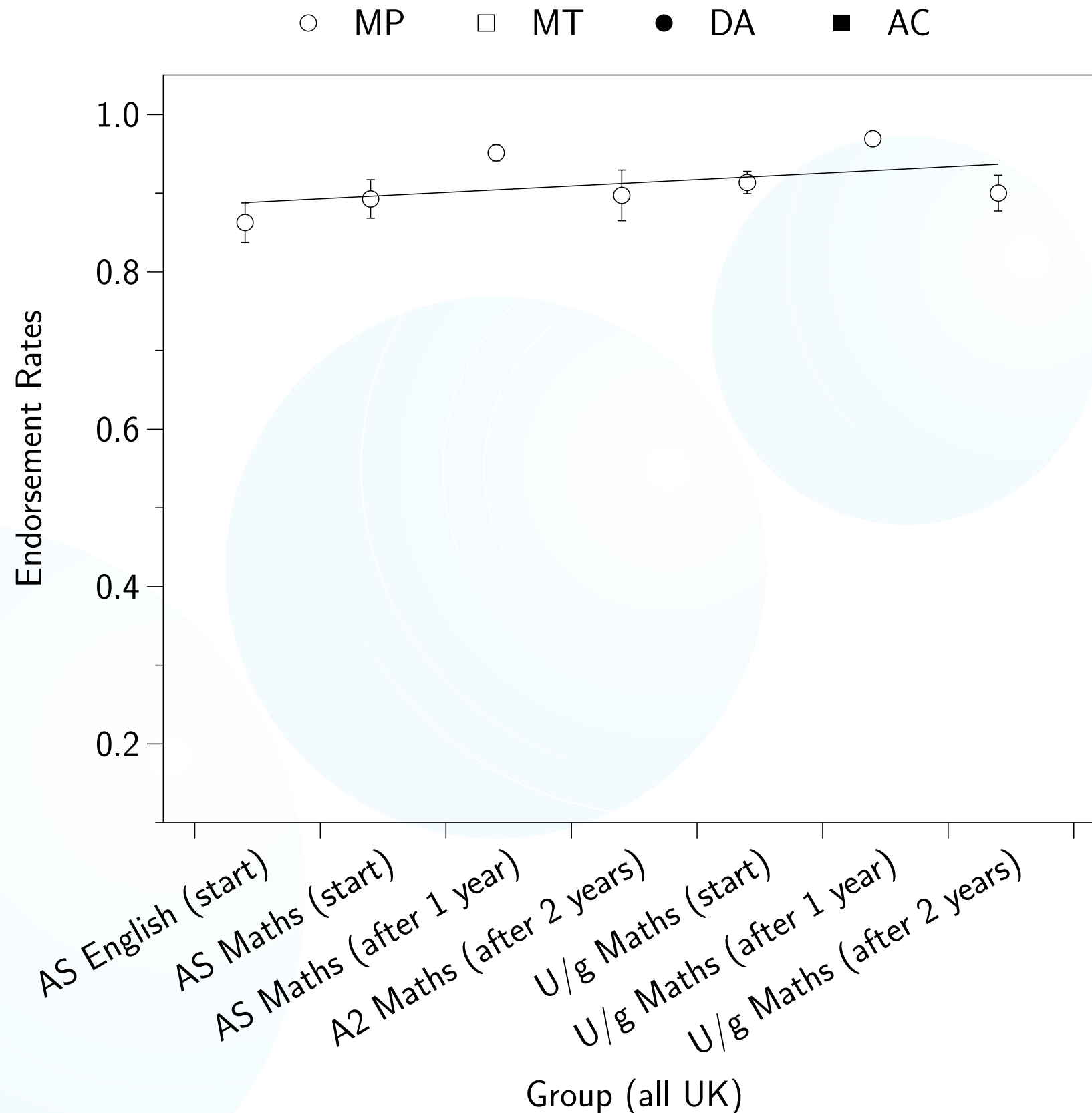
- There is an association between post-compulsory mathematical study and the development of conditional reasoning skills.
- But this appears to be towards a defective conditional interpretation rather than the normatively correct material conditional.
- **You can think about this as being increased scepticism of deductions: does studying mathematics make you better at spotting flaws in arguments?**
- Not caused by development in intelligence or thinking disposition, or by explicit curriculum content.

Summary of Lots of Similar Studies

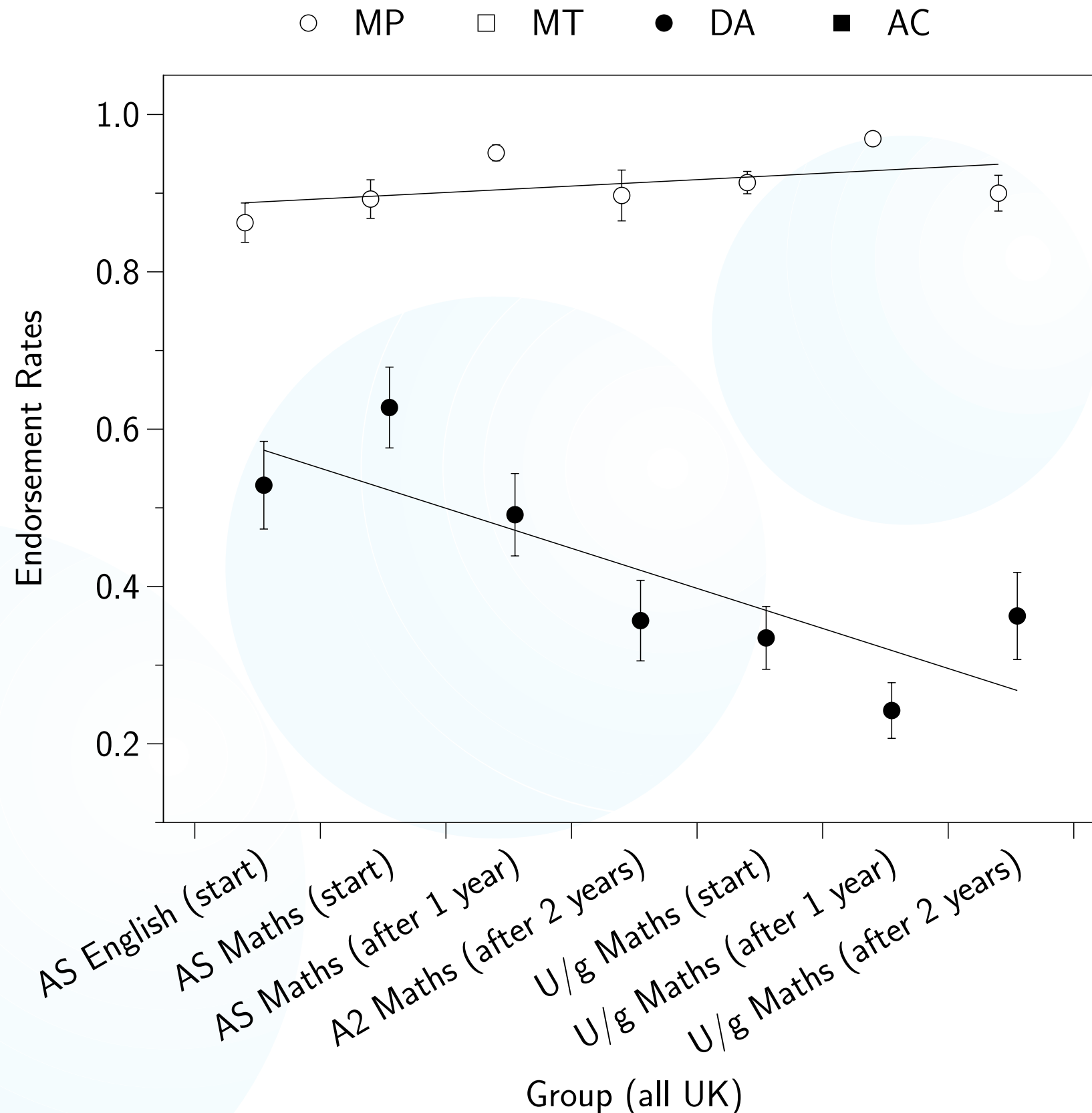
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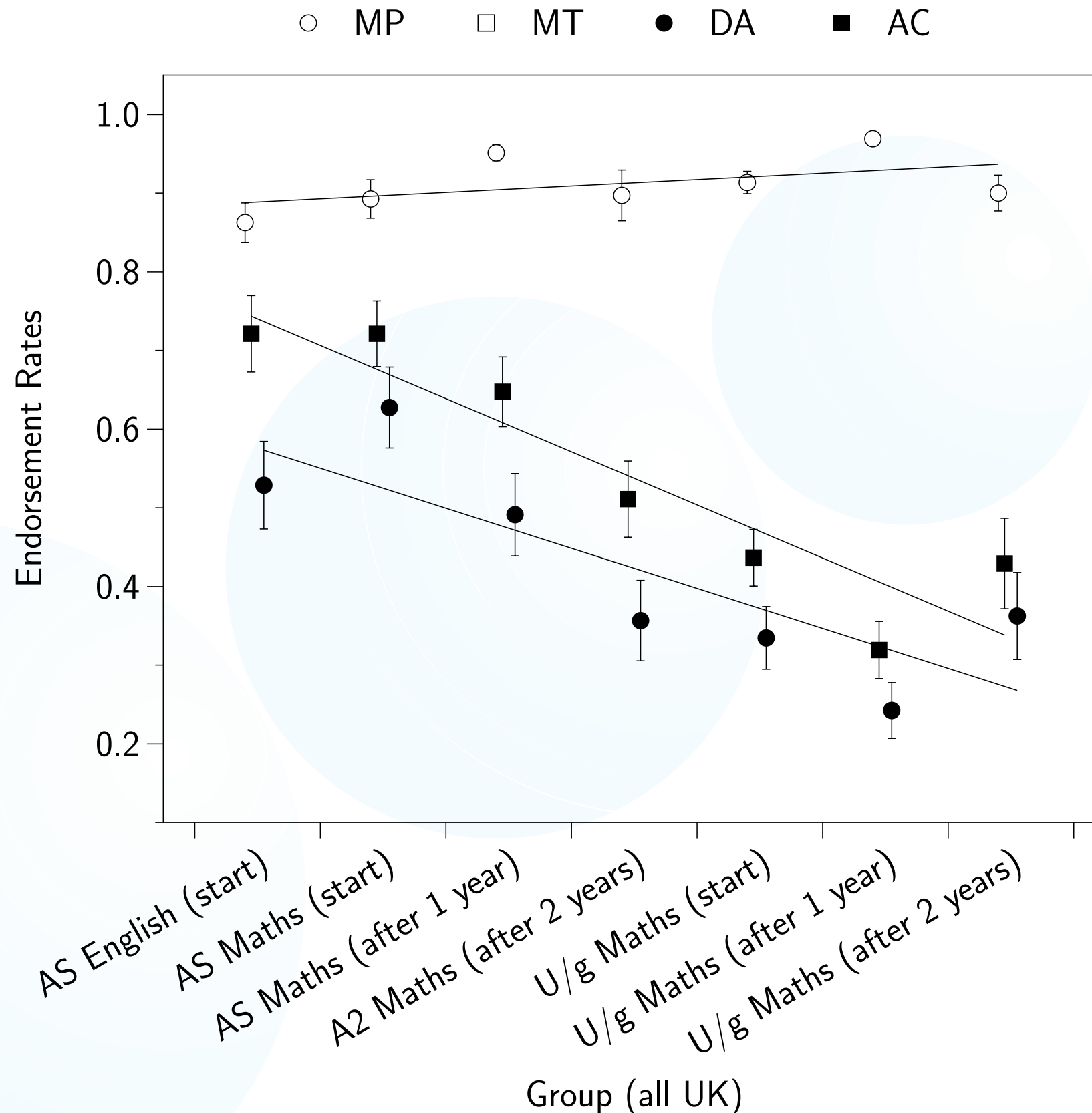
Summary of Lots of similar studies



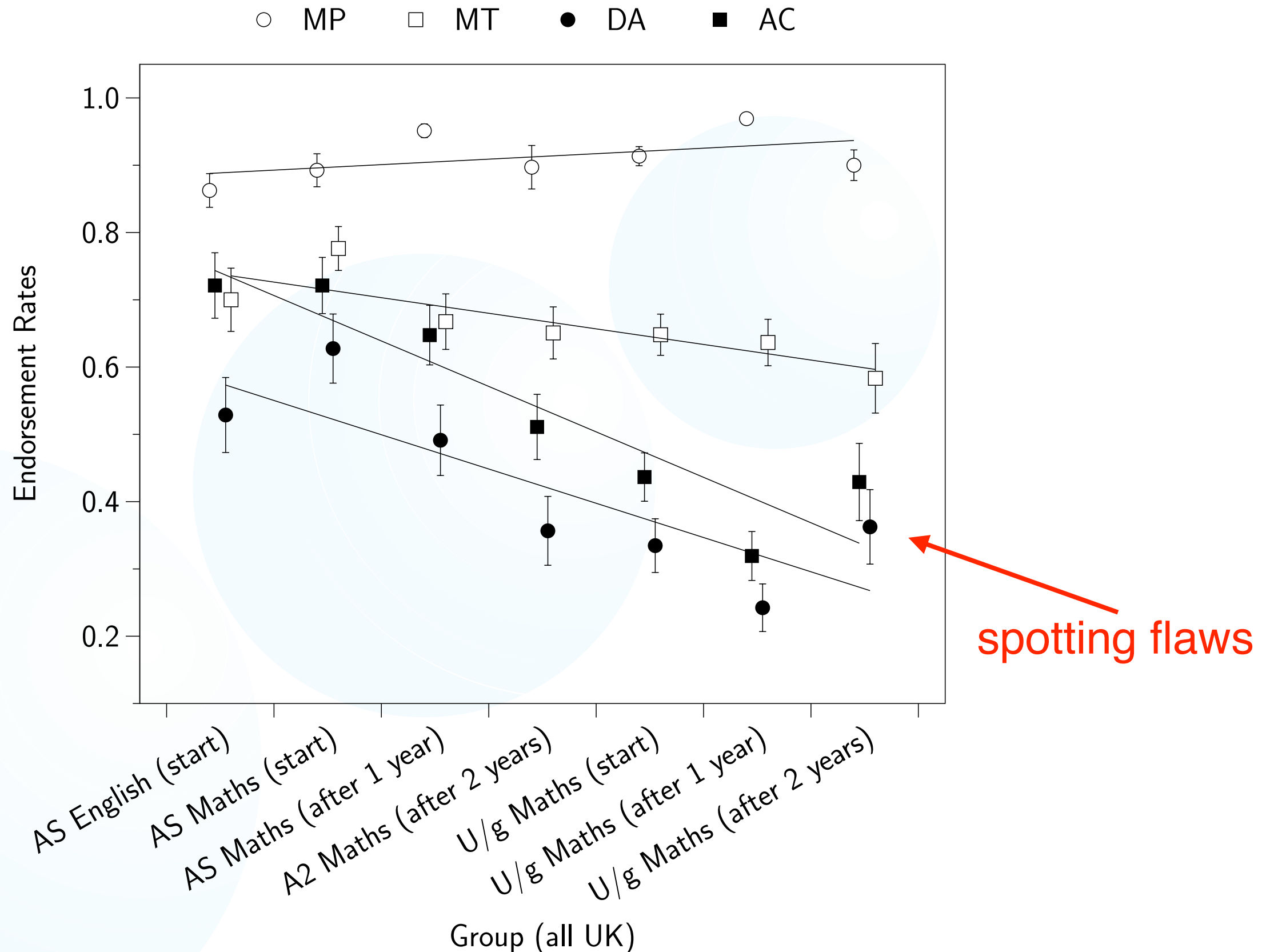
Summary of Lots of similar studies



Summary of Lots of similar studies



Summary of Lots of similar studies



Research Strategy

1. ~~How can we measure reasoning performance?~~
2. ~~Do mathematicians “reason differently” to non-mathematicians?~~
3. ~~Are such differences developmental?~~
4. Does the curriculum matter?

Cypriot Comparison

- To investigate the curriculum question, I needed to look at the same issues in a different context.
- Repeated this study in Cyprus.
- Were able to run the study over two years.
- Cypriots can study “high intensity” or “low intensity” mathematics from 16-18.
- In this sense it is a more typical country than England (Hodgen et al., 2011).

Αν πιστεύετε ότι το συμπέρασμα συνεπάγεται αναγκαία παρακαλώ βάλτε νι (ν) στο κουτί που λέει ΝΑΙ, διαφορετικά βάλτε νι (ν) στο κουτί που λέει ΟΧΙ. Μην επιστρέψετε σε κάποιο πρόβλημα εάν το έχετε τελειώσει και προχωρήσατε ήδη στο επόμενο πρόβλημα. Απαντήστε σε όλες τις ερωτήσεις.

1. Αν το γράμμα είναι το Π τότε ο αριθμός δεν είναι το 2.

Ο αριθμός είναι το 7.

Συμπέρασμα: Το γράμμα είναι το Π.

☐ ΝΑΙ

☐ ΟΧΙ

23

2. Αν το γράμμα δεν είναι το Α τότε ο αριθμός δεν είναι το 1.

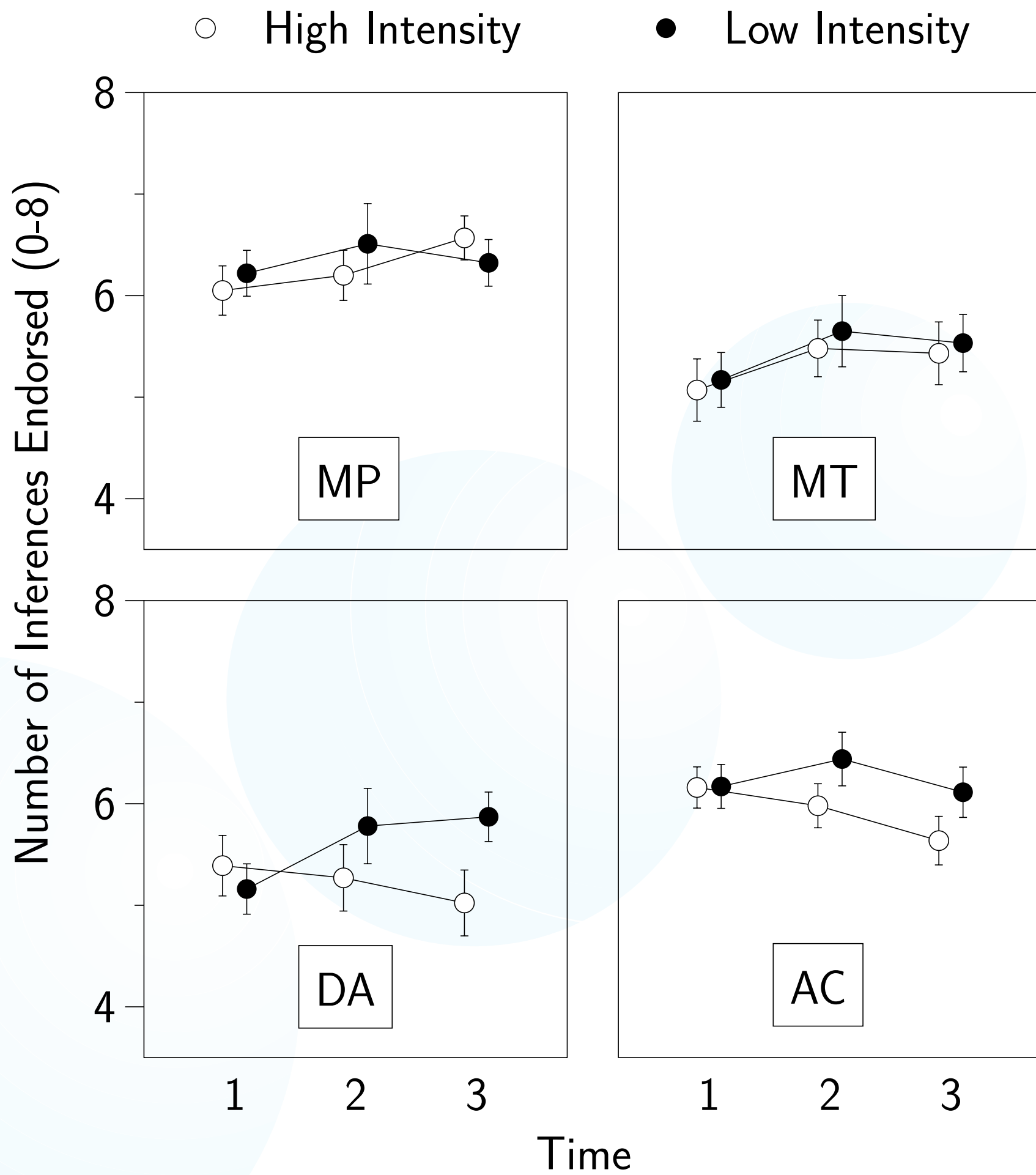
Το γράμμα είναι το Ν.

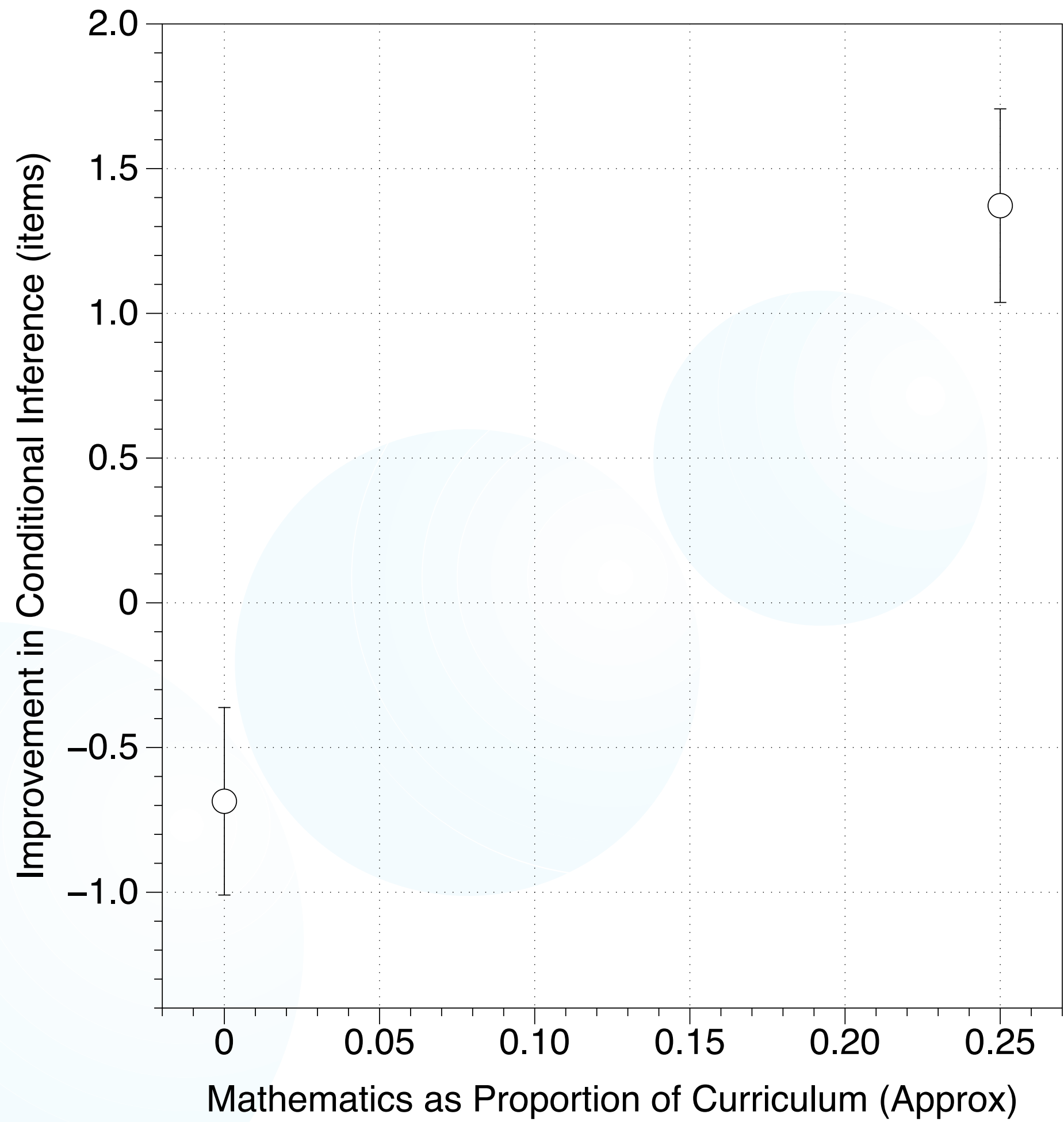
Συμπέρασμα: Ο αριθμός δεν είναι το 1.

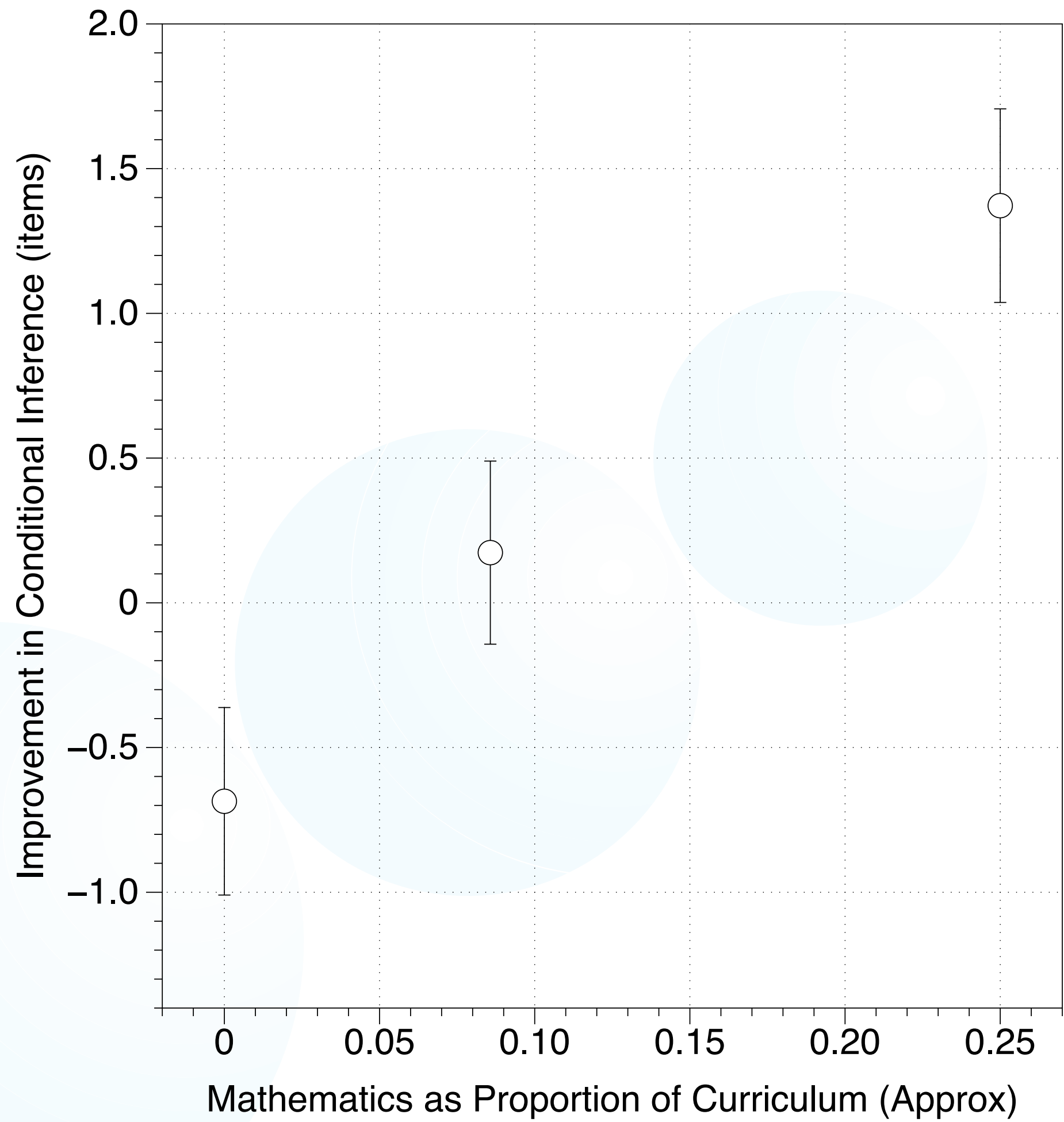
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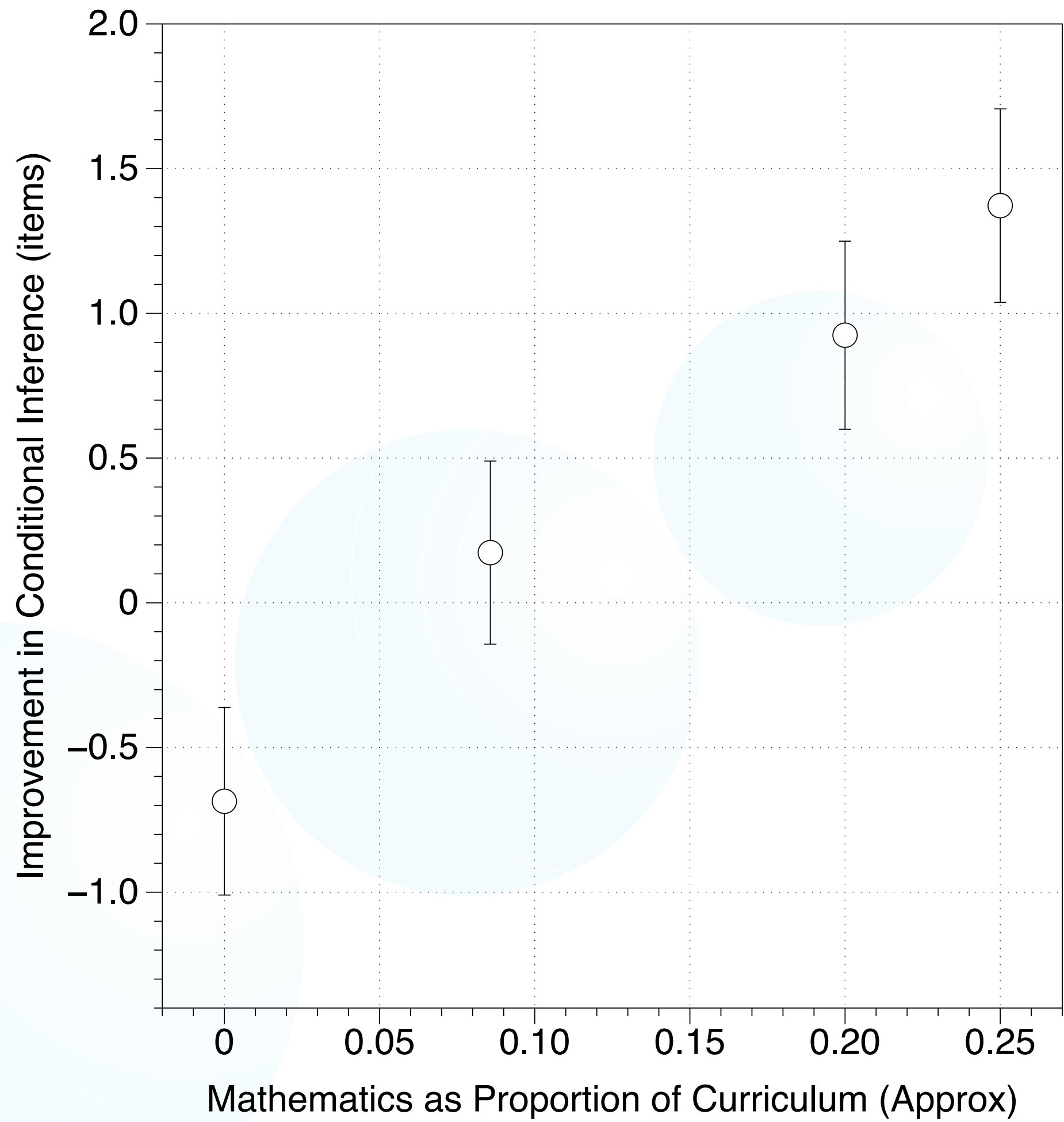
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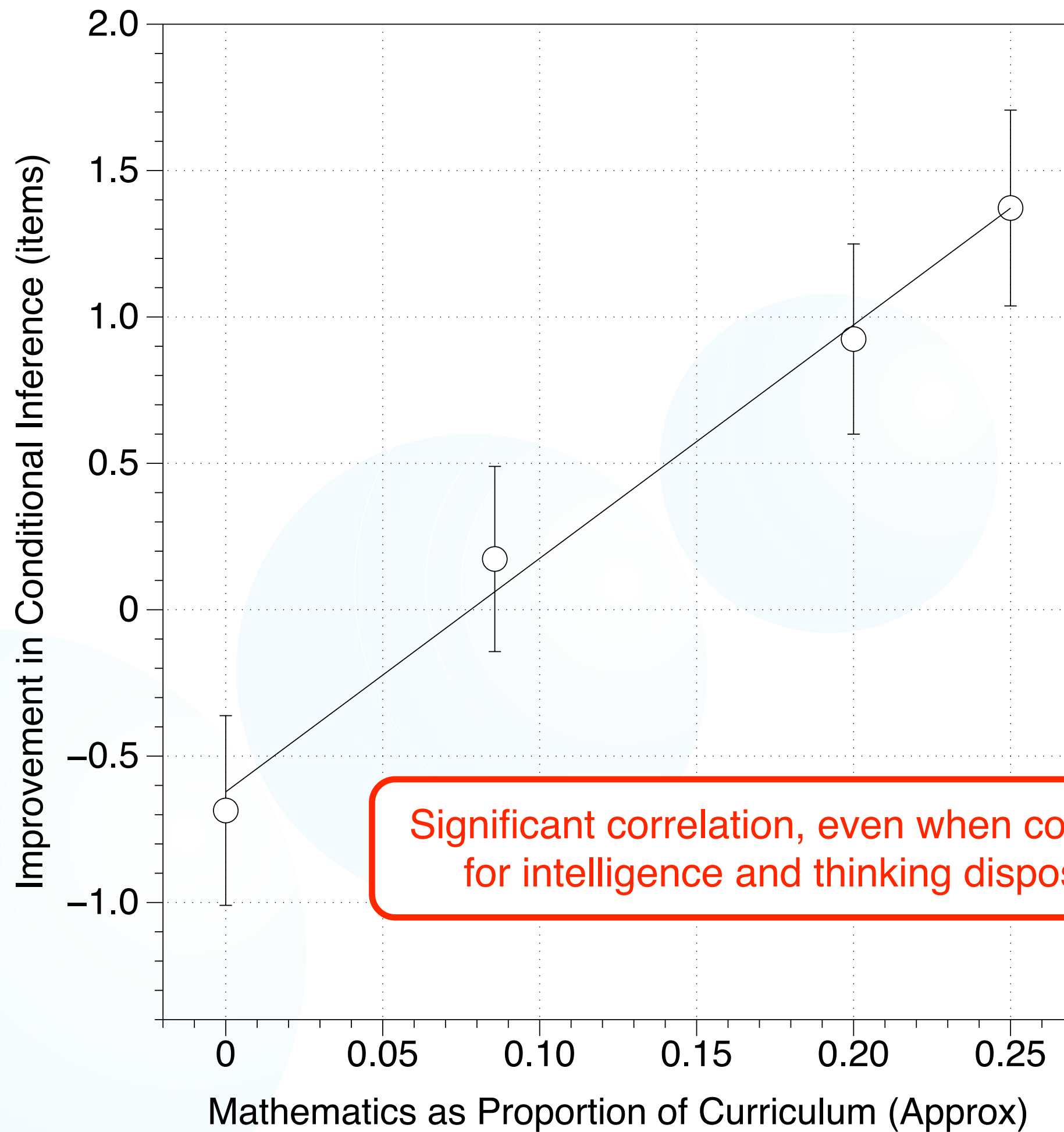
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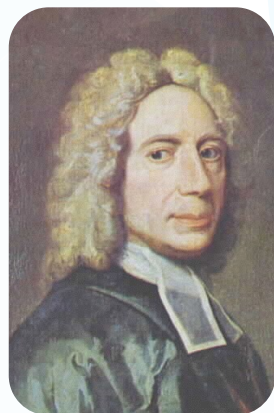
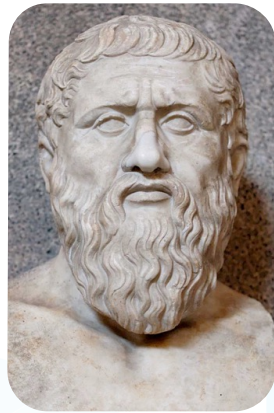


Summary

- It seems that studying mathematics may be associated with the development of a defective conditional, at least for abstract “if p then q ” statements, and the reduced influence of the biconditional.
- Good news for Plato/Vorderman: inconsistent with Thorndike, Piaget, Newell etc.

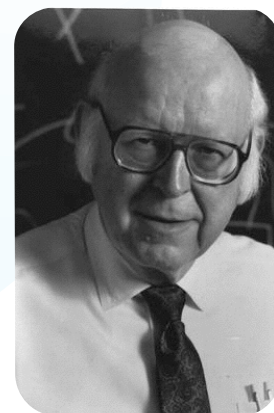
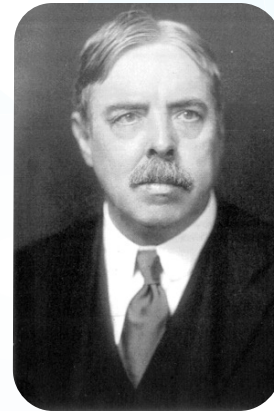
Summary

There is an fundamental (but under-debated) disagreement between people who claim that studying mathematics develops reasoning skills, and those who don't.

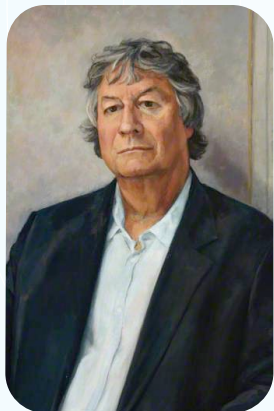
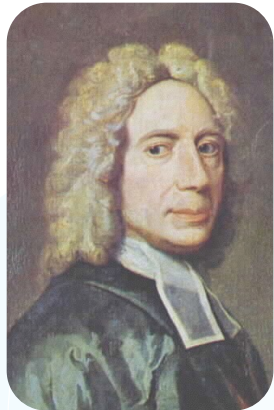
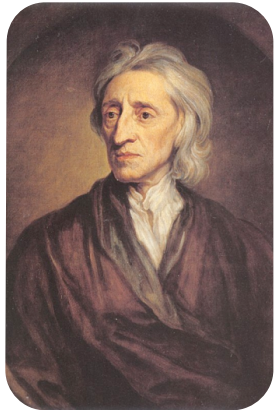
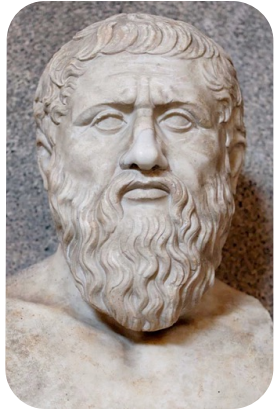


Plato, John Locke,
Isaac Watts, Adrian Smith

v

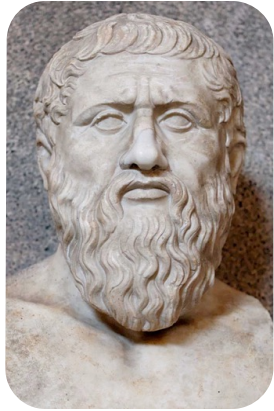


Edward Thorndike, Jean Piaget,
Alan Newell, William James

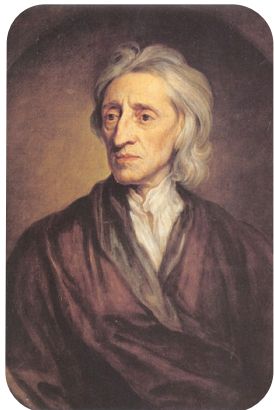


Summary

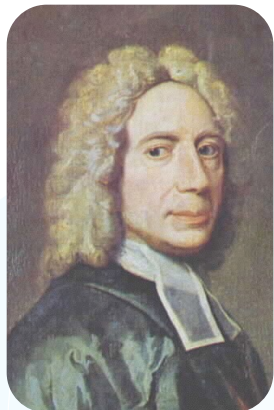
- These data are consistent with the suggestion that mathematics **is** associated with the development of conditional reasoning skills.
- Using modern psychology of reasoning measures allows for a more sensitive design than Thorndike's (1924) study.
- However: the development appears not to be towards the normative model of the conditional, but towards the defective conditional.
- Can conceptualise this as a tendency to be more sceptical of deductions than the general population.



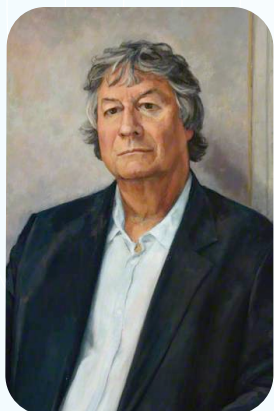
Summary



Was Plato right?



I think so: but it's a bit more nuanced than he thought.



Acknowledgements

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(Warwick)

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Worshipful Company
of Actuaries

