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Dynamic Asset Allocation to Hedge Cash Guarantees

using

Revised Option Based Portfolio Insurance

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ABSTRACT

This paper introduces a dynamic asset allocation for life funds which aims to maximise equity exposure and to provide a safety net in adverse market conditions.

Unlike other papers on dynamic hedging, this analysis assumes there is no extra funding to support it. This assumption leads to a revised formulation of the hedging technique using options, usually called option based portfolio insurance (OBPI), to achieve a self-funded market protection. The adjustment to the OBPI is inspired by investment banking techniques used to construct the asset backing the Guaranteed Equity Bonds (GEBs) taking into account the market price of the matching assets.

The matching asset backing a GEB is a zero coupon bond and a proportion of a Call option strategy. The adjusted OBPI clearly identifies the nature, size and characteristics of the option providing financial protection to be considered in the fund. Decomposing Call option strategies into equity holdings and cash bond borrowing leads to a dynamic investment strategy aiming to achieve similar capital protection to GEBs. The proposed formula for the equity exposure in this investment strategy is very similar to the Black & Scholes Call option formula.

In the context of managed funds, the equity backing ratio (EBR) could be managed dynamically to minimise the cost of the guarantees and to maximise the equity exposure. The key drivers of the recommended EBR are the level of the guarantees, the duration of the liability, equity volatility and risk free rates. The dynamic asset allocation is extended to with-profits business by deriving the EBR from the level of the asset shares, the Bonus Reserve Valuation (BRV), the duration of the guarantees and equity volatility.

The paper concludes that the theoretical justification of any asset allocation is to maximise the risky assets exposure and to achieve portfolio insurance targeting a certain level of capital protection. Targeting a constant level of protection requires the adoption of this dynamic asset allocation as investment strategy. In the context of the investment theory, this paper proposes a new asset allocation methodology based on the option pricing theory.

KEYWORDS

Option Based Portfolio Insurance, Guaranteed Equity Bond, Replicating portfolio, Delta hedging, Cash guarantees; Call option; Call spread strategy, Black & Scholes model; Equity backing ratio.

Dynamic asset allocation to hedge cash guarantees using revised option based portfolio insurance

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1 Introduction

Background

Life offices wanting to hedge market risk exposure arising from maturity guarantees might buy matching assets to meet their liabilities. A Guaranteed Equity Bond (GEB) is a good example, where the exact policy benefits are backed by matching assets provided by an investment bank. Alternatively, a life office might use a delta hedging technique to replicate the matching asset in-house and minimise the cost of the guarantees. The dynamic asset allocation is obtained by substituting the replicating portfolio to the option-based derivatives involved in the matching assets backing the GEB.

The dynamic asset allocation aims to provide equity exposure with downside protection by actively managing the EBR. The theoretical development of dynamic asset allocation is discussed in detail, along with the practical issues of its implementation, in the context of the life assurance business.

In terms of life funds with embedded guarantees, the dynamic asset allocation has two objectives:

- minimise the cost of the guarantees
- maximise the level of equity exposure

The asset allocation in this paper could be seen as a broader investment methodology filling the gap in the portfolio theory and ALM techniques on how to derive the risky assets exposure from a targeted minimum return and option replication technique.

Paper layout

The dynamic asset allocation will be constructed by means of the following steps:

- illustrate the asset structure backing GEBs
- revision and generalisation of the OBPI
- conversion of the revised OBPI to a dynamic asset allocation by replacing the option-based derivatives by the replicating portfolio

The paper starts with a brief description of the asset structure backing GEBs, supported by numerical illustrations.

Section 3 covers the revision of OBPI based on both the Call and the Put approach and proposes a general formulation based on a ZC bond and a Call spread strategy.

Section 4 shows how to construct the replicating portfolio of the Call option strategy using delta hedging techniques and the B&S Call option formula.

Section 5 describes the theoretical framework of the dynamic asset allocation concluding that any asset allocation could be expressed as an OBPI using a replicated Call option.

Section 6 discusses how the theoretical EBR of with-profits funds could be derived from the asset shares and the bonus reserve valuation.

The paper concludes with an illustration of how dynamic asset allocations may work in practice, when used to provide a downside protection of a unit-linked contract.

2 Guaranteed Equity Bonds

2.1 Concept

Guaranteed Equity Bonds provide exposure to equity performance by linking policy benefits to an equity index, while offering a downside protection. Investment banks provide the asset backing the new generation of the GEBs (offered in the UK since 1989). The matching asset backing a typical GEB is a combination of a zero coupon (ZC) bond and a proportion of Call option on the FTSE-100 index. This has been a popular asset structure backing GEBs since it is possible to define more flexible and innovative protected equity exposure, with reduced administration costs.

In the early years of these products, the participation in equity growth was very attractive. This was due to high interest rates and low equity volatility. In market conditions where equity volatility is high and interest rates are low, GEBs offer a modest participation in the equity growth. Investment banks have been very creative in finding features to reduce the equity volatility in order to offer a higher participation in the equity growth. The popular methods to reduce equity volatility are to link the performance to a basket of equity indices and to average out the equity returns. This produces a higher participation in the equity growth to attract more investors.

The GEB considered in this paper is also called a Growth Structured Bond within the life assurance industry. Cliquet products and high-income bonds, which are also known as structured products, will not be discussed since their payoff profile is not comparable to life funds with embedded guarantees.

The following is a description of a GEB offered by a building society in March 2004 to UK investors:

- three-year term (Bond matures on 30 April 2007)
- performance linked to the FTSE 100™ Index with 100% participation in growth
- no limit on the growth potential of your investment
- guaranteed minimum return of 5% (1.64% AER)
- no charges or management fees
- final level of the FTSE 100™ Index will be averaged out over the last six months

The detail of this investment offer is shown in Appendix A.

The asset backing this product is a ZC bond to provide the guaranteed minimum return and a Call option providing the participation to the upside return of the FTSE-100 index averaged out over the last six months. The difference between the amount invested and the cost of the matching asset provided by an investment bank, is an implicit charge paid by the policyholders.

The main features of a GEBs are:

- the term of the benefit with a specified start date and maturity date
- the guaranteed minimum return
- the level of participation in the equity growth
- the equity index (or indices) representing the equity exposure
- any averaging of the equity performance
- any limit on the equity performance

These features define the matching asset and hence the upfront charge applied to the policy. Putting a limit on the policy return decreases the cost of the matching asset and increases the linkage to the equity growth. The Call spread strategy is used instead of the Call option strategy in order to give up some of the upside equity return. The Call strategy will be used in this paper to illustrate the GEB basic design. The general framework of the dynamic asset allocation will based on the Call spread strategy.

2.2 Equity Participation

The level of equity participation is derived from the cost of the matching asset. The Call proportion in the matching asset provides the participation in the growth of an equity index. It is the balancing item in the following portfolio:

Policyholder's Investment = ZC Bond + ***Call-Proportion*** ** Call-Option + Charges + Tax*

Call-Proportion is the proportion of the option strategy that the fund can afford to buy after allowing for the cost of the ZC bond, charges and related tax. It is an important parameter linking the policy return to the equity performance, reflecting the market condition, taxation and the implicit charges. At maturity, the policyholder will receive the guaranteed minimum amount plus any growth in the equity index (excluding any dividend income) times the ***Call-Proportion.***

The Call proportion is obtained as follows:

$$\textbf{Call-Proportion} = \frac{\textit{Policyholder's Investment} - (\textit{ZC Bond} + \textit{Charge} + \textit{Tax})}{\textit{Call-Option}}$$

ZC Bond is the cost of a zero coupon bond with a maturity value equal to the cash guarantee. Investment banks use swap rates to provide the payments from the ZC bonds. The ZC bond providing the guarantee is the largest investment in the matching asset.

Charge is implicitly deducted at the outset from the policyholder's investment to cover the distribution cost, initial and renewal expenses and a profit margin. The expenses are considered net of any tax relief. The GEBs involve very little administration cost and no fund management. The main expense is the distribution cost representing the commission paid.

Tax represents the cost of the assets backing the tax liability arising from the ZC bond and the option strategy. The tax liability arising from a Call option is usually covered by the purchase of an extra Call option with the appropriate strike price. The taxation increases the cost of the matching asset, leading to a lower participation in the equity growth. The tax assumptions and

related cash flow arrangements are agreed in advance with the provider of the matching asset including the tax liability. The exact tax treatment of GEBs is beyond the scope of this paper. The Inland Revenue website is a useful initial source for the current tax treatment for structured products.

The ***Call-Option*** is the market price of a European Call written on an equity index to provide the potential growth of the investment. It provides a geared equity exposure with no downside risk. The FTSE-100 index is a popular equity index used in the GEBs distributed in the UK market. The maturity of the Call option should be equal to the term of the guarantee and the strike price represents the level of the guarantee. The implied market equity volatility matrix (varying by term and strike price) and the market swap rates are used to calibrate the B&S Call option formula to mid-market prices. Call spread strategy is used to give up some of the equity return in order to increase the participation in the equity growth. The maximum return of the GEB based on Call spread is equal to the difference between the strike prices times the Call spread proportion.

More detail about option strategies could be found in Hull (2000) Chapter 8.

Sale and distribution risk

In the UK, GEBs are mostly distributed by building societies since these products can offer a high level of capital protection despite the linkage to the equity performance. The particularity of GEBs is that the purchase of the matching asset is agreed before being offered to the potential investors. Before launching a GEB, the policy benefit is defined and the profitability is assessed based on the cost of the matching asset quoted by an investment bank.

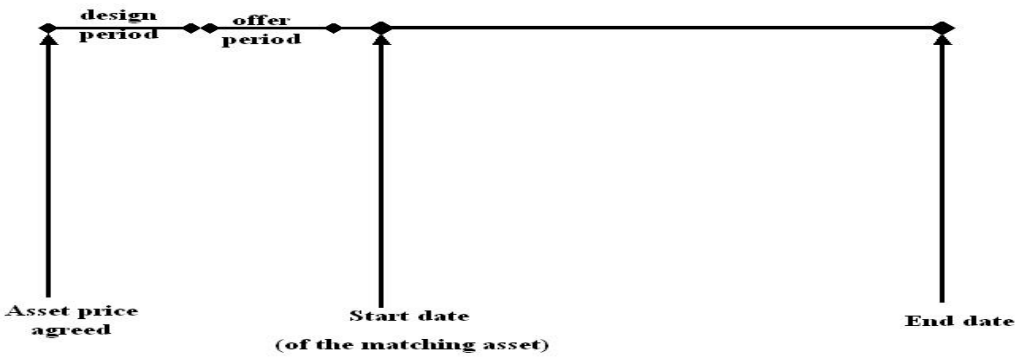
The cost of the matching asset changes on a daily basis mainly due to the movements of the swap rates and the implied equity volatility. The size of the matching asset has an impact on the level of the bid-offer-spread applied by investment banks. The bigger the asset, the more competitive the price, as the bid-offer-spread is lower.

The life office needs to estimate the level of new business that it will be able to write during the offer period (generally a few weeks) in order to reduce the risk of the asset price moving

adversely. The longer the offer period, the higher the risk that the market price of the asset will move adversely before being allocated to unit-linked policies.
Reducing the length of the offer period reduces the ability to attract sufficient new business. The GEBs are generally tranche-based products to support a relative continuous availability of the products and reduce the risk of adverse market movements. The provider of the GEBs tends to keep the benefit of successive tranches identical. The objective during the offer period is to attract enough investors to match the size of the matching asset. The product could be withdrawn from the market before the end of the offer period to avoid new business exceeding the matching assets. If the cost of the matching asset moves significantly between two tranches, the company may redefine the linkage to equity growth to reflect market movements rather than adjusting the upfront charges.

The following chart shows the design and distribution process contributing to the perfect match between the assets and the liabilities. Policies with the same guarantee maturing at the same date make it possible to buy exact matching assets from an investment bank.
This process should inspire the design of life funds with embedded guarantees. Tranche-based life funds will have the ability to reduce the cost of guarantees without a significant cross-subsidy between policyholders. The practical solution for life funds would be a compromise between an open-ended structure and the very strict process of structured products.

The following chart shows the distribution process of structured products.



2.3 Numerical illustrations

The objective of this section is to illustrate how the linkage to the equity exposure of GEBs is calculated using numerical examples based on the market data at 12th January 2004 shown in the Appendix B. The participation in the equity index growth is calculated for different maturities and guarantees. The equity participation is illustrated in two different contexts:

i. Participation in the growth of the FTSE-100 index for an offshore bond (before tax)
ii. Participation in a notional FTSE-100 index including dividend income (before tax)

The participation in the equity growth including dividend income could be compared to the asset allocation of a life fund investing the underlying shares and receiving the dividend income. The importance of the participation in the equity growth (including the dividend income) in deriving the dynamic asset allocation will be shown later in this paper.

The following tables show the necessary steps to derive the participation in the equity growth by calculating the price of the ZC bond and the Call option price.

Table 2-1: Cost of the ZC Bond

Capital Guaranteed	Maturity (in years)							
	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
90%	78%	75%	71%	68%	64%	61%	58%	55%
95%	83%	79%	75%	71%	68%	64%	61%	58%
100%	87%	83%	79%	75%	71%	68%	64%	61%
105%	92%	87%	83%	79%	75%	71%	67%	64%
110%	96%	91%	87%	83%	78%	74%	71%	67%

The ZC bond price decreases with the maturity date but increases with the level of the guarantee.

Table 2-2: Call option price for FTSE-100 index

Capital Guaranteed	Maturity (Years)							
	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
90%	17%	20%	21%	23%	24%	26%	27%	29%
95%	14%	17%	18%	20%	22%	23%	25%	27%
100%	11%	14%	16%	18%	19%	21%	23%	24%
105%	9%	11%	13%	15%	17%	19%	20%	22%
110%	7%	9%	11%	13%	15%	16%	18%	20%

Unlike the ZC bond, the cost of the Call option increases with maturity and decreases with the level of the guarantee.

Table 2-3: Participation in the FTSE-100 index growth (excluding the dividend income)

Capital Guaranteed	Maturity (Years)							
	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
90%	101%	106%	115%	119%	124%	127%	130%	133%
95%	92%	100%	111%	117%	122%	127%	130%	133%
100%	77%	90%	105%	113%	120%	125%	130%	133%
105%	50%	74%	95%	107%	116%	123%	129%	133%
110%	1%	46%	79%	97%	110%	120%	127%	132%

The participation in the FTSE-100 index growth will be received at maturity on top of the minimum capital guaranteed before any taxation.

This participation is obtained by deducting the initial charges in Appendix B and the ZC bond cost from the investment divided by the cost of the Call option.

For short maturity, the cost of ZC bond dominates the level of equity exposure. The higher the guarantee, the lower is the equity participation. As maturity increases, the marginal cost of the ZC bond for higher guarantee is offset by a lower Call option cost. The typical maturity of the GEBs in the UK is a 5-year period. It is not surprising to see participation in the growth of an equity index such as the FTSE-100 index above 100% since the equity index performance excludes the dividend income.

The following table shows participation in the growth of the FTSE-100 index including the dividend income.

Table 2-4: Participation in equity growth including the dividend income (e.g. FTSE All-share index)

Capital Guaranteed	Maturity (Years)							
	3	4	5	6	7	8	9	10
90%	72%	73%	76%	76%	77%	78%	78%	79%
95%	63%	66%	71%	72%	74%	75%	76%	77%
100%	50%	57%	64%	67%	70%	72%	74%	75%
105%	31%	45%	55%	61%	65%	68%	70%	72%
110%	1%	26%	44%	53%	59%	63%	67%	69%

Participation in equity growth including the dividend income is much smaller, due to a higher theoretical Call price. A notional Call option based on the FTSE All-Share index would be more expensive because the dividend income will be taken into account in calculating the payout. The indices including dividend income are more suitable to benchmark the equity investments in the managed funds. In the GEBs, the dividend income is not received since investment banks only offer derivatives based on equity indices excluding the dividend income.

In developing the asset allocation, a notional Call option including the dividend income is used. This reflects the fact that life funds invest directly in equities. The participation in the equity growth including the dividend income is always below 100%. This is a fundamental conclusion that will contribute to the development of an asset allocation consistent with the investment strategies in the life assurance funds. It will be shown later that the Call proportion including the dividend income is a major parameter of the recommended level of EBR.

3 Revised Option Based Portfolio Insurance

3.1 Introduction

The portfolio insurance technique is described in the financial mathematics literature as a tool to protect the value of a well-diversified equity portfolio from falling below a certain level over a specified period of time. There are several portfolio insurance methods available in the literature, in particular the OBPI, which is the focus of this paper.

The following are the portfolio insurance strategies available in the portfolio management theory:

1. Stop loss strategy
2. Buy and hold strategy
3. Constant Proportion Portfolio Insurance (CPPI)
4. Options (OBPI using traded options)
5. Option replication (OBPI using synthetic options)

OBPI was introduced by Leland and Rubinstein (1976) and consists of protecting a risky investment from downside market movements using a Put option, while allowing participation in the growth of the equity market. More simply, it is an equity investment covered by a Put option held in the same fund, to provide protection from adverse equity market movements. Alternatively, OBPI could be achieved using a ZC bond to provide the required minimum return and Call option to provide participation in the equity growth. As seen in the Section 2, investment banks prefer to use the Call option approach to provide the matching asset of the GEBs. OBPI may involve either traded or synthesised options using an option replication technique.

This section proposes a revised formulation of the OBPI taking into account the fund value and the market cost of the matching asset.

3.2 The Option Based Portfolio Insurance

The following two investment strategies are the definition of the OBPI extracted from the portfolio management theory textbooks:

- Hold the portfolio and buy a put option with strike price X
- Hold the risk-free asset and buy a call option with strike price X

Where X is the guaranteed minimum value (or floor).

The mathematical definition of the OBPI in the textbooks is:

$$\Pi = S + Put = ZC + Call \quad \textbf{(3:1)}$$

Where:

- Π is an investment portfolio
- ZC is a ZC bond representing the present value of the guarantee
- T is the guarantee maturity date
- X is the minimum capital guaranteed at maturity, such $ZC = X \cdot e^{-r\cdot(T-t)}$
- r is the risk free rate for the outstanding term (T-t)
- S represents a well-diversified equity portfolio such as an equity index
- Call and Put are European equity options written on the underlying S with strike price X and maturity date T

For consistency with managed funds investing directly in equities, the dividend income from the underlying risky assets should be ignored in calculating the option prices. The Call/Put parity applies without the dividend income adjustment. The Call and Put are notional options since no investment banks write option-based derivatives including the dividend income.

Formula **(3:1)** gives the amount to be invested to achieve the OBPI. In life funds, the size of the assets is known. This formula does not tell us how the funding available would meet the cost of the assets in the OBPI. Assuming no extra funding is available in the Put formulation, the risky asset exposure should be reduced to raise enough cash to finance the Put option. The formulation of the OBPI needs to be adjusted to take into account the market cost of the matching asset and the money available for investment. Investment banks implicitly use the

correct formulation of the OBPI in structuring the asset backing the GEB using the Call option approach.

3.3 Revised OBPI using Call option approach

The asset backing the GEB described in Section 2 is the appropriate formulation of the OBPI if no external funding is available. The OBPI implicitly used in the asset of the GEBs (ignoring taxation and charges) is as follows:

$$\Pi = ZC + \lambda \cdot Call \qquad \textbf{(3:2)}$$

Where**:**

- Lambda is a Call proportion calculated as follows:

$$\lambda = \left(\frac{\Pi - ZC}{Call} \right)^{+} \qquad \textbf{(3:3)}$$

- $Call$ is the market price of an European Call option with strike price X.

After investing in the ZC bond to meet the guarantee, any money left is invested in the Call option. The parameter λ represents the proportion of the Call option that the fund can afford to buy after allowing for the cost of the guarantee.

For a no-dividend paying asset the Call proportion has value between 0 and 1.

$$0 \le \lambda \le 1$$

Propriety of the Call option formula proves this statement. Using the B&S Call option formula gives the following expression for lambda:

$$\lambda = \frac{(\Pi - ZC)^{+}}{Call} = \frac{(\Pi - ZC)^{+}}{\Pi \cdot N(d_1) - ZC \cdot N(d_2)} \qquad \textbf{(3:4)}$$

Where:

- $N(.)$ is the standard normal cumulative distribution
- $$d_1 = \frac{\ln(\Pi / ZC) + (\sigma^2/2) \cdot (T - t)}{\sigma\sqrt{(T - t)}} \qquad \textbf{(3:5)}$$
  $$d_2 = d_1 - \sigma\sqrt{(T - t)}$$

The derivation of the Call option formula in **(3:4)** is shown in Appendix C.

The lower Bound property of the Call option for no-dividend paying stock indicates that:

$$Call \geq \max(\Pi - ZC, 0)$$

Hence:

$$\Pi \cdot N(d_1) - ZC \cdot N(d_2) \geq (\Pi - ZC)^+ \quad \textbf{(3:6)}$$

The derivation of the lower bound property of a Call option could be found in Hull (2000).

$\Pi < ZC$ is an extreme situation where the affordable Call proportion is nil $(\lambda = 0)$. Investing the total fund in ZC bond would not be sufficient to meet the required minimum guaranteed.

OBPI using Put approach

Using the Call/Put parity, the equivalent formulation of the revised OBPI using the Put strategy is as follows:

$$\Pi = \lambda \cdot (S + Put) + (1 - \lambda) \cdot ZC \quad \textbf{(3:7)}$$

This formulation clearly indicates that the equity exposure needs to be reduced and reinvested in ZC bond in order to support the self-funded portfolio insurance. The formula highlights that the Put option should be on the equity holding only. The equity volatility should be used to calculate the market cost of the Put option.

The Call and Put option formulations are equivalent, but developing and manipulating the Call option strategy is a more straight forward way of providing the required level of the downside protection. The equivalent OBPI using Put option asset is less pleasant formulation as it involves more parameters that the Call approach.

Generalisation of the OBPI

The Call approach provides more flexibility in defining different styles of equity exposure while providing the full downside protection. Using a Call option strategy in the GEBs could be treated as a special case of the asset structure using a Call spread strategy. Using a Call spread instead of a Call strategy provides a general expression of the revised OBPI. A Call spread is constructed by buying a Call option with strike price X and selling a Call option with higher strike price Y (X<Y).

The Call spread strategy has the following payout at maturity.

- 0 for $S_T \leq X$
- $S_T - X$ for $X \leq S_T \leq Y$
- $Y - X$ for $S_T \geq Y$

Where S_T is the risky asset price at maturity of the option strategy.

The Call spread has a lower cost than the Call option, leading to a higher participation in the equity growth. The formulation of OBPI using a Call spread strategy is as follows:

$$\Pi = ZC + \lambda \cdot Call\text{–}Spread$$

or

$$\Pi = ZC + \lambda \cdot \left(Call_X - Call_Y\right) \quad \textbf{(3:8)}$$

Where:

- $Call_X$ is a Call option with strike price X
- $Call_Y$ is a Call option with strike price Y with Y>X
- $\lambda = \dfrac{\Pi - ZC}{Call_X - Call_Y}$ **(3:9)**

To make the Call spread structure more general than the derivative textbooks, the following Call strategy will be considered:

$$Call_X - k \cdot Call_Y$$

Where k is a positive parameter such as $0 \leq k \leq 1$

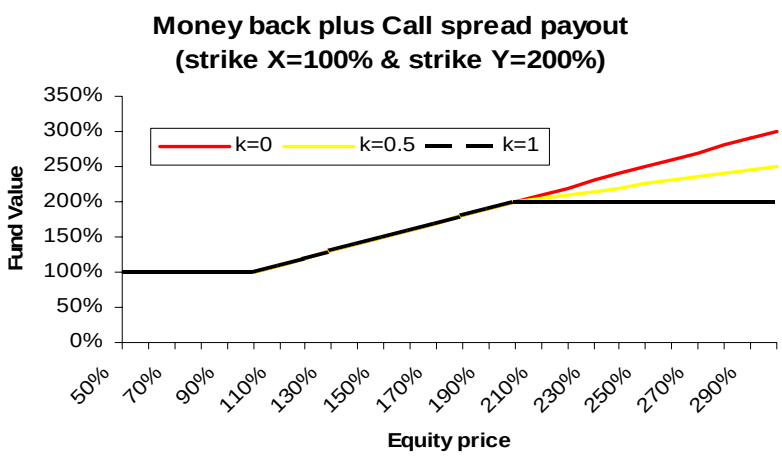
Introducing this formulation of the Call spread strategy is aimed to give a wider range of dynamic asset allocation tailored to the investment objective of a life fund.

The maturity payout an OBPI using this generalisation the Call spread strategy is as follows:

- $X + \lambda \cdot (S_T - X)^+$ for k=0
- $X + \lambda \cdot \left((S_T - X)^+ - (S_T - Y)^+\right)$ for k=1
- $X + \lambda \cdot \left((S_T - X)^+ - k \cdot (S_T - Y)^+\right)$ for 0<k<1

The chart **(3:1)** shows the payout of three Call spread strategies depending on the risky asset price at maturity where k=0, k=0.5, k=1 and $\lambda = 1$.

Figure 3-1: Maturity payout from Call spread strategy for k=0, k=0.5 and k=1



When $S_T \geq Y$ the following statements could be made:

- participation in the equity growth is reduced to nil for k=1
- participation in the equity growth is reduced to 50% from 100% for k=0.5
- participation in the equity growth is unaffected as k=0 represents a Call option

Using the general expression of Call spread strategy will introduce additional features to the dynamic asset allocation. This will be highlighted in Section 8, where investment strategies considered are derived from this new Call strategy.

4 Call option replication and delta hedging

Section 3 shows the correct derivation of the OBPI using Call strategy. Replacing the Call strategy by its replicating portfolio will lead to the dynamic asset allocation. The mathematical tool required to convert the revised OBPI using traded options into a dynamic asset allocation is the option replication technique described in the B&S framework.

Delta hedging is very closely related to option replication, as hedging an option dynamically is equivalent to replicating the opposite position. The delta of a Call option is the second main parameter driving dynamic asset allocation. Providing the sensitivity analysis of the delta of the Call option strategy to the market movements is the aim in this section. Throughout this paper, the delta will refer to the delta of a Call option unless otherwise stated.

Readers not very familiar with the delta hedging could read Appendix D, which is an attempt to demystify the concept behind option replication.

4.1 Decomposing a Call option into its Constituents

The market price of an option is sometimes described in the derivatives textbooks as the discounted value at risk-free rate (risk neutral valuation) of the expected payout. It is important to stress that this 'expectation' does not use the real world probabilities and that the actual drift is irrelevant in the B&S options formulae. The main assumption leading to the B&S option formulae is the arbitrage pricing rather than the expectation pricing. In the B&S model, the payoff of any option could be synthesised using the replicating portfolio, which is a self-financing strategy under B&S assumptions. The no arbitrage argument tells us that the market value of any option should be equal to the cost of the replicating portfolio. The B&S Call option formula gives the cost and the components of the replicating portfolio. The B&S option formulae reflects the make-up of the replicating portfolio.

The following expression is B&S Call option formula for a no-paying dividend stock:

$$c = S \cdot N(d_1) - X \cdot e^{-r\cdot(T-t)} N(d_2) \quad \textbf{(4:1)}$$

Where:

- $$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)\cdot(T-t)}{\sigma\sqrt{(T-t)}}$$ **(4:2)**
  $$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

- S is the underlying risky asset
- σ is the volatility of the underlying risky
- T is the maturity of the Call option
- $N()$ is the standard normal cumulative distribution.

The Call option formula could also be described as a portfolio holding the underlying asset S and cash bond borrowing. It is more obvious in the following expression, which is more meaningful:

$$c = S \cdot N(d_1) - ZC \cdot N(d_2) \qquad \textbf{(4:3)}$$

Where:

- $$\Delta = \frac{\partial c}{\partial S} = N(d_1)$$ **(4:4)**
- $$ZC = X \cdot e^{-r\cdot(T-t)}$$ **(4:5)**

The Call option formula has the following proprieties:

$$0 \le N(d_1) \le 1, \qquad 0 \le N(d_2) \le 1 \qquad N(d_2) \le N(d_1)$$

$N(d_2)$ represents the probability of the Call option maturing in-the-money, so the strike price will be paid to the writer for delivering the underlying asset. The B&S Call option formula is consistent with arbitrage pricing principle and the delta hedging process. The second term of the Call option price (and the replicating portfolio) is the cash bond borrowing that is necessary to finance the replicating portfolio. The net value of this portfolio is the market price of the Call option. The replicating portfolio needs cash borrowing to increase the risky asset exposure in order to achieve equivalent exposure than the Call option. This is a confirmation that a Call option provides a geared exposure to the underlying risky asset.

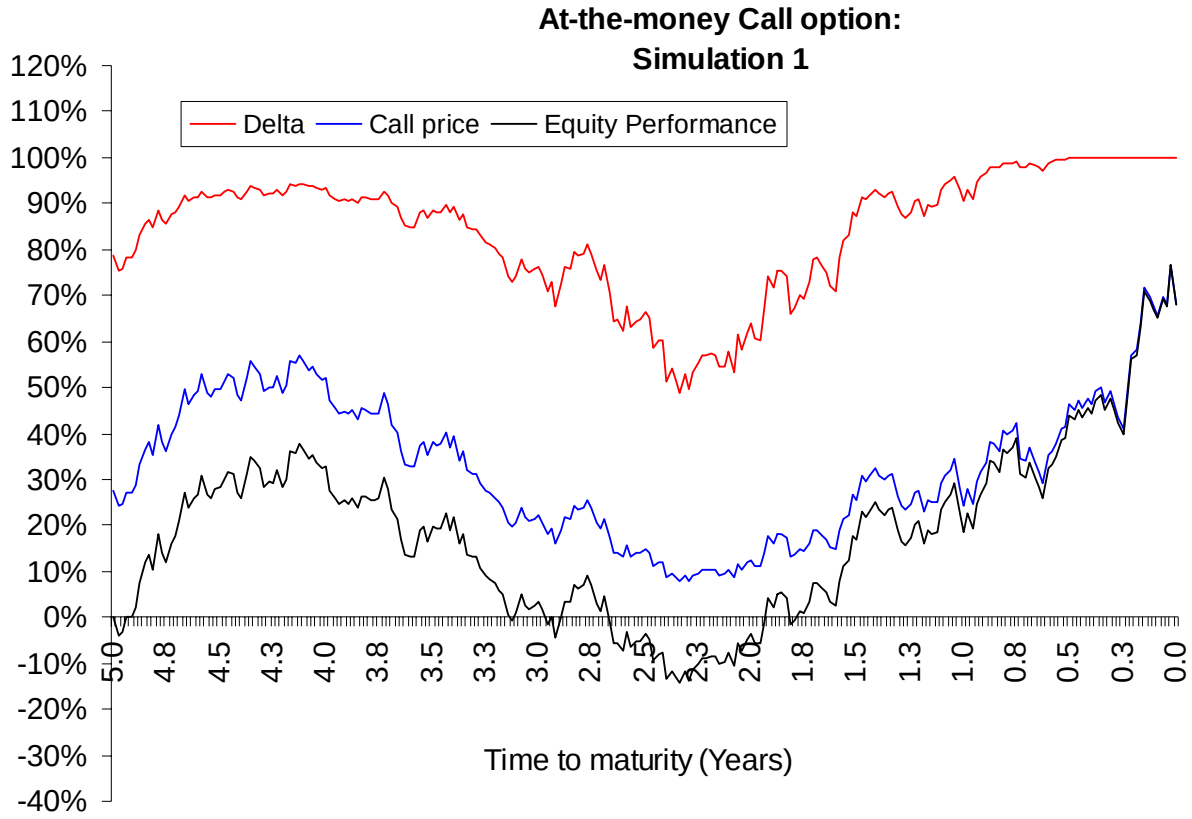
4.2 Replicating a Call option

This section shows two examples of the delta replication of a Call option.

Stochastic simulations illustrate the process of the delta hedging and how the replicating portfolio may work in practice. Simulations combining the random movements of the underlying asset prices and the passage of time are used to illustrate the dynamic of the Call replication.

The charts below show the delta of a Call option in one particular scenario of an equity index. The delta represents the proportion of the replicating portfolio invested in the risky asset. Holding a delta position in the underlying asset is equivalent to holding a Call option. At the start of the projection, delta and the Call price are respectively equal to 80% and 30% of the price of the risky asset. The initial replicating portfolio should hold 0.8 units of the underlying asset, where 0.5 units are funded through cash borrowing. The proportion of the risky asset in the replicating portfolio moves in the same direction as the risky asset price.

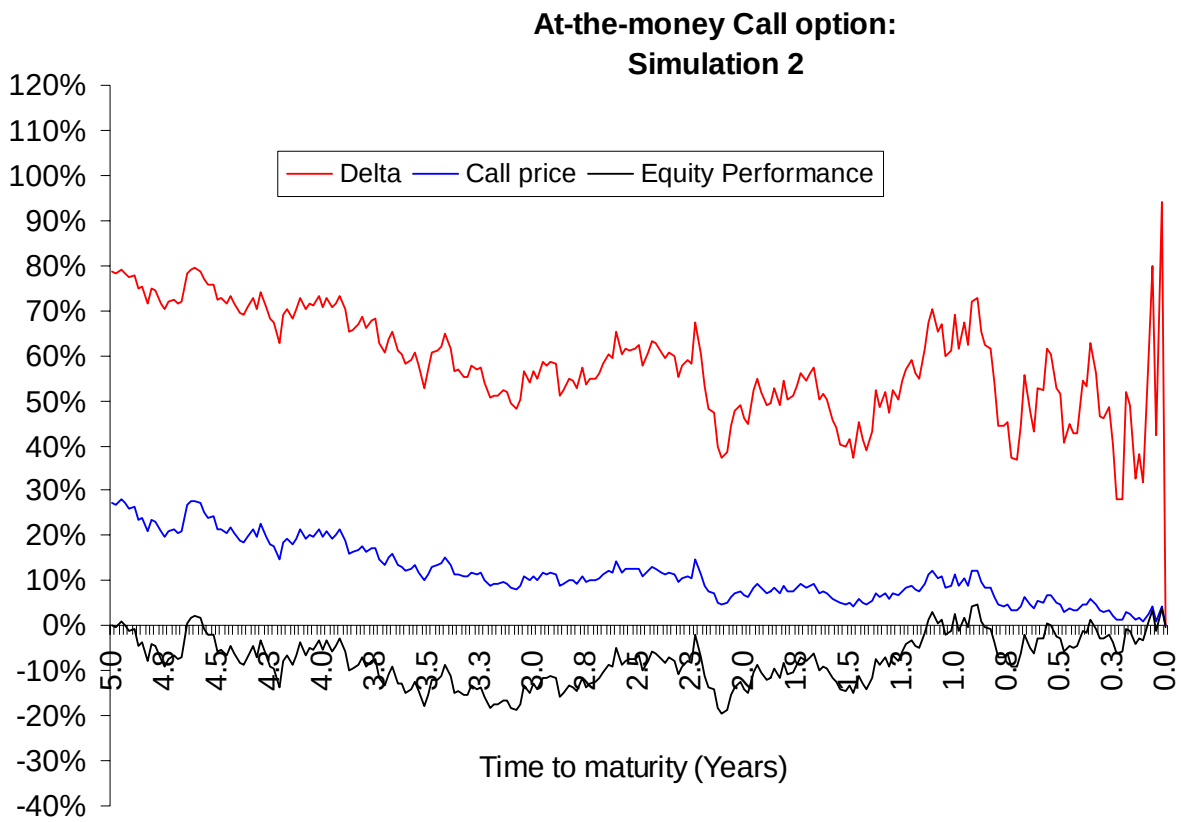
Figure 4-1: Delta hedging process (scenario 1)



In the chart above the equity performance represents the intrinsic value of the Call option subject to a zero minimum value. The time value of the Call option is nil near maturity in this scenario as the price of the Call and the equity performance become equal. This represents 100% probability that the Call option will mature in-the-money. Holding a deep-in-the-money Call option is equivalent to holding one unit of the underlying risky asset.

The following chart represents a second simulation showing the instability of delta near maturity as the asset price is around the strike price.

Figure 4-2: Delta hedging process (scenario 2)



Unlike the first scenario, the time value of the Call is positive even when the maturity date is close.

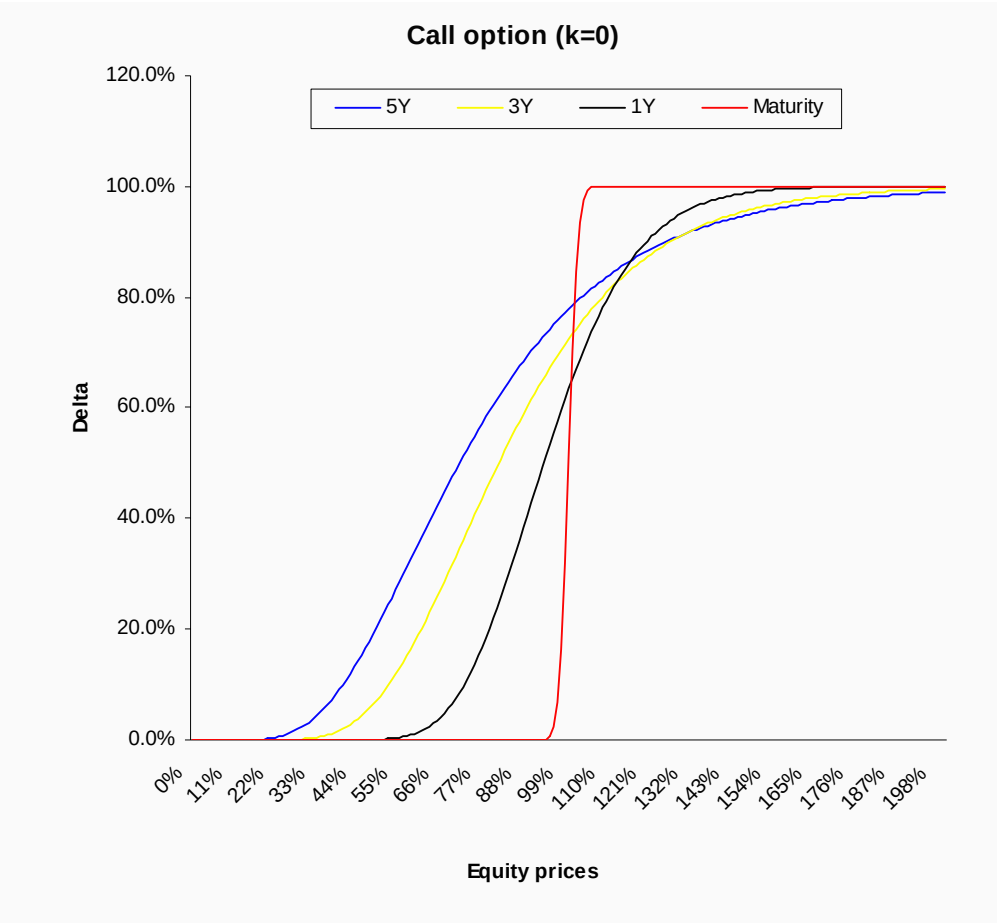
4.3 Impact of equity price movements

The major parameter driving the change to the delta is the change to the underlying risky asset price. This is described in the derivatives textbooks as *Gamma*. Understanding how the delta and the replicating portfolio change with the underlying asset price should help to identify suitable option strategies in developing the dynamic asset allocation.

Call option

The chart below shows how the delta varies with underlying asset prices for different maturities. Near maturity, the delta becomes very sensitive to the movements of the underlying asset prices as a small change in the asset price gives large change in delta.

Figure 4-3: Sensitivity of $N(d_1)$ to equity movements (Call spread where k=0)

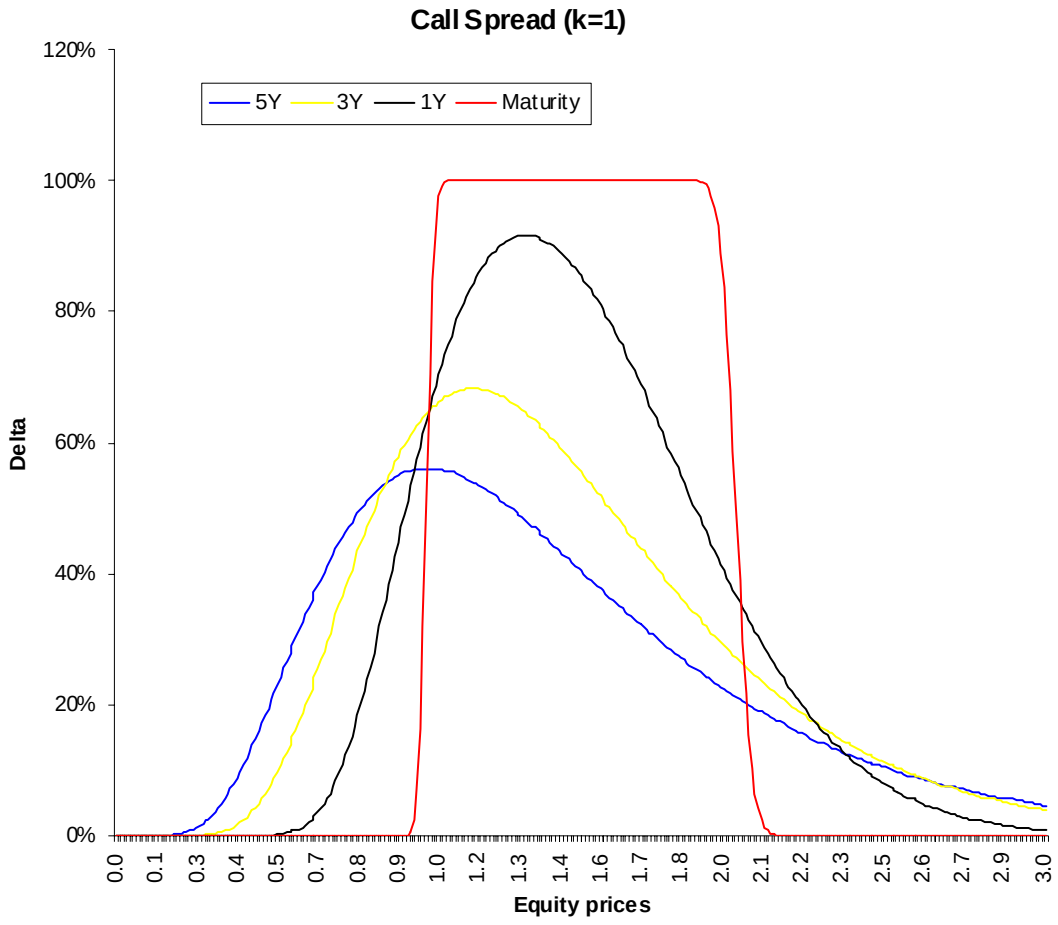


If the Call option matures in-the-money, delta is equal to 1, otherwise is equal to 0.

Call spread

The chart below shows the delta of a Call spread strategy, where the strike prices are respectively 100% and 200% of the underlying risky asset price. The most interesting feature of the Call spread is that after a good performance of the underlying asset, the replicating portfolio exposure to the risky assets reduces to nil.

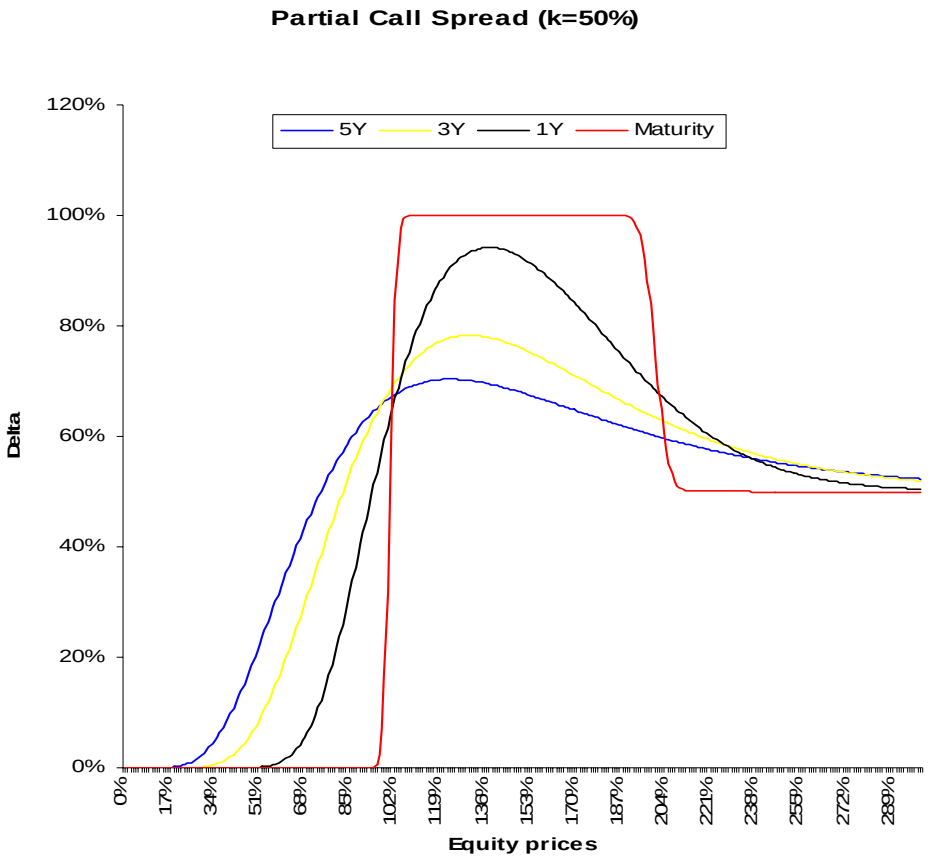
Figure 4-4: Sensitivity of $N(d_1)$ to equity movements (Call spread where k=1)



Partial Call pread (k=0.5)

The figure below shows the delta of an option strategy consisting of one position of at-the-money (ATM) Call option (strike price equals to 100%), and –0.5 position in a deep-out-of-the-money Call option (strike price equals 200%). The underlying risky asset exposure in the replicating portfolio is reduced to 50% after a strong performance.

Figure 4-5: Sensitivity of delta to the risky asset price for a Call spread strategy with k=0.5

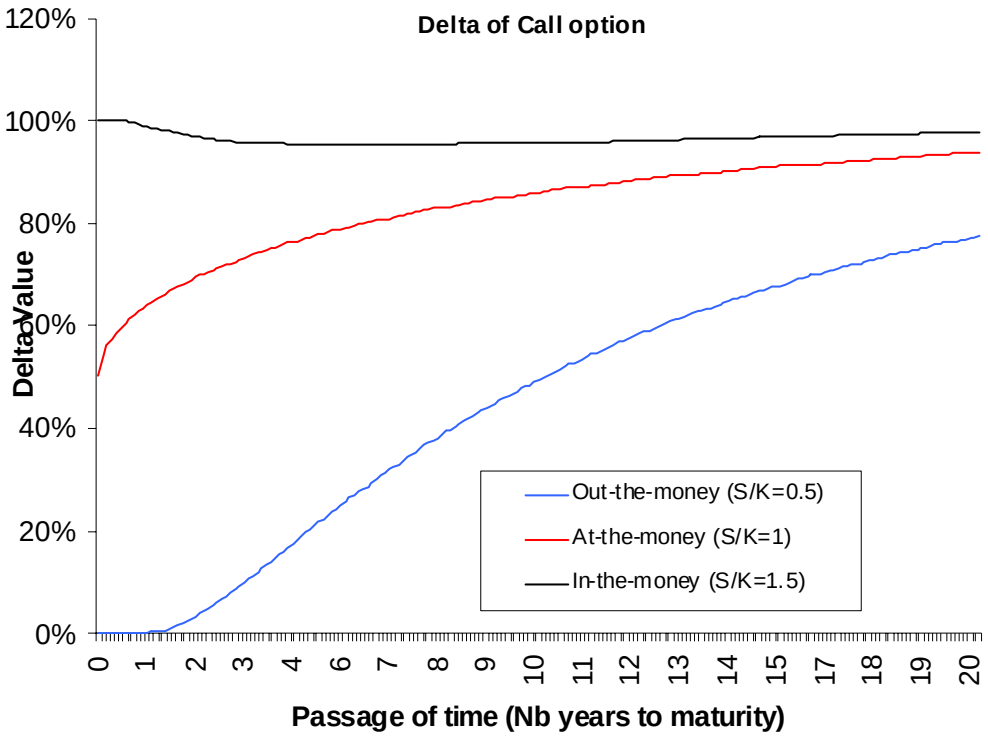


The objective in analysing different Call strategies is to come up with features that may fit different life funds by addressing some unsuitable features of a simple Call option. Call spreads are good strategies for funds willing to reduce equity exposure after an equity market boom. This will be highlighted in the case study in Section 7.

4.4 Passage of time

Passage of time is another parameter that has a major impact on delta. The chart below shows how the delta changes with the passage of time for three different levels of strike price, representing scenarios where the Call option is in-the-money, at-the-money and out-of-the-money.

Figure 4-6: Sensitivity to the passage of time (Call spread where k=0)



The Delta of out-of-the money Call option is very sensitive to the passage of time. The chance of such a Call option maturing worthless increases with the passage of time, leading to a continuous reduction of the equity exposure implied the delta movements.

The consequence of this feature is that the level of recommended EBR depends not only on the level of the guarantee but also the duration of the guarantees. This will be illustrated in Section 5.

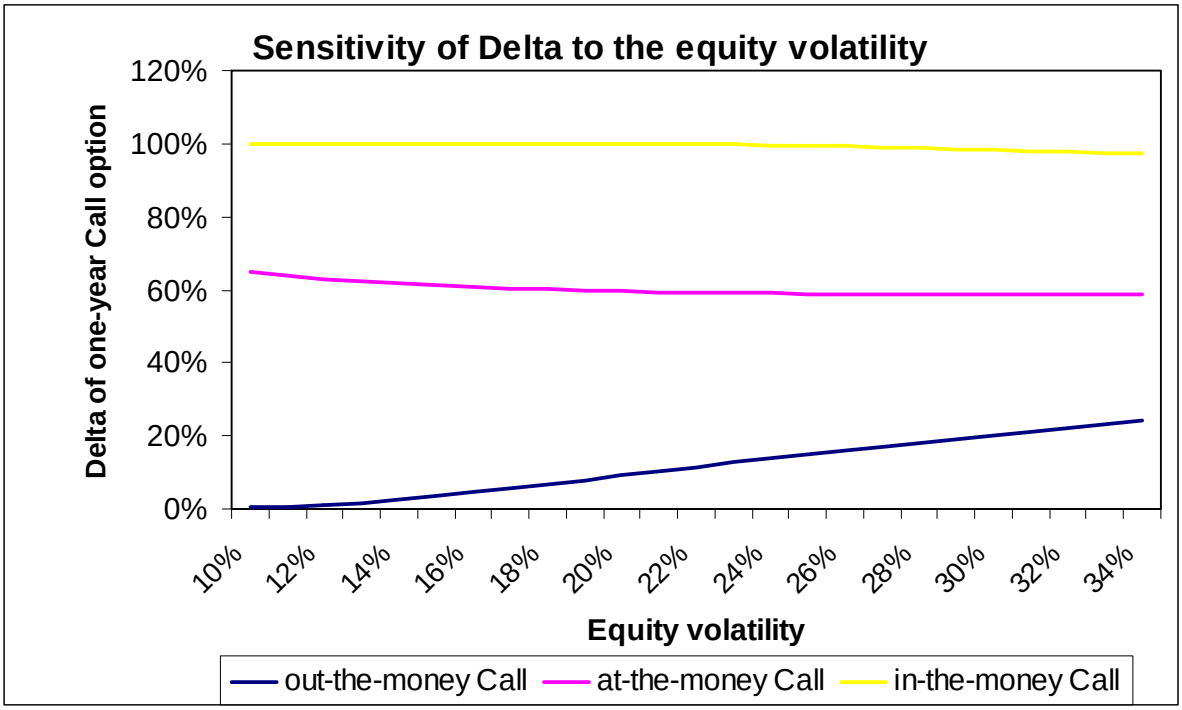
4.5 Sensitivity to equity volatility and interest rates

In the B&S world, the equity volatility and the fixed interest rates are deterministic but in the real world they move unpredictably. This implies that the replicating portfolio using the B&S framework leads to a replication error. Sensitivity analysis of delta to the risk-free rates and equity volatility should give an idea of the materiality of the replication error.

Equity volatility

The level of volatility has a consistent impact on the price of a Call option. The higher the volatility, the higher the price. The sensitivity of delta to the volatility depends on the relative position between the strike price and the asset price. When a Call option is out-of-the-money, the delta increases with the equity volatility. The delta of an in-the-money option is not very sensitive to the risky asset volatility. The following chart shows how delta moves with the underlying asset volatility for a one-year option. The chart suggests that delta is less sensitive to the volatility compared with the sensitivity to the asset price movements.

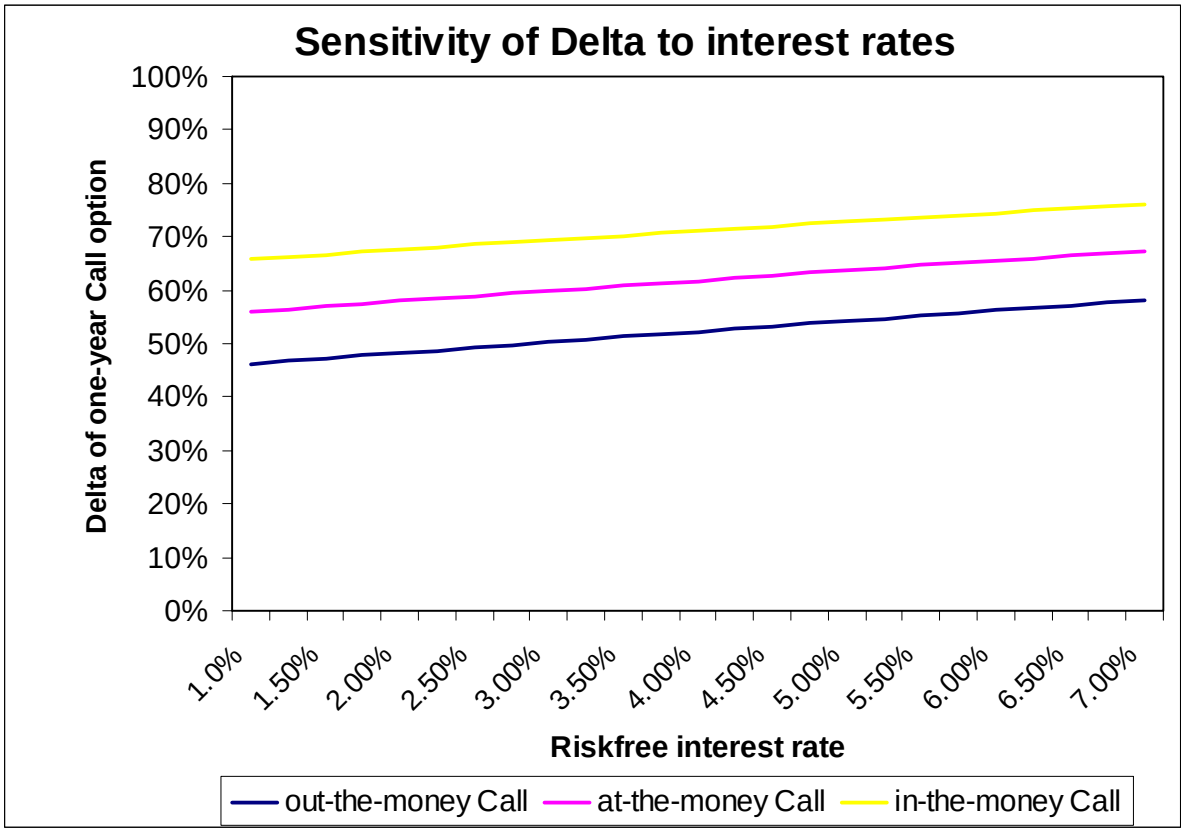
Figure 4-7: Sensitivity to the equity volatility



Sensitivity to fixed interest rates

The Call price and delta increase with the level of the swap rate. The higher the interest rate, the higher the delta. The following chart shows that there is a positive linear relationship between delta and the level of interest rates regardless of the value of strike price.

Figure 4-8: Sensitivity to fixed interest rates



The sensitivity analysis confirms that the risky asset price is the major parameter deriving delta movements.

5 Dynamic asset allocation

5.1 Introduction

The dynamic asset allocation introduced in this paper refers to the split between risky assets (e.g. equity indices) and risk-free assets (e.g. Government Bonds), aiming to provide the same downside protection as the GEBs. The dynamic asset allocation could be applied to life funds where the provision for the active management of the level of equity exposure has been made in the policy literature. The dynamic asset allocation could invest in much broader assets than the GEB. More diversified equity exposure and fixed interest assets is possible in the framework of this dynamic asset allocation. This contrasts with the GEBs offering exposure to a single ZC bond and few equity indices.

Replacing the notional Call option in the revised OBPI by its replicating portfolio is the final step leading to this dynamic asset allocation. The dynamic EBR will follow the equity exposure of the replicating portfolio guided by delta calculation. Using a simple Call option strategy in the OBPI will imply that the recommend EBR increases with the equity price and vice versa in order to achieve the downside protection dynamically.
In order to maintain or reduce the level of EBR after strong equity performance the targeted minimum return could be increased or replace the Call strategy by a Call spread strategy. In the context of life assurance business, this aim of dynamic asset allocation between risky and risk-free assets is to minimise the cost of the cash guarantees in adverse market conditions and maximise the EBR.

To put the dynamic asset allocation of this paper in a broader context than the life assurance business, it is useful to mention that Capital Asset Line (CAL) is the only asset allocation methodology available in the investment theory. The portfolio theory describes the CAL as a statistical tool proposing an asset allocation for a given expected return or expected volatility. This method requires the Utility function of an investor, while the OBPI involves the minimum return required and the option pricing theory. Detail about CAL and Capital Market Line (CML) could be found in Essentials of Investments, Bodie, Kane, Marcus (2001)

5.2 Initial asset allocation

To derive the EBR from the revised OBPI, the Call option needs to be replaced by its replicating portfolio. The Call option formula implied by the OBPI formulation is:

$$c_0 = \left(\Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_2)\right) \quad \textbf{(5:1)}$$

Where

$$d_1 = \frac{\ln(\Pi_0 / X) + (r + \sigma^2/2) \cdot T}{\sigma\sqrt{T}} \quad \textbf{(5:2)}$$
$$d_2 = d_1 - \sigma \cdot \sqrt{T}$$

This formula is consistent with GEB pricing technique and its derivation is shown in Appendix C. To differentiate between the underlying price and the underlying holding, S^H and S^P are assumed to be respectively the risky asset price and the risky asset holding.

Replacing the Call option by its replicating portfolio gives the following portfolio involving only the risky asset and a ZC bond:

$$\Pi = ZC_0 + \lambda_0 \cdot \frac{\Pi_0}{S_0^P} \cdot \left[S_0^H \cdot N(d_1) - \frac{S_0^p}{\Pi_0} \cdot ZC_0 \cdot N(d_2) \right] \quad \textbf{(5:3)}$$

This formula shows that on one-hand there is a significant amount invested in the ZC bond and on the other-hand there is a cash bond borrowing to finance the delta replication. The negative position in the ZC bond could be offset against the ZC bond providing the guarantee as they have the same maturity date.

Aggregating the two ZC bonds gives the following portfolio:

$$\Pi_0 = \left[1 - \lambda_0 \cdot N(d_2)\right] \cdot ZC_0 + \left[\frac{\Pi_0}{S_0^P} \cdot \lambda_0 \cdot N(d_1) \right] \cdot S_0^H \quad \textbf{(5:4)}$$

The portfolio obtained is now made up of a positive holding of the risky asset and a positive holding of ZC bond. Because both $N(d_1)$ and λ_0 have a minimum value of 0 and maximum value of 1, it follows that:

$$0 \le \lambda_0 \cdot N(d_1) \le 1$$

The recommended EBR is:

$$\lambda_0 \cdot N(d_1) = \frac{(\Pi_0 - ZC_0)^+}{c_0} \cdot N(d_1) = \frac{(\Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_1))^+}{\Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_2)} \quad \textbf{(5:5)}$$

$\lambda_0 \cdot N(d_1)$ represents the initial EBR of a dynamic investment strategy aiming to provide a capital protection equal to X.

Numerical Example (excluding tax and management charge)

Inputs

$\Pi = 100 \quad X = 100 \quad T = 5$

$r = \ln(1 + 4.82\%)$

Results

$ZC = 100 * (1.0482)^{-5} = 79.03$

$$d_1 = \frac{\ln\left(\frac{100}{79.03}\right) + \left(15.9\%^2 / 2\right) * 5}{15.9\% * \sqrt{5}} = 0.83979$$

$d_2 = d_1 - 15.9\% * \sqrt{5} = 0.484255$

$N(d_1) = 79.9\%$

$N(d_2) = 68.6\%$

$$\lambda = \left(\frac{100 - 79.03}{79.9\% * 100 - 68.6\% * 79.03}\right) = \frac{20.97}{25.74} = 81\%$$

$EBR = 81\% * 79.9\% = 65\%$

5.3 Numerical illustrations

This section shows numerical examples of calculated EBRs for different level of guarantees (X) and different terms (T), assuming S is the FTSE-All-shares index and $\Pi_0 = 1$. The market data in Appendix B has been again used to derive the EBR, ignoring taxation and the management charges.

Table 5-1: **ZC Bond** = ZC_0

	Maturity (in years)							
Guarantee (X)	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
90%	78%	75%	71%	68%	64%	61%	58%	55%
95%	83%	79%	75%	71%	68%	64%	61%	58%
100%	87%	83%	**79%**	75%	71%	68%	64%	61%
105%	92%	87%	83%	79%	75%	71%	67%	64%
110%	96%	91%	87%	83%	78%	74%	71%	67%

Table 5-2: **Call proportion** = λ_0

	Maturity (in years)							
Guarantee (X)	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
90%	88%	89%	89%	90%	91%	92%	93%	93%
95%	82%	84%	86%	88%	89%	90%	91%	92%
100%	73%	78%	**81%**	84%	87%	88%	90%	91%
105%	59%	69%	75%	80%	83%	86%	88%	90%
110%	35%	55%	66%	74%	79%	83%	86%	88%

The option prices on the FTSE All-Shares index used to derive these proportions are shown in Appendix B.

Table 5-3: **Delta** = $N(d_1)$

	Maturity (in years)							
Guarantee (X)	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
90%	84%	85%	86%	87%	88%	89%	90%	91%
95%	80%	82%	83%	85%	87%	88%	89%	90%
100%	75%	78%	**80%**	83%	85%	86%	88%	89%
105%	69%	73%	77%	80%	82%	84%	86%	88%
110%	62%	68%	73%	77%	80%	82%	84%	86%

Table 5-4: **Recommended EBR=** $\lambda_0 \cdot N(d_1)$

	Maturity (in years)							
Guarantee (X)	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
90%	74%	75%	77%	79%	80%	82%	83%	85%
95%	65%	68%	72%	74%	77%	79%	81%	83%
100%	54%	60%	**65%**	69%	73%	76%	78%	81%
105%	40%	50%	57%	63%	68%	72%	75%	78%
110%	22%	37%	48%	56%	63%	68%	72%	75%

As expected, the recommended EBR increases with the duration of the guarantees but decreases with the size of the guarantee. This is consistent with the linkage to the equity performance in GEBs. The variation of the EBR by maturity and guarantee highlights the difficulty of an open-ended life fund achieving the portfolio insurance. Averaging out the individual EBRs will involve a replication error and increase the cost of the guarantees and cross-subsidy between policyholders. The cost of the shortfalls depends on the equity price movements and the averaging method. Averaging the individual EBRs is required to derive a single EBR for a life fund. The averaging techniques are discussed in Section 6 treating with-profits business.

The risk neutral valuation has been adopted through this paper. Life offices may use their own views on equity volatility and equity drift to calculate the EBR without generating any arbitrage opportunity in the market. Stochastic simulation and stress testing on a market consistent basis will be required to measure the adequacy of the capital supporting the market risk.

Adjusting the initial EBR

The theoretical EBR is derived from the profile of the guarantees and the asset modelling. For a new fund, the life office could set the initial EBR consistent with the guarantee charge. To achieve this objective, the life office may choose to implement an initial EBR different from the theoretical one. For an existing fund, this may be justified to take into account the fund history or a higher appetite for equity exposure. This could be achieved by changing artificially the level of the guarantee. The second way of setting the initial EBR is to adjust the proportion invested in the ZC. This method has the advantage of not impacting the strike price of the Call strategy.

The following formula is an OBPI where the proportion invested in the ZC bond could be different from 100%:

$$\Pi_0 = \beta \cdot ZC + \lambda_0 \cdot Call \quad \textbf{(5:6)}$$

where:

- β is a positive parameter
- $\lambda_0 = \dfrac{(\Pi_0 - \beta \cdot ZC)}{Call}$

In the GEB the parameter β is always equal to 1. Increasing the parameter beta will decrease the initial EBR and vice versa. The adjusted asset allocation to accommodate the new parameter β is as follows:

$$\Pi_0 = [\beta - \lambda_0 \cdot N(d_2)] \cdot ZC_0 + \left[\frac{\Pi_0}{S_0} \cdot \lambda_0 \cdot N(d_1)\right] \cdot S \quad \textbf{(5:7)}$$

This is a more general formulation of the dynamic EBR giving a freedom to set the initial EBR to a desirable level. The parameter β could be reset during the investment period to reflect the change to the dynamic investment strategy.

To illustrate the sensitivity of the EBR to the parameter β, the following table shows how the EBR changes due to reduction in the proportion invested in the ZC bond.

Table 5-5: Increase in the EBRs (in absolute term) by investing only 95% into ZC bond.

	Maturity (in years)							
Guarantee (X)	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
90%	13%	11%	9%	8%	7%	6%	6%	5%
95%	16%	13%	11%	9%	8%	7%	6%	6%
100%	18%	15%	12%	10%	9%	8%	7%	6%
105%	22%	17%	14%	12%	10%	9%	8%	7%
110%	26%	20%	16%	13%	11%	10%	9%	8%

Mismatch between the asset and liability cash flows may suggest using a parameter β above 100%. This leads to higher proportion invested in the fixed interest assets providing a margin to cover the mismatch of fixed interest cash flows.

Taking into account the taxation and annual management charge leads to a higher proportion invested in the ZC bond. Appendix E shows how the proportion invested in the fixed interest assets implied by the dynamic asset allocation could be increased to cover the income tax and the annual management charge.

Constant Proportion Portfolio Insurance (CPPI) is another dynamic asset allocation aiming to deliver the equity exposure with a downside protection with a complete freedom in setting the initial EBR. The following formula is the EBR implied by CPPI, where k is a positive factor chosen by the life office:

$$EBR_t = (\Pi_t - ZC_t) \cdot k$$

The CPPI could be treated as a mechanical asset allocation, which does not pay much attention to the outstanding term of the guarantees and the equity volatility. The CPPI could be seen as a special and a basic case of the dynamic OBPI. The dynamic OBPI is a generalisation of the CPPI formulation with k having the following stochastic expression:

$$k = \frac{N(d_1)}{(\Pi_t \cdot N(d_1) - ZC_t \cdot N(d_2))}$$

5.4 Projecting the EBR

The initial theoretical EBR is as follows:

$$\textbf{EBR} = \lambda_0 \cdot N(d_1) = \frac{\left(\Pi \cdot N(d_1) - ZC \cdot N(d_1)\right)^+}{\Pi \cdot N(d_1) - ZC \cdot N(d_2)} \quad \textbf{(5:8)}$$

The dynamic asset allocation implies that the equity exposure needs to be reviewed on a regular basis to take into account the equity price movements, the passage of time and the change to the calibration of the asset models in order to maximise the probability of meeting the guarantees. There are two possible ways of recalculating the equity exposure for *t>0*.

The first method is to keep λ_0 constant and the EBR will follow the changes to delta $N(d_1)$ depending on the risky asset price. The second method is to reset the OBPI and recalculate the Call proportion λ_t and delta taking into account the change to the fund value.

Constant Call proportion method

To be consistent with GEB benefits, delta hedging implies that the EBR for *t>0* should follow the changes to the delta, keeping the Call proportion constant (λ_0). The Call proportion is calculated at the start of the investment period and kept constant during the investment period. The payout targeted by the dynamic asset allocation with a constant Call proportion could be compared to the payout of the GEB giving a measure of the performance of the dynamic hedging. Implementing and monitoring the EBR based on a constant Call proportion require the availability of the risky asset price **S**. The change to the value of the fixed interest portfolio and total fund will not have a direct impact on the EBR. The change to the delta will be mainly due to the movement of risky asset prices. The change to equity volatility and the fixed interest rates should be taken into account as part of the recalibration process of the asset models.

The proposed formula for recalculating the EBR is as follows:

$$EBR_t = \lambda_0 \cdot N\left(d_1^t\right)$$

Where : $d_1^t = \dfrac{\ln\left(\dfrac{\Pi_0}{X}\cdot S_t\right)\cdot\left(r+\sigma^2/2\right)\cdot(T-t)}{\sigma\sqrt{(T-t)}}$

The second approach, to recalculate the EBR is more convenient as it does not require the risky asset prices.

Floating Call proportion

The value and the performance of the life funds are more readily available than the proportion invested in equities. It would make sense to reset the portfolio insurance by liquidating the replicating portfolio and using the new fund value to redesign the OBPI. By resetting the portfolio insurance, the Call proportion λ will change mainly due to the fund performance and the passage of time.

For traded options, this is equivalent to closing-out the Call strategy before its maturity. Any realised profit or loss will be used to increase or decrease the Call proportion. Floating Call participation leads to more active and aggressive asset allocation, which may produce more frequent switches between equities and fixed interest assets and increase the cost of the dynamic hedging.

The proposed formula for the EBR is:

$$EBR_t = \lambda_t \cdot N(d_1) = \frac{\left(\Pi_t \cdot N(d_1) - ZC_t \cdot N(d_1)\right)^+}{\Pi_t \cdot N(d_1) - ZC_t \cdot N(d_2)} \cdot \quad \textbf{(5:9)}$$

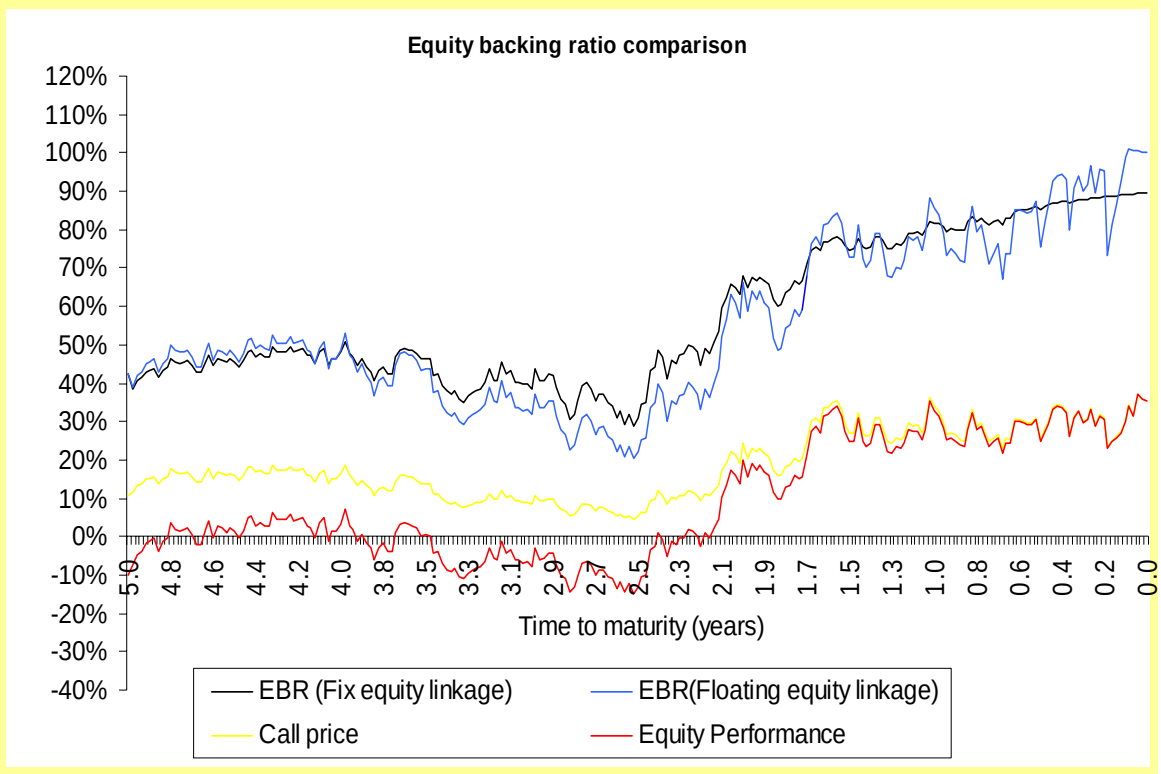
Where :

$$d_1 = \frac{\ln\left(\dfrac{\Pi_t}{X}\right)\cdot\left(r+\sigma^2/2\right)\cdot(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \quad \textbf{(5:10)}$$

The figure below shows the projected EBR derived from a floating and a constant Call proportion for one particular simulation over a 5-year period.

Figure 5-1: Comparison of EBR derived from Floating and constant Call proportion



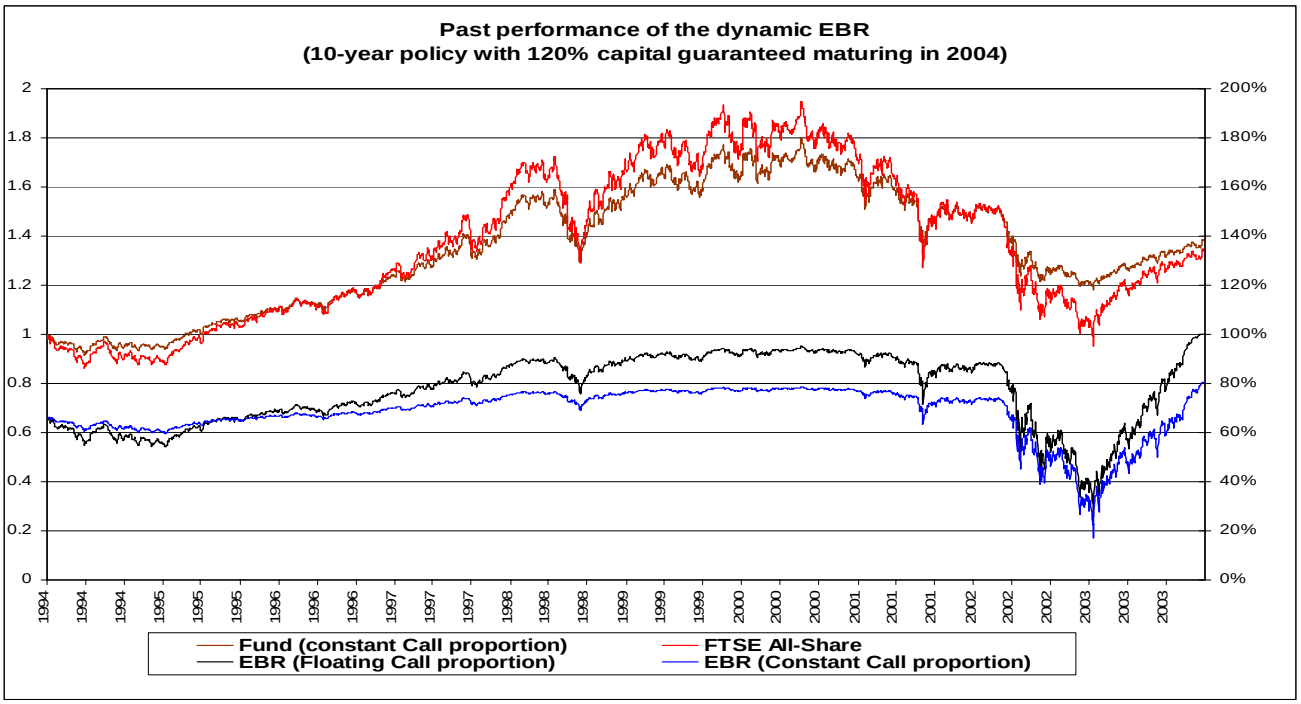
Floating Call proportion implies that the underlying policy benefit is redefined, which makes the comparison to GEB payout more difficult. The EBR in the two methods starts with the same level, but then they diverge as equity prices move away from the initial position. Strategy 2 is likely to generate more switches and increase the cost of the hedge due to unnecessary switches. Rebalancing the asset allocation in the opposite direction increases the cost of the dynamic hedging. The floating Call proportion approach tends to move out from equities faster than the second approach when equities are performing very poorly. Past performance based in recent market experience suggests that dynamic asset allocation with a floating Call proportion gives better performance than with a constant Call proportion.

5.5 Past performance of the dynamic EBR

Past performance has a number of attractions. It provides the performance the dynamic hedging over a period where life office had number of investment choice to make. It allows the comparison of the performance of different investment strategies over real FTSE All-shares data rather than using stochastic simulation.

The following chart illustrates the performance and the benefit of the dynamic hedging targeting 20% minimum return maturing in February 2004, along with the performance of the FTSE All-shares index. The fixed interest assets returns have been assumed to be equal to 6%.

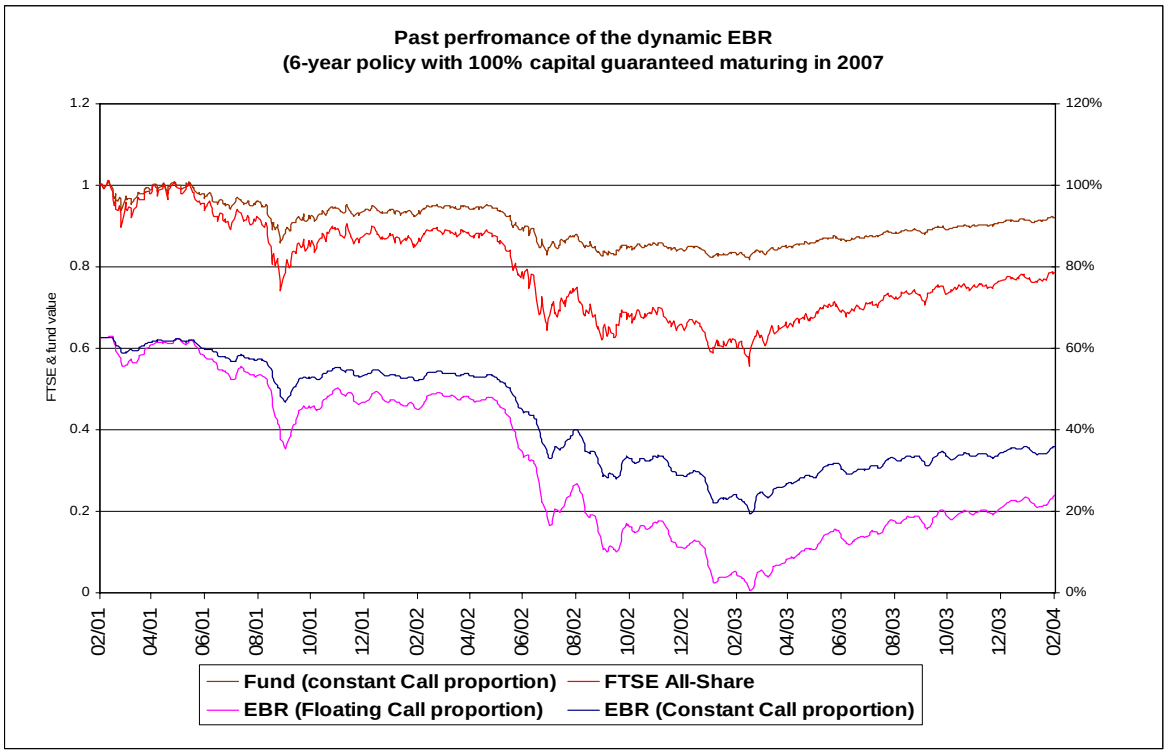
Figure 5-2: Performance of the dynamic asset allocation targeting 120% capital guaranteed maturing in 2004.



The chart above illustrates the benefit of the dynamic asset allocation. The targeted minimum return is met and the delivered fund volatility is 10% over the 10-year period. By contrast, the delivered volatility over the same period of the FTSE All-Share is 19%.

The chart below is another illustration of the dynamic asset allocation to smooth the FTSE All-Share index returns. The illustration represents a 6-year investment period with money back guarantee starting in 2001 and maturing in 2007. The fixed interest rate has been assumed to be equal to 5%.

Figure 5-3: Dynamic asset allocation for a money back guarantee maturing in 2007



The delivered volatility of the FTSE All-shares over the three-year period in this chart is 21%, while the dynamic hedging delivered 9% volatility. The asset allocation in this example shows that the recommended EBR is below 40% at February 2004. This is not a complete shock as the chosen period represents the worst equity performance for decades but the cost of the guarantee is under control.

5.6 Performance of the dynamic hedging

There are two major assumptions in B&S framework that are not realistic. Interest rates are assumed to be deterministic, while equity volatility is assumed to be constant. In the real world, the future changes to the yield curve and equity volatility are not known in advance.

In the derivative textbooks, there are some adjustments to B&S work to take into account market reality, but it is not clear how those adjustments impact the delta hedging and the replicating process. This section discusses the drivers of the hedging cost and the impact of a rise in the equity volatility over the calculated EBR. The hedge performance using the dynamic asset allocation depends on the pattern of future equity returns. The replication of a Call option may cost more than the theoretical cost.

The following table shows the sensitivity of the calculated EBRs due to equity price movement based on a constant Call proportion. The calculated EBRs after 10% drop in equity price are shown in Appendix F.

Table 5-6: Reductions in the EBR (as %) due to 10% drop equity prices

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	-9%	-7%	-6%	-5%	-4%	-4%	-3%	-3%
95%	-10%	-8%	-7%	-6%	-5%	-4%	-4%	-3%
100%	-10%	-8%	-7%	-6%	-5%	-5%	-4%	-4%
105%	-9%	-8%	-7%	-7%	-6%	-5%	-5%	-4%
110%	-6%	-7%	-7%	-7%	-6%	-6%	-5%	-5%

Equity volatility increase

To give an idea of the impact of the equity volatility movements, the following table shows the reduction in the EBR due to an increase in the implied equity volatilities by 25%. The equity volatility in the 1st quarter of 2003 had been more than 25% higher compared to early 2004. Sudden change to the equity volatility is extremely rare and suggests a major crash in the equity market.

Table 5-7: Reductions in the EBRs (as %) due to 25% increase in the equity volatility

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	-11%	-11%	-10%	-10%	-9%	-9%	-8%	-8%
95%	-12%	-12%	-11%	-10%	-10%	-9%	-9%	-8%
100%	-14%	-13%	-12%	-11%	-11%	-10%	-9%	-9%
105%	-15%	-14%	-13%	-12%	-11%	-11%	-10%	-9%
110%	-17%	-15%	-14%	-13%	-12%	-11%	-11%	-10%

The table shows the new EBRs divided by the initial EBRs as in Table 5-4. The recalculated EBRs are shown in Appendix F. The impact on the EBR highlights the importance of a correct modelling of the equity volatility. Equity volatility tends to increase when equity prices are falling. In practise the actual drop in recommended EBR due to an increase in the equity volatility could be much higher.

Regime switching is an extension of the lognormal model. It assumes that equity prices move randomly between two states. The first state is a lognormal model with relatively high positive equity drift and low equity volatility. The second state is a lognormal model with a negative drift and higher volatility. Using a regime-switching model implies that the dynamic asset allocation will produce a higher EBR when equity volatility is low and vice versa. Observations of equity prices in the last couple of years may suggest that the equity prices move randomly between three different states. Equity prices without a clear downward or upward trend may suggest the possibility of a third state close to the risk neutral valuation.

Fall in swap rates

The cost of replicating portfolio arises from the cost of borrowing. The higher the interest rate, the higher the Call price. The levels of interest rates have a major impact on the initial EBR. The higher the interest rates, the higher the calculated EBR. In an economy where the interest rates and inflation are high, the asset allocation gives a higher EBR and vice versa. The EBR is derived from the calculation of the Call proportion and the delta. Both parameters increase with the level of interest rate. Section 4 shows the sensitivity of delta to interest rates.

The table below shows that despite the major impact of the level of interest rates on the initial EBR; the change to the interest rates has only a limited impact on the implemented EBR. This is due to two offsetting effects. A drop in interest rates would suggest an important drop in the

EBR. But the capital gain made on the fixed interest assets would give a higher EBR. The overall impact of interest rates movement over the EBR is reduced.

The following table shows the reduction in the calculated EBRs due to 25% drop in the swap rates.

Table 5-8: Reductions in the EBRs (as %) due to 25% drop in the swap rates

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	-6%	-7%	-7%	-7%	-8%	-8%	-8%	-8%
95%	-8%	-8%	-9%	-9%	-9%	-9%	-9%	-9%
100%	-10%	-10%	-11%	-11%	-11%	-11%	-11%	-10%
105%	-13%	-13%	-13%	-13%	-13%	-12%	-12%	-12%
110%	-16%	-16%	-16%	-15%	-15%	-14%	-14%	-13%

The table shows the new EBRs divided by the initial EBRs as in table 5-4. The recalculated EBRs are shown in Appendix F**.**

5.7 Managing the fixed interest portfolio

The dynamic asset allocation implies that the proportion invested in Bonds is the aggregation of the ZC bond to match guarantees and the cash bond borrowing involved in the replicating portfolio of the Call. This leads to a lower proportion of the fund being invested in the fixed interest assets compared to the asset backing the GEB. The recommended EBRs are more in line with the levels observed in the life funds in the UK.

Duration

The main assumptions made to immunise the fixed interest portfolio from adverse market movements during the investment period are:

- the fixed interest portfolio is invested in risk-free bonds
- the bonds mature around the same date as the guarantees

This is an investment constraint for European guarantee style as assumed throughout this paper. American guarantee style where the guarantee could be exercised at any time suggests cash and money market instruments should be used instead of Bonds.

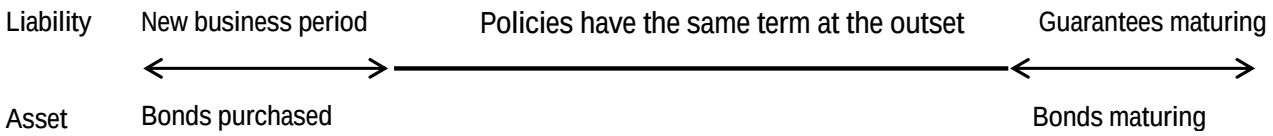
The cash flow matching of the liabilities is not a sufficient condition to lock in the fixed interest rates for all policies. To achieve a complete immunization, ideally all guarantees should mature at the same time covered with a single dated bond. In such a case every policy would be accredited the return from a bond maturing around the guarantee date. It does not matter how the yield curve changes in the future as long as the ZC bond is kept to maturity.

In the GEB, policies in the same tranche have the same start-date and end-date. The individual guarantees are met with a single ZC bond. Life funds are generally open-ended. Typically the guarantee dates are spread over a long period. The open-ended form is not the best way to achieve a high degree of immunisation of the fixed interest portfolio. A tranche-based life fund will allow a greater immunization. Every tranche should have a dedicated fixed interest portfolio with all bonds maturing around the same time as the guarantees. The second argument in favour of the tranche-based fund is the reduction of the cross subsidy

between policies in deciding the adequate level of EBR. It provides the ability to implement an efficient dynamic hedging for every single tranche without impacting policyholders in other tranches. The length of the period where the tranche could be open to new business is a trade off between the desire to have a sizeable life fund and the desire to achieve a perfect immunization.

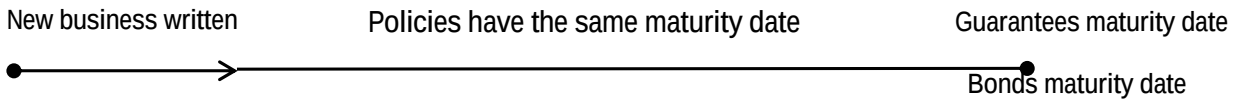
Below are two illustrations showing two possible ways of structuring the tranche-based managed fund open to new business during a limited period of time.

Figure 5-4: Single tranche illustration



In this illustration, all policies have the same term at the outset but the maturity dates will spread over a period equal to the new business period. The materiality of the risk arising from the asset & liability mismatch will depend on the outstanding duration of the fixed portfolio at the guarantee date of every policy. Any material mismatch risk could be covered by MGWP maturity guarantee reserves or hedged by investing a higher proportion of the fund in the fixed interest assets.

Figure 5-5: Single tranche illustration (copied from GEB)



This fund design is inspired from the GEB. All policies have the same maturity date regardless of the start-date of the policies. The maturity date of the guarantees should be carefully chosen to match the maturity date of bonds available in the market with very good market liquidity and the appropriate credit risk. This may give the opportunity to leave the

tranche open to new business over a longer period offering an exact cash flow matching of the liabilities.

Using Corporate Bonds & Gilts

ZC government bonds are quite expensive and less liquid in comparison to the Gilts and swap rates market. Using coupons bearing Bonds to match the guarantees introduces a mismatch and reinvestment risk. However the coupons may be used to match the liability cash flows such as lapses, death benefit, management charge and future changes to the asset allocation. The matching fixed interest portfolio should have a maturity date consistent with the maturity date of the guarantees and a duration reflecting other liability cash flows.

The dynamic hedging gives the opportunity to take some credit risk to boost the fixed interest portfolio returns by investing part of the fund in a diversified corporate bonds portfolio. Cash flow matching of the liability should reduce the risk of spreads rising. This implies that corporate bonds should be held to their maturities. Credit risk optimisation and portfolio diversification is consistent with the dynamic asset allocation but they are beyond the scope of this paper.

5.8 Dynamic hedging versus GEBs

The aim of this section is to compare and contrast the benefits and the features of the dynamic hedging strategy versus buying a matching asset from an investment bank providing equity exposure with a downside protection.

Active fund management and portfolio diversification:

The dynamic asset allocation is consistent with the active fund management usually adopted in life funds. Fund managers will have the ability to apply their stock-selection skills to both equity and fixed interest portfolios. A life office would have strong control over the EBR, the duration and the maturity date of the bonds portfolio.

This contrasts with GEB where the market exposure is achieved through an equity index and a single ZC bond. The main advantage of the dynamic hedging is that the fund manager has access to the whole range of financial instruments while OTC derivatives are generally based on few indices. A high degree of equity portfolio diversification reduces the volatility and increases the EBR.

Liquidity and flexibility:

The liquidity and depth of the cash equity market tend to be higher than the OTC equity market. The valuation of a fund using the dynamic asset allocation is straight forward and transparent. The valuation of a fund invested in the OTC options is more difficult to obtain and less transparent due to the lack of liquidity.

The dynamic hedging gives also the flexibility to write new business over a longer period compared with GEBs and to review the targeted minimum return. Reviewing the objective of the dynamic hedging requires a change to the EBR. By contrast, reviewing the objective of a hedging strategy involving OTC options requires the liquidation of the existing hedge instruments and opening a new option-based derivative position.

The guarantees in life funds using the dynamic hedging could be rolled over at a guarantee date for another period by resetting the level of the guarantee. This will encourage policyholders to stay invested in the fund after they have reached a guarantee date.

Payout and charges

The policy benefit based on traded options is clearly defined because it is linked to equity index performance. The final payout of the dynamic hedging is less clear as the EBR is subject to changes. The actual performance depends on the market movements, the rebalancing policy and the accuracy of the asset modelling.

High equity volatility, provoking up and down equity prices movements with no clear trend, may lead to the under-performance of the dynamic hedging relative to a buy and hold strategy. Averaging out the equity performance and putting in place a switch trigger should reduce the number of transactions in a high equity volatility environment.

The life office needs to set up reserves for the guarantees based on the maximum cost for a given confidence level using Monte Carlo simulation. A passive investment strategy with an equivalent EBR and no allowance for management actions needs to be supported by higher reserves. Monte Carlo simulation is required to calculate the distribution of the shortfall and the cost of capital allowing for the dynamic EBR.

This allows the life office to test the adequacy of guarantee charge and the projected cost as no closed form solution is available. For beta equal to 100%, the modelling will show a small shortfall. Life offices wishing to offer a higher equity exposure compared to the neutral position may reduce the initial proportion invested in the ZC Bond and increase the proportion invested in the Call option. A higher initial EBR leads to a higher cost of the guarantees.

Derivative expertise:

Expertise in derivatives, delta hedging and financial engineering systems are required in order to monitor and implement the dynamic asset allocation. Monitoring equity volatility and developing an understanding of the drivers of the market movements may help the company to increase the benefit of the hedging. Life office with a strong and correct view on future equity volatility would increase the performance of dynamic hedging and the fund performance in general.

Using traded option-based derivatives within the life companies also requires option pricing and derivatives expertise to deal with investment banks more effectively.

6 With-Profits business

6.1 Introduction

Many papers discussing the realistic liabilities of with-profits business emphasise the lack of methodology in the investment theory to help the life industry to derive the EBR from the level of the assets and the liabilities. This section discusses how the dynamic hedging derived from the GEB pricing technique could be extended to with-profits funds. The revised OBPI based on the profile of the liabilities could provide life offices with a framework to guide the investment strategy for with-profits business. Dullaway & Needleman, in a discussion paper presented to the Institute of Actuaries in November 2003, made a good case for the Call option approach and discussed the benefits of the Bonus Reserve Valuation (BRV). This section complements their work by deriving the EBR from the Call approach, the asset shares and the BRV.

With-profits are pooled funds providing policyholders with smoothed equity exposure with downside protection. In the past, life offices have focused on adopting competitive EBRs in their with-profits funds to attract new business assuming that smoothing payouts to policyholders is sufficient to sustain market falls and maintain the concept of the with-profits fund. Equity volatility in the last couple of years has defeated the resilience of the smoothing rules, as with-profits assets have shrunk without a corresponding reduction in their liabilities. The smoothing rules have not been sufficient to support a high EBR combined with high equity volatility. The financial strength of with-profits funds has weakened especially for those with a significant level of contractual guarantees.

Recent market experience highlights the need for life offices to smooth the performance of the assets by actively managing the equity exposure in order to minimise the risk of insolvency and to meet the guarantees. Stress-testing is the first risk management tool that should be used to underline any inadequacy between the investment strategy, the level of guarantees and the level the Estate. A sensible investment strategy would move away from the practice that does not pay much attention to the profile of the liabilities. Deriving the investment strategy from

the asset shares and the level of the guarantees should in theory increase the prospect of meeting the guarantees and maximising the affordable EBR.

6.2 Theoretical EBRs

The initial concept of with-profits business implies that a single investment strategy could fit all products and all generations of policies. This is not an ideal concept for meeting the guarantees and maintaining a decent level of EBR. The fall in the free assets within with-profits funds implies that the mismatch between the asset shares and the guarantees needs to be managed more carefully. Applying the dynamic portfolio insurance to with-profits' assets should reduce the cost of guarantees and reduce the claims on the Estate.

Calculating a single EBR

The difficulty facing the implementation of the dynamic hedging to with-profits funds is to decide the nature of the liabilities that should be covered by the ZC bonds in the OBPI.

The BRV method used to calculate with-profits reserves seems to be a good candidate for this modelling if the cost of GAO and the time value of the guarantees are excluded. The following list presents the liabilities taken into account in the BRV approach (excluding GAO reserve):

- Intrinsic value of the contractual guarantees, future regular bonus and cost of smoothing
- Tax on shareholder transfers, charges and investment expenses
- Future contractual premiums

The following OBPI could be used as an investment strategy for the assets supporting the policyholders' benefits:

$$Assets = \beta \cdot \mathrm{BRV} + \lambda \cdot Call \quad \textbf{(6:1)}$$

Where:

- *Assets* represent the amount of money the life office is willing to use to support the realistic liabilities. These assets could be expressed follows

$$\mathrm{Assets} = \mathrm{Assets\,Shares} + \mathrm{k} \cdot \mathrm{Estate} \qquad \text{where } 0 \le \mathrm{k} \le 1.$$

- β is the proportion of the BRV to be met by the fixed interest assets. The appropriate level of the BRV proportion depends on the projected distribution of the free Estate and the probability of ruin.

- λ is a Call option proportion, such $\lambda = \dfrac{\text{Assets - } \beta \cdot \text{BRV}}{\text{Assets} \cdot N(d_1) - \beta \cdot BRV \cdot N(d_2)}$ **(6:2)**

Replacing the Call option by its replicating portfolio and the Call proportion by its expression give the following portfolio:

$$Assets = \beta \cdot \text{BRV} + \frac{(Assets - \beta \cdot BRV)^+}{Assets \cdot N(d_1) - \beta \cdot BRV \cdot N(d_2)} \cdot [Assets \cdot N(d_1) - \beta \cdot BRV \cdot N(d_2)] \quad \textbf{(6:3)}$$

Where:

- $d_1 = \dfrac{Ln\left(\dfrac{\text{Assets}}{\beta \cdot BRV}\right) + (\sigma^2/2) \cdot T}{\sigma\sqrt{T}}$ **(6:4)**
- $d_2 = d_1 - \sigma\sqrt{T}$
- σ is the volatility of the assets representing the EBR
- T is the average outstanding term for guarantees
- r is the risk-free rate for the maturity T
- N(.) is standard normal cumulative distribution
- BRV is the bonus reserve valuation excluding GAOs costs.

Aggregating the ZC bond backing the BRV proportion and the cash bond borrowing gives the following portfolio:

$$Assets = \left(1 - \frac{Assets \cdot N(d_2) - \beta \cdot BRV \cdot N(d_2)}{Assets \cdot N(d_1) - \beta \cdot BRV \cdot N(d_2)}\right) \cdot \beta \cdot BRV + \frac{Assets \cdot N(d_1) - \beta \cdot BRV \cdot N(d_1)}{Assets \cdot N(d_1) - \beta \cdot BRV \cdot N(d_2)} \cdot Assets \quad \textbf{(6:5)}$$

The assets supporting the policyholders' benefits are split into two parts:

- $\left(1-\frac{Assets\cdot N(d_2)-\beta\cdot BRV\cdot N(d_2)}{Assets\cdot N(d_1)-\beta\cdot BRV\cdot N(d_2)}\right)\cdot\beta\cdot BRV$ is the amount to be invested in a matching fixed interest portfolio.

- $\frac{Assets\cdot N(d_1)-\beta\cdot BRV\cdot N(d_1)}{Assets\cdot N(d_1)-\beta\cdot BRV\cdot N(d_2)}\cdot Assets$ is the amount to be invested to get exposure to equities and property.

The recommended EBR is equal to: $\frac{(Assets\cdot N(d_1)-\beta\cdot BRV\cdot N(d_1))^+}{Assets\cdot N(d_1)-\beta\cdot BRV\cdot N(d_2)}$

Numerical examples:

The table below shows the calculated EBR for different level of β (BRV proportion) and equity volatility, assuming Assets =£6b, BRV = £5b and the outstanding term T = 9:

Table 6-1: Calculated EBRs with different equity volatility and BRV proportion

Guarantee backed by FI		Volatility					
		16%	18%	20%	22%	24%	26%
BRV proportion	90%	61%	57%	54%	51%	48%	46%
	92%	58%	54%	51%	48%	46%	44%
	93%	56%	52%	49%	46%	44%	42%
	94%	54%	51%	48%	45%	43%	41%
	95%	53%	49%	46%	43%	41%	39%
	96%	51%	47%	44%	42%	40%	38%
	97%	49%	46%	43%	40%	38%	36%
	98%	47%	44%	41%	39%	37%	35%
	99%	46%	42%	40%	37%	35%	34%
	100%	44%	41%	38%	36%	34%	32%
	101%	42%	39%	36%	34%	32%	31%
	102%	40%	37%	34%	32%	31%	29%

This table shows that the EBR decreases with the level of equity volatility and the BRV proportion backed by the fixed interest assets.

The following tables show the sensitivity of an initial EBR of 46% to an increase and a decrease in the asset shares. It is assumed that BRV is less sensitive to the market movements.

Table 6-2:Change 46% EBR due to asset shares and BRV movements

	Asset shares changes	
BRV change	+5%	+10%
0%	53%	60%
1%	52%	58%
2%	50%	57%
3%	49%	56%
4%	47%	54%
5%	46%	53%

	Asset shares changes	
BRV changes	-5%	-10%
0%	37%	26%
-1%	39%	28%
-2%	40%	30%
-3%	42%	32%
-4%	44%	34%
-5%	46%	36%

Imperfect correlation between the assets and the BRV means that the EBR needs to be managed dynamically in order to achieve the portfolio insurance and meet the guarantees.

Such a framework provides the life office with a dynamic investment strategy designed to set the EBR to the level of the protection required, which is expressed as a proportion of the BRV. The level of the free Estate is likely to be the main driver in deciding of the level of the BRV to be backed by fixed interest assets. Stochastic projection using Monte Carlo simulation should be carried out to establish the relationship between the distribution of the actual cost of the guarantees and the BRV proportion.

In calculating the BRV, the intrinsic value of the following guarantees could be taken into account:

- No-mvr guarantees
- Guaranteed minimum bonuses
- Glidepath
- Guaranteed Minimum Pension (GMP)
- Mortgage promises (ROME)

If some guarantees are not taken into account in the BRV calculation, the free Estate should be reduced to allow for the realistic reserving of those guarantees.

The guaranteed annuity options (GAOs) are options on the policyholders' payout and the annuity cost. Maximising the policyholders' payout will increase the cost of the GAOs. The Estate should have its own hedging instruments and investment strategy to minimise the cost of the GAOs.

In GMP, life offices are exposed to an increase of the cost of annuities. Unlike the GAO, maximising the policyholders' payouts reduces the cost of the GMP. The cost of the GAO and GMP increases with falls in interest rates. Falling interest rates could be hedged by holding a Call option on the appropriate bond or a Put on the appropriate swap. The dynamic replication of these hedging instruments involves holding long dated swaps or bonds in the Estate for GMP and hypothetical asset shares for GMP.

Deriving the global EBR

With-profits funds are pooled funds with policies having different start dates, maturity dates and maturity benefits. In theory, a single EBR could be calculated for every sub-fund by entry year and maturity year. To calculate the global EBR for the asset shares supporting the whole fund, the following techniques could be used:

- Calculating a single EBR by using the aggregated the asset shares, BRV and average out the outstanding term of the guarantees. Such an EBR implies a high degree of cross-subsidy between policyholders that may lead to an excessive cost of the guarantees (depending on the profile of the guarantees).
- The second approach is to average out the individual EBRs calculated for policies grouped by entry year, maturity date and product type.

Calculating the individual EBRs provides the life office with valuable information to assess the materiality and breakdown of the costs of guarantees in the sub-funds. Averaging the individual EBRs could be based on the materiality of the cost of the guarantees. Put option prices are a good measure of the materiality the guarantees and could be used as weighting factors. This leads to the following formula:

$$Global\_EBR = \frac{\sum Put_i \cdot EBR_i}{\sum Put_i} \quad \textbf{(6:6)}$$

The following expression is also possible but it is likely to provide a higher EBR:

$$\frac{\sum Asset\_Share_i \cdot EBR_i}{\sum Asset\_Share_i} \quad \textbf{(6:7)}$$

The following formulae may give a much lower EBR:

- $Global\_EBR = Min(EBR_i)$
- $Global\_EBR \equiv \overline{EBR} - k \cdot S$

Where S is the standard deviation of the individual EBR_i and k is a positive parameter.

The aim of dynamic EBR is to invest the assets in order to meet with-profits realistic liabilities (excluding GAOs). The theoretical investment strategy where every single policy may have its own EBR and dedicated ZC Bond should reduce the cost of the contractual guarantees to the minimum. With-profits funds by nature are at the extreme end of the principle behind the investment strategy of GEBs. The ability of a single EBR to control the cost of the guarantees is reduced with high variability between the individual EBRs. The free assets represent the cushion supporting any shortfalls between the asset shares and the maturity payouts.

6.3 Guidance for financial management

The discretion in managing with-profits business gives some flexibility in moving towards the portfolio insurance. Implementing the dynamic asset allocation combined with the following set of risk reduction measurements should move the assets closer to the matched position:

- Adopt the appropriate EBR for different sub-funds when it is possible
- Consider tranche-based with-profits fund when writing new business
- Apply a guarantee charge to asset shares to finance the Estate assuming 100% asset shares payout is targeted. This is a more transparent smoothing policy.
- Invest the fixed interest assets in order to achieve the cash flow match liability.
- Allocate to asset shares the returns on bonds whose duration matches the duration of the guarantees.
- Reduce the EBR in the Estate in order to reduce the correlation between asset shares' returns and the Estate's returns
- Adopt a sensible regular bonus strategy with the appropriate grouping to support the assets and liabilities matching.

Dynamic reversionary bonuses

The Call option strategy could be extended to link the asset and liabilities to the regular bonus strategy. The delta of a call option gives a good measure of the fund ability to meet the contractual guarantees and to pay terminal bonuses. A high delta value (and EBR) indicates that the assets are in a strong position relative to the realistic cost of guarantees suggesting there is an ability to increase the discretionary payments. Conversely, the opposite of this, a very weak fund or sub-fund with a zero delta, would suggest that the asset shares should be invested in risk-free assets and supported by the free Estate in order to meet the guarantees.

7 Case study

The benefits of the dynamic asset allocation derived from OBPI and the practical issues surrounding its implementation are best demonstrated through an example. This section shows a life assurance case where the dynamic asset allocation is adopted as an investment strategy. The life assurance example is a managed fund backing a unit-linked pension contract with the following two objectives:

- maximise the equity exposure
- produce a safety net at the retirement date in the case of adverse market conditions

The provision of the safety net could be either an objective of the fund (without a guarantee charge and reserves) or a contractual guarantee. The dynamic asset allocation could be used as an investment strategy for any managed fund with or without contractual cash guarantees.

7.1 Modelling

The life assurance example is based on the following liability assumptions:

- single premium contribution (e.g. Trustee Investment Bond)
- the outstanding term for the guarantee is 10 years
- 15% minimum return at maturity (net of amc)
- the fund value is at 95% discount of the face value of the liability
- the annual management charge is 1%
- no taxation
- no allowances are made for lapse, mortality and new business

Investment strategies

Different dynamic asset allocations have been tested to emphasise the relationship between the Call option strategy and the projected EBR. Analysing the features and the performance of different option strategies using stochastic projections should help tailoring the dynamic hedging to a specific fund's objectives. The selected option strategy could reflect the fund objectives, the policyholders' expectations and the company's appetite for equities. The risk

outcomes measuring the performance of each investment strategy are the projected EBR, the distribution of the shortfall and the total fund returns.

Six different investment strategies have been considered. The first strategy is a floating EBR representing a passive investment strategy with no allowance for management action. This strategy is useful in assessing the performance of the active management of the EBR. The second strategy is based on the concept of a floating Call proportion. Strategies 3, 4 and 5 are based on a constant Call spread proportion approach where k=0, k=20% and k=50%. Strategy 6 is based on a floating Call proportion targeting 95% of the capital guaranteed. This strategy will show the sensitivity to the initial EBR of a dynamic asset allocation. The rebalancing in the dynamic asset allocations occurs on a quarterly basis.

The investment strategies are summarised in the following table:

Table 7-1: Investment strategies

Strategy	Asset allocation	Call spread **K** factor	Option proportion λ	Initial EBR
1	Floating	-	-	54%
2	Dynamic	0%	Floating	54%
3	Dynamic	0%	Constant	54%
4	Dynamic	20%	Constant	52%
5	Dynamic	50%	Constant	48%
6	Dynamic	0%	Floating	61%

The dynamic EBRs are reduced to take into account the amc on the fixed interest assets.
Call spread strategies where $k \neq 0$ are designed to reduce EBR near maturity, after a strong equity performance to achieve the lifestyle concept usually sought in pension funds. The second way of achieving the lifestyle concept is to lock-in the fund performance by increasing the required minimum guaranteed.

Asset model

The asset modelling used in this example is based on two asset classes, equity asset and risk-free bonds. The equity price follows a lognormal model (no mean reversion). The Bond price is projected with 0% volatility. The equity asset is projected stochastically until maturity using the Monte Carlo technique in order to generate the distribution of the EBR, the shortfalls and the fund performance over the term. For consistency with the option replication assumption in B&S, the equity model calibration has been set according to the risk neutral valuation with a zero equity risk premium. The shortfalls are discounted using risk free rates. The monitoring and rebalancing of the asset allocation is assumed to take place on a quarterly basis with no transaction cost or market impact. The volatility for the fixed interest asset is assumed to be zero for cash flow matched liabilities.

The assumed parameters for asset models are:

- The risk free rate equal to 5.12%
- Equity volatility equal to 19%

For sensitivity purposes, table 7-10 shows the distribution of the cost of the guarantees assuming the following:

- ZC bond with 11-year maturity is backing 10-year guarantees
- Fixed interest rate is stochastic following Heath-Jarrow-Merton one factor model with 16% volatility and 75% correlation with equity price

Rebalancing the EBR and hedging error

Implementing the switches using financial futures is recommended in order to minimise the transaction cost. Financial futures have the following benefits over the spot market:

- they involve a lower transaction cost, (up to 20 times lower than the cash market)
- they could be used to reduce the market impact
- they have a higher liquidity with an immediate implementation
- their daily settlement should improve the risk monitoring and highlight the appropriateness action to be taken
- they can be used to avoid unfavourable tax treatment.

It is likely that the financial futures will need to be rolled over, but this is consistent with regular monitoring of the asset allocation.

To avoid frequent small changes to the EBR, a switch trigger could be put in place. If the calculated EBR differs from the actual EBR by less than the switch trigger, the asset allocation remains unchanged. Floating EBR moves in the same direction as the dynamic asset allocation. Adopting a floating EBR between switches will reduce the size of the transaction and its associated cost.

The uncertainty about future equity volatility and interest rates movements may lead to an increase in the cost of the dynamic hedging. The dynamic asset allocation may generate an hedge error due to the following reasons:

- The option replication technique assumes a continuous rebalancing between risky and risk-free assets, which is not realistic due to the transaction cost. A more pragmatic and less frequent rebalancing lead to an additional replication error
- The transaction cost and taxation may have an adverse effect on the hedge cost
- Modelling error as the lognormal model does not capture all uncertainty about the equity price movements
- Managed funds have a single equity exposure, and it is not possible to implement a different EBR for every policy. Performance of the policies are linked to a single unit price reflecting the performance of the whole fund

7.2 Projected outcomes

The percentiles in this section reflect the equity price performance. The distributions of the risk outcomes are shown relative the equity price performance. This presentation will show that after a strong equity performance, dynamic EBRs using Call spread strategy are reduced.

The Monte Carlo technique has been used to generate 10,000 simulations with quarterly steps to derive the distribution of the fund performance and the projected EBRs for each investment strategy.

Equity performance

The following table shows the equity performance based on the risk neutral valuation and 1% annual management charge.

Table 7-1: The distribution of the equity performance net of 1% amc.

Percentile	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10
1st	-34%	-44%	-49%	-54%	-58%	-60%	-63%	-65%	-67%	-68%
5th	-25%	-33%	-37%	-40%	-44%	-46%	-49%	-50%	-51%	-53%
25th	-10%	-13%	-15%	-15%	-16%	-16%	-17%	-17%	-18%	-17%
40th	-3%	-3%	-2%	-1%	0%	1%	2%	4%	6%	7%
50th	2%	4%	6%	9%	11%	14%	16%	19%	22%	25%
60th	7%	12%	16%	20%	24%	28%	32%	38%	41%	46%
75th	16%	25%	33%	41%	48%	55%	64%	73%	81%	89%
90th	40%	63%	83%	103%	125%	146%	170%	195%	215%	239%
99th	59%	94%	132%	166%	202%	238%	271%	321%	363%	403%
Mean	4%	8%	13%	17%	22%	27%	32%	39%	45%	51%

Cost of the guarantees

The following table shows the distribution of the shortfall discounted at the risk-free rate as a proportion of the fund.

Table 7-2: Present value of the guarantees cost.

Percentile	Strategy_1	Strategy_2	Strategy_3	Strategy_4	Strategy_5	Strategy_6
1st	**-21%**	-2%	-5%	-5%	-5%	-7%
2nd	**-19%**	-3%	-8%	-8%	-8%	-10%
5th	**-16%**	-2%	-7%	-7%	-7%	-9%
25th	**-5%**	-1%	-4%	-3%	-3%	-5%
40th	0%	-2%	-6%	-6%	-6%	**-7%**
50th	0%	0%	0%	0%	0%	0%
Mean	-3.2%	-0.7%	-2.0%	-2.1%	-2.1%	-2.8%

The cost of the guarantees at 50th percentile and higher is nil.

Fund performance

The following table shows the fund performance net of annual management charge.

Table 7-3: Fund performance @ maturity

Percentile	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5	Strategy 6
1st	**-14%**	**19%**	13%	12%	12%	10%
5th	**-6%**	**18%**	9%	9%	9%	6%
25th	13%	**19%**	15%	15%	16%	12%
50th	**36%**	24%	27%	27%	25%	26%
75th	71%	55%	70%	72%	**74%**	71%
95th	152%	154%	156%	156%	154%	**165%**
99th	240%	**305%**	249%	227%	190%	270%
Mean	50%	50%	50%	50%	50%	50%

The strategies produce the same expected returns because the risk neutral valuation has been adopted in setting the parameters of the equity model. The green shading highlights the highest performance for every percentile. The floating EBR outperforms other dynamic strategies only in 50th percentile. The dynamic asset allocations give up some of the returns at the mid-percentile to improve the returns in other scenarios.

EBRs Projection

The following tables show the projected EBRs for the six investment strategies. The red shading represents EBRs below 10%, while green shading highlights EBRs above 75%.

Table 7-4: Projected EBRs based on Strategy 1

Percentile	t=0	T=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	54%	43%	38%	35%	31%	29%	27%	25%	23%	21%	20%
5th	54%	46%	42%	40%	37%	35%	33%	31%	30%	28%	27%
25th	54%	50%	49%	47%	46%	45%	44%	42%	41%	40%	39%
50th	54%	53%	53%	53%	52%	52%	51%	51%	50%	50%	50%
75th	54%	57%	57%	58%	59%	59%	59%	59%	60%	60%	60%
95th	54%	61%	64%	66%	67%	68%	69%	71%	72%	72%	73%
99th	54%	64%	68%	71%	73%	74%	76%	77%	78%	79%	80%
Mean	54%	53%	53%	53%	52%	52%	51%	51%	51%	50%	50%

Table 7-5: Projected EBRs based on Strategy 2

Percentile	t=0	T=1y	T=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	54%	22%	12%	7%	1%	1%	0%	0%	0%	0%	0%
5th	54%	28%	17%	14%	8%	2%	0%	0%	0%	0%	0%
25th	54%	43%	37%	27%	27%	14%	14%	9%	2%	0%	0%
50th	54%	53%	50%	50%	47%	42%	38%	40%	35%	28%	100%
75th	54%	62%	65%	65%	69%	71%	76%	79%	89%	87%	100%
95th	54%	76%	84%	88%	91%	95%	98%	99%	100%	100%	100%
99th	54%	82%	91%	95%	98%	99%	100%	100%	100%	100%	100%
Mean	54%	52%	51%	49%	47%	45%	43%	42%	40%	39%	46%

This table shows that the natural movements of the EBRs of strategy 2. It is possible to put an artificial limit o the EBR in this strategy but the aim is to highlight the volatility of the calculated EBR using the floating Call proportion method.

Table 7-6: Projected EBRs based on Strategy 3

Percentile	T=0	T=1y	T=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	54%	32%	19%	11%	3%	0%	0%	0%	0%	0%	0%
5th	54%	39%	29%	22%	14%	7%	2%	0%	0%	0%	0%
25th	54%	48%	44%	40%	37%	32%	27%	20%	13%	3%	0%
50th	54%	54%	53%	53%	52%	51%	50%	49%	48%	47%	52%
75th	54%	59%	60%	62%	63%	65%	66%	68%	71%	73%	74%
95th	54%	64%	67%	69%	70%	72%	72%	73%	73%	74%	74%
99th	54%	67%	69%	71%	72%	72%	73%	73%	73%	74%	74%
Mean	54%	53%	51%	50%	48%	47%	45%	43%	42%	40%	38%

Table 7-7: Projected EBRs based on Strategy 4 based on a Call spread

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	T=10y
1st	52%	32%	20%	12%	3%	0%	0%	0%	0%	0%	0%
5th	52%	39%	30%	23%	15%	8%	3%	0%	0%	0%	0%
25th	52%	47%	45%	41%	38%	34%	29%	21%	13%	3%	0%
50th	52%	52%	52%	52%	52%	52%	52%	52%	51%	51%	56%
75th	52%	56%	58%	59%	61%	63%	65%	68%	72%	76%	79%
95th	52%	59%	62%	63%	64%	64%	65%	65%	64%	63%	63%
99th	52%	61%	62%	63%	63%	63%	62%	62%	62%	63%	63%
Mean	52%	51%	50%	49%	48%	46%	45%	43%	42%	40%	39%

At maturity the EBRs for high equity percentiles are lower than the middle-percentiles.

Table 7-8: Projected EBRs based on Strategy 5 based on a Call spread

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	T=10y
1st	48%	34%	22%	13%	4%	0%	0%	0%	0%	0%	0%
5th	48%	40%	32%	25%	17%	10%	3%	0%	0%	0%	0%
25th	48%	46%	45%	43%	41%	37%	32%	24%	15%	4%	0%
50th	48%	49%	51%	52%	53%	54%	56%	57%	57%	56%	62%
75th	48%	51%	53%	55%	58%	61%	64%	68%	73%	81%	87%
95th	48%	51%	52%	52%	52%	52%	51%	50%	48%	45%	44%
99th	48%	50%	49%	48%	46%	45%	44%	43%	43%	43%	44%
Mean	48%	48%	48%	47%	46%	45%	44%	43%	42%	41%	39%

Table 7-9: Projected EBRs based on Strategy 6

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	61%	35%	21%	12%	4%	0%	0%	0%	0%	0%	0%
5th	61%	42%	32%	24%	16%	8%	3%	0%	0%	0%	0%
25th	61%	52%	48%	44%	40%	35%	29%	21%	14%	3%	0%
50th	61%	58%	58%	57%	56%	55%	54%	53%	52%	51%	56%
75th	61%	63%	65%	67%	68%	70%	71%	73%	76%	78%	79%
95th	61%	69%	72%	74%	76%	77%	78%	78%	79%	79%	79%
99th	61%	72%	75%	76%	77%	78%	78%	78%	79%	79%	79%
Mean	61%	57%	56%	54%	52%	51%	49%	47%	45%	43%	41%

Table 7-10: The distribution of the cost of guarantees assuming fixed interest rates follow a simple stochastic model (HJM one factor).

Percentile	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5	Strategy 6
1st	-21%	-15%	-11%	-11%	-11%	-14%
2nd	-20%	-10%	-5%	-5%	-5%	-6%
5th	-17%	-5%	-7%	-7%	-7%	-9%
25th	-5%	-1%	-3%	-3%	-4%	-5%
40th	0%	0%	0%	0%	0%	-2%
Mean	-3%	-2%	-2%	-2%	-2%	-2%

Table 7-11: The distribution of the fund performance assuming fixed interest rates follow a simple stochastic model (HJM one factor).

Percentile	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5	Strategy 6
1st	-15%	-3%	2%	2%	2%	-2%
2nd	-11%	5%	13%	13%	14%	11%
5th	-6%	12%	10%	10%	9%	7%
25th	13%	19%	16%	15%	15%	13%
40th	26%	23%	21%	21%	21%	19%
50th	36%	72%	25%	25%	24%	23%
60th	46%	37%	38%	39%	40%	37%
75th	69%	70%	71%	72%	74%	73%
90th	149%	51%	136%	137%	139%	144%
98th	197%	228%	205%	193%	173%	219%
99th	237%	172%	233%	221%	198%	252%
Mean	49%	49%	49%	49%	49%	49%

8 Summary and conclusions

The paper introduces a dynamic asset allocation built from the OBPI taking into account the market cost of the matching asset. Life offices and pension funds have been using common sense and resilience tests in setting and adjusting the asset allocation. This paper provides a market consistent framework to the active management of the asset allocation. The analysis shows that level of theoretical EBR increases with the level of interest rates and decreases with the level of the guarantee, equity volatility, taxation and the policy charges. Any mismatch in the fixed interest assets and the liability implies a lower EBR.

The optimal asset allocation is derived from the OBPI. This leads to the conclusion that any asset allocation at a given time could be described as portfolio insurance with a certain level of downside protection. The OBPI with 0% capital protection recommends 100% EBR, while a minimum guaranteed return equals to the free-risk rate implies 0% EBR. Any other capital protection levels give an EBR between these two extremes. The theoretical justification of an asset allocation could be expressed on the basis of this paper as the willingness to maximise exposure to a diversified equity portfolio and to achieve certain level of downside protection.

The dynamic asset allocation is aiming to achieve a given level of the downside protection, while a constant asset allocation implies that the protection level is adjusted implicitly with the fund performance.

The implication of the dynamic hedging is that a life office should abandon setting a targeted EBR. Instead, the matched position and close monitoring of EBR depending on the asset and the liability should be the norm. The level of EBR and the fixed interest portfolio benchmark should be set to maximise the chance of meeting the guarantees based on the option replication technique. The benchmarks should be reviewed to reflect the required changes to the EBR.

The calculated EBR depends on the level of the guarantees and the outstanding term. This implies that open-ended managed funds could not achieve an efficient dynamic hedging without a large cross-subsidy between policyholders. Tranche-based funds allow greater control over the market risk arising from the guarantees and give future investors the chance not to be impacted by the past performance. Tranche-based life funds could be seen as a compromise between structured products backed by a perfectly matching asset and the traditional open-ended managed fund.

9 Further research & references

Further research

The asset modelling supporting the dynamic hedging is based on the assumptions of the Black & Scholes option-pricing model. Changes to the asset allocation due to equity volatility and interest rates movements were shown as a sensitivity analysis. Further research is welcome to investigate the impact of less frequent switches between asset classes and stochastic behaviour of interest rates and equity volatility on the option replication technique. These challenges have been facing option-based derivatives experts since the initial Black, Scholes and Merton work. The benefit of actuaries carrying out such research is that they can lead to more practical applications for life offices and pension funds.

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I am very grateful to the Institute of Actuaries for the formal CPD offering actuaries the possibility to develop an expertise in derivatives. The materials supporting the CID and ACID and my knowledge of Guaranteed Equity Bonds inspired the development of the dynamic asset allocation presented in this paper.

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APPENDICES

A: Example of a Guaranteed Equity Bond

3-Year Guaranteed Equity Bond

Unlimited growth potential, with a minimum guarantee of 5% (1.64% AER)

Extracted from a Building Society Website.

When reading about this investment product you must read the Legal Information which contains all Terms and Conditions

Harness the growth of the stock market without any risk to your initial capital

- Three-year term
- Performance linked to the FTSE 100™ Index with 100% participation in growth
- No limit on the growth potential of your investment
- Guaranteed minimum return of 5% (1.64% AER)
- No charges or management fees
- Minimum investment only £500

An investment with unlimited potential for growth

Have you ever wanted to invest in the stock market but were afraid of the risks? Well now you can benefit from rises in the market and have the reassurance of a minimum 5% (1.64% AER) return on your investment .

The 3-Year Guaranteed Equity Bond is linked to the FTSE 100™ Index

Your investment will grow at the same rate as the Index and there's no restriction on the level of gains you can make.

Plus a minimum return of 5% over three years

What's special about this Bond is that even if the market falls over the three-year period, you will still receive a guaranteed return of 5% on your original capital. So you get the best of both worlds – the strength and security of a building society account and the exciting growth potential of the stock market.

And no charges or management fees

Unlike investing in the stock market itself, the Bond has no up-front charges or management fees to pay. So, whatever amount you invest, you can be sure that all of your money will be working for you.

How the 3-Year Guaranteed Equity Bond Works...

The 3-Year Guaranteed Equity Bond requires an investment over three years and returns are dependent on the performance of the FTSE 100™ Index. A minimum return of 5% interest on your investment is guaranteed, so you will either receive 100% of the growth in the FTSE 100™ Index or 5% interest (1.64% AER) whichever is higher.

Over the last few years share prices have followed a general downward trend. However, in recent months the market has begun to recover and there is potential for it to rise to previous levels. A stock market linked product such as the 3-Year Guaranteed Equity Bond would

allow you to capitalise on any future gains.
Limited offer

This exciting investment opportunity is available until the close of business on 16 April 2004. However, it may be withdrawn before this date should demand be higher than expected. So you should act quickly to avoid disappointment.

The start date for the Bond – i.e. the date on which the FTSE-linked performance becomes active – will be close of business on 30 April 2004. Up until midnight 29 April 2004, any money you invest will be placed in a feeder account and will receive interest at 3.50% gross p.a./AER. The capital plus interest will be transferred into the Bond on the start date. All transactions should be made by cheque or transfer. After the start date you will receive a certificate detailing your total investment amount.
What happens on maturity?

The Bond matures on 30 April 2007. The initial reading will be taken at close of business of the London Stock Exchange on the start date Friday 30 April 2004. For the final reading the level of the FTSE 100™ Index will be averaged out over the last six months of the three-year term. The final six month average is designed to provide you with additional protection on your investment, so that any potential fluctuations during this final period are taken into consideration.

Fourteen days after maturity you will have access to your capital and the interest the Bond has earned up to maturity. We require this time to calculate your interest payment and during these 14 days you will receive a variable rate equivalent to that of the Maturity Account (or equivalent account applicable at that time).

B: Assumption for Pricing GEB

Assumed implied volatility for FTSE-100 at 12th January

Strike	Y=3	Y=4	Y=5	Y=6	Y=7	Y=8	Y=9	Y=10
90%	16.8%	17.2%	17.4%	17.6%	17.8%	18.0%	18.2%	18.4%
95%	16.0%	16.4%	16.6%	16.8%	17.0%	17.2%	17.4%	17.6%
100%	15.3%	15.7%	15.9%	16.1%	16.3%	16.5%	16.7%	16.9%
105%	14.5%	14.9%	15.2%	15.4%	15.6%	15.8%	16.0%	16.2%
110%	13.8%	14.3%	14.5%	14.7%	14.9%	15.1%	15.3%	15.5%

Source UBS Investment Research with an interpolation of the assumption between 6 and 10 years.

The FTSE-100 implied volatility has been used as an approximation of FTSE All-share volatility. Historical comparison shows that the FTSE All-Share has a slightly lower volatility than the FTSE-100 index.

Swap rates and FTSE-100 dividend yields at 12th January 2004

Term	Dividend yield	Swap Rate
3	3.14%	4.67%
4	3.03%	4.76%
5	2.96%	4.82%
6	2.89%	4.88%
7	2.82%	4.94%
8	2.75%	5.00%
9	2.68%	5.06%
10	2.61%	5.12%

Source UBS Investment Research with an interpolation of the assumption between 6 and 10 years.

Assumed initial charges for pricing GEB.

Policy Term	Initial Charge
3	4.00%
4	4.50%
5	4.50%
6	5.00%
7	5.50%
8	6.00%
9	6.50%
10	7.00%

Call option prices

The following table shows the Call price based on the FTSE-100 for different terms and different exercise prices.

Call option price on the FTSE-100 using B&S formula

Guarantee	Maturity							
	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
90%	17%	20%	21%	23%	24%	26%	27%	29%
95%	14%	17%	18%	20%	22%	23%	25%	27%
100%	11%	14%	16%	18%	19%	21%	23%	24%
105%	9%	11%	13%	15%	17%	19%	20%	22%
110%	7%	9%	11%	13%	15%	16%	18%	20%

Call option price on the FTSE All-shares using B&S formula

Guarantee	Maturity							
	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
90%	24%	29%	32%	36%	39%	42%	46%	49%
95%	21%	25%	29%	33%	36%	39%	43%	46%
100%	18%	22%	26%	29%	33%	37%	40%	43%
105%	14%	19%	23%	26%	30%	34%	37%	40%
110%	11%	16%	20%	23%	27%	31%	34%	38%

C: Derivation of the Call option formula

Deriving appropriate nominal and the strike price of the Call option is the preliminary step before tackling the development of the asset allocation. The expression of the revised OBPI using traded Call option is as follows:

$$\Pi_0 = ZC_0 + \lambda_0 \cdot Call_0$$

Let us assume that:

$Call\_BS\_European(S_0, \frac{X_0}{\Pi_0} \cdot S_0)$ represents the B&S Call option formula with underlying asset price S_0, strike price $\left(\frac{S_0 \cdot X_0}{\Pi_0}\right)$ and maturity T.

To achieve the participation in the equity growth consistent with the GEB, the nominal of the Call option should be equal to $\frac{\Pi_0}{S_0}$. The payout at maturity needs to be checked to make sure the definition of the Call option gives the expected payout profile. The explicit B&S formula of such a Call option is:

$$Call_0 = \frac{\Pi_0}{S_0} \cdot Call\_BS\_European(S_0, \frac{X_0}{\Pi_0} \cdot S_0)$$

Writing the B&S formula explicitly gives the following expression:

$$Call_0 = \frac{\Pi_0}{S_0}\left(S_0 \cdot N(d_1) - \frac{ZC_0}{\Pi_0} \cdot S_0 \cdot N(d_2)\right)$$

Where:

$$d_1 = \frac{\ln\left(\frac{\Pi_0}{ZC_0}\right) + \left(\sigma^2/2\right) \cdot T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The Call option formula could be simplified to:

$$Call_0 = \Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_2)$$

At maturity, the payout of such a portfolio is as follows:

$$\Pi_T = X + \lambda_0 \cdot \frac{\Pi_0}{S_0} \cdot Max(S_T - \frac{X}{\Pi_0} \cdot S_0, 0)$$

$$\Pi_T = X + \lambda_0 \cdot \left(\Pi_0 \cdot \frac{S_T}{S_0} - X \right)^+$$

The payout of such GEB is:

- $\Pi_T = X + \lambda_0 \cdot (\Pi_0 \cdot \frac{S_T}{S_0} - X)$ for $\frac{S_T}{S_0} \succ \frac{X}{\Pi_0} \cdot$
- $\Pi_T = X$ otherwise.

As expected, the maturity value gives a payout consistent with the benefit of the GEB, indicating clearly the linkage to equity performance. For a money back guarantee $(X = \Pi_0)$, the maturity payout is:

$$\Pi_T = \Pi_0 + \lambda_0 \cdot \left(\Pi_0 \cdot \frac{S_T}{S_0} - \Pi_0 \right)^+ = \Pi_0 \cdot \left(1 + \lambda_0 \cdot \left(\frac{S_T}{S_0} - 1 \right)^+ \right)$$

D: Introduction to delta hedging

The delta (Δ) of an option is the most important parameter in hedging and replicating options. It is the rate of change in the option price with respect to the price of the underlying asset. The change in the price of an option due to a small movements in the price of the underlying asset should be equal to the change in price of the risky asset times the delta. The B&S analysis shows that it is possible to set up a risk-free portfolio consisting of a short position in a derivative and a delta position in the underlying risky asset over a small period of time. This could be expressed as follows:

$$\Pi = -f + \Delta \cdot S$$

Where:

- S is a risky asset
- f is a derivative based on underling asset S
- Δ is the delta of f relative to S, such $\Delta = \frac{\partial f}{\partial S}$

Within a short period of time, the change to the derivative value is offset by the change to the price of the underlying risky asset. The stochastic element of the derivative f and the underlying asset S cancel out. Assuming the no-arbitrage argument, the return of the portfolio Π should be equal to the risk-free interest rate r. This assumption leads directly to the process of replicating options using the delta position of the risky asset and risk-free bond. To maintain the portfolio risk-free (or delta neutral) over a longer period, the position in the risky asset needs to be adjusted on a continuous basis, in order to reflect the change to the delta.
Ito lemma gives the following equation:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

Where:

- $dS_t = u \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz$

$$df = (\frac{\partial f}{\partial S} u \cdot S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 \cdot S^2) \cdot dt + \frac{\partial f}{\partial S}\sigma \cdot S \cdot dz$$

- z is a stochastic variable following a Brownian motion
- u the drift of S and σ is the volatility

Hull (2000) states that the equation above has no stochastic variable assuming that $\frac{\partial f}{\partial t}$ and $\frac{\partial^2 f}{\partial S^2}$ are deterministic functions. The no-arbitrage argument gives the following equation:

$$\Delta\Pi = r \cdot \Pi \cdot \Delta t$$

Replacing Π and $\Delta\Pi$ by its equation leads directly to Black-Scholes-Merton differential equation for any derivative:

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 \cdot S^2\right) = r \cdot \left(-f + \frac{\partial f}{\partial S} \cdot S\right) \Rightarrow \left(r \cdot \frac{\partial f}{\partial S} \cdot S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \cdot \sigma^2 \cdot S^2\right) = r \cdot f$$

The B&S option formulae are obtained by solving of this general equation for a specific derivative. Detailed analysis of the process of the replicating portfolio and delta hedging technique based on the PDF approach and no arbitrage argument can be found in Hull (2000).

An alternative and more elegant technique to the PDF approach can be found in Baxter & Rennie (1996) who derive B&S option formulae and the replicating portfolio using Martingales.

E: Adjustment for income tax and AMC

In the life assurance business, there are two major features to be taken into account in developing the dynamic asset allocation to hedge the cash guarantee. These two features are the taxation and the annual management charge. The new requirement of the OBPI is that ZC bond should be increased to meet the guarantee, the management charge and the income tax. In the GEB, the linkage of equity performance is net of tax for onshore products but this rule does not apply to managed funds. This section shows how the ZC bond could be increased to accommodate the income tax and the management charge. In deriving the EBR, the adjustment for tax and amc will be applied to the aggregated ZC bond exposure.

To allow for income tax, the following parameters will be used:

- X_0^{Tax} is the amount invested in the ZC bond at time zero to meet the guarantee and the income tax at maturity
- *Tax* is the rate of the income tax rate

At maturity the ZC bond needs to cover the guarantee amount and the income tax. This could be expressed as follows:

$$X_T^{Tax} = X + Taxation = X + \left(X_T^{Tax} - X_0^{Tax}\right) \cdot Tax$$

Expressing the initial guarantee, as a discounted maturity payout gives the equivalent formula

$$X_T^{Tax} = X + \left(X_T^{Tax} - X_T^{Tax} \cdot e^{-r \cdot T}\right) \cdot Tax$$

The value of the guarantee including tax is derived as follows:

$$X_T^{Tax} = \frac{X}{\left(1 - \left(1 - e^{-r \cdot T}\right) \cdot Tax\right)}$$

The amount of cash to be invested in ZC bond is increased to:

$$ZC_0^{\ Tax} = \frac{X \cdot e^{-r \cdot T}}{\left(1 - \left(1 - e^{-r \cdot T}\right) \cdot Tax\right)} = \frac{ZC_0}{\left(1 - \left(1 - e^{-r \cdot T}\right) \cdot Tax\right)}$$

Allowance for income tax increases the proportion of the asset invested in the ZC bond. This leads to a lower Call proportion and to a lower linkage to equity performance:

$$\lambda_0^{Tax} = \frac{\left(\Pi_0 - ZC_0^{Tax}\right)^+}{\Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_2)} \leq \frac{\left(\Pi_0 - ZC_0\right)^+}{\Pi_0 \cdot N(d_1) - ZC_0 \cdot N(d_2)} = \lambda_0$$

The following expression $\frac{1}{\left(1-\left(1-e^{-r\cdot T}\right)\cdot Tax\right)}$ is greater than or equal to 1 leading to higher proportion invested in the ZC bond $\left[ZC_0^{Tax} \geq ZC_0\right]$.

The annual management charge has a similar effect to the income tax. The investment in the ZC bond needs to be higher to cover the contractual guarantees and the annual management charge. The following expression of the ZC bond could be used to allow for the AMC:

$$ZC_0^{(amc,tax)} = \frac{ZC_0 \cdot (1+m)^T}{\left(1-\left(1-e^{-r\cdot T}\right)\cdot Tax\right)}$$

Where m is the annual management charge.

In theory the AMC and tax should have an impact on the delta hedging process similar to the dividend income. The dividend income of the Call replication should be set equal to the AMC leading to lower EBR. The impact of the tax and the AMC on delta hedging has been ignored in calculated EBR in the table below. Taking into accounts the AMC and tax in the replication gives a reduced initial EBR allowing for the future charges. Alternatively, ignoring the AMC and tax on the delta hedging means that the EBR will be reduced over time.

The following tables show the reduction in the EBR due to income taxation and AMC. The taxation and the AMC are taken into account in the aggregate ZC bond holding.

EBR (excluding income tax and AMC)

	Maturity (in years)							
Guarantee (X)	**T=3Y**	**T=4Y**	**T=5Y**	**T=6Y**	**T=7Y**	**T=8Y**	**T=9Y**	**T=10Y**
90%	74%	75%	77%	79%	80%	82%	83%	85%
95%	65%	68%	72%	74%	77%	79%	81%	83%
100%	54%	60%	65%	69%	73%	76%	78%	81%
105%	40%	50%	57%	63%	68%	72%	75%	78%
110%	22%	37%	48%	56%	63%	68%	72%	75%

EBR with 20% income tax

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	68%	68%	69%	70%	70%	71%	71%	72%
95%	60%	62%	64%	65%	67%	68%	69%	70%
100%	49%	53%	57%	60%	62%	64%	66%	67%
105%	35%	43%	49%	54%	57%	60%	62%	64%
110%	17%	31%	40%	47%	52%	56%	59%	61%

This table indicates that the calculated EBRs are reduced due to the income tax. The reduction in the EBR is higher with longer maturities. The level of the guarantee has a little effect on the reduction of the EBR. The reduction in the EBR is about 5% for three-year guarantee and increases gradually to 14% for ten-year guarantee.

EBR with 1% AMC

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	68%	68%	69%	70%	70%	71%	71%	71%
95%	60%	61%	63%	65%	66%	67%	68%	69%
100%	48%	53%	57%	60%	62%	64%	65%	67%
105%	35%	43%	49%	53%	57%	60%	62%	64%
110%	17%	30%	39%	46%	51%	55%	58%	61%

The reduced EBRs due to 1% AMC are very similar to the EBRs with 20% income tax.

EBR adjusted with 20 income tax with 1% AMC

	Maturity (in years)							
Guarantee (X)	T=3Y	T=4Y	T=5Y	T=6Y	T=7Y	T=8Y	T=9Y	T=10Y
90%	67%	67%	68%	68%	68%	69%	69%	69%
95%	58%	60%	62%	63%	64%	65%	66%	67%
100%	47%	51%	55%	57%	60%	61%	63%	64%
105%	33%	41%	47%	51%	54%	57%	59%	61%
110%	15%	28%	37%	43%	48%	52%	55%	57%

The reduction of the EBR due to the AMC and the income tax combined is much higher.

The EBR is on average 7% lower for three-year guarantee and 17% lower for ten-year guarantee.

E: Market impact on the theoretical EBRs

	EBRs with initial assumption							
Guarantee (X)	**T=3Y**	**T=4Y**	**T=5Y**	**T=6Y**	**T=7Y**	**T=8Y**	**T=9Y**	**T=10Y**
90%	74%	75%	77%	79%	80%	82%	83%	85%
95%	65%	68%	72%	74%	77%	79%	81%	83%
100%	54%	60%	**65%**	69%	73%	76%	78%	81%
105%	40%	50%	57%	63%	68%	72%	75%	78%
110%	22%	37%	48%	56%	63%	68%	72%	75%

	EBRs with 10% drop in the equity prices							
Guarantee (X)	**T=3Y**	**T=4Y**	**T=5Y**	**T=6Y**	**T=7Y**	**T=8Y**	**T=9Y**	**T=10Y**
90%	67%	70%	72%	75%	77%	79%	81%	82%
95%	59%	63%	67%	70%	73%	76%	78%	80%
100%	49%	55%	60%	65%	69%	72%	75%	78%
105%	37%	46%	53%	59%	64%	68%	72%	75%
110%	21%	35%	45%	52%	59%	64%	68%	72%

The change to the EBRs due equity price movements is based on floating delta and constant Call proportion. The sensitivity to asset parameters is based on floating Call proportion by resetting the OBPI with the adjusted fund value.

	EBRs with 25% in increase to equity volatility							
Guarantee (X)	**T=3Y**	**T=4Y**	**T=5Y**	**T=6Y**	**T=7Y**	**T=8Y**	**T=9Y**	**T=10Y**
90%	65%	67%	69%	71%	73%	75%	77%	78%
95%	57%	60%	64%	67%	69%	72%	74%	76%
100%	47%	52%	57%	61%	65%	68%	71%	74%
105%	34%	43%	50%	56%	60%	64%	68%	71%
110%	18%	32%	41%	49%	55%	60%	64%	68%

	EBRs with 25% drop in the yield curve							
Guarantee (X)	**T=3Y**	**T=4Y**	**T=5Y**	**T=6Y**	**T=7Y**	**T=8Y**	**T=9Y**	**T=10Y**
90%	69%	70%	71%	73%	74%	76%	77%	78%
95%	60%	63%	65%	68%	70%	72%	74%	75%
100%	49%	54%	58%	62%	65%	68%	70%	72%
105%	35%	44%	50%	55%	59%	63%	66%	69%
110%	18%	31%	41%	48%	53%	58%	62%	65%

F: Fund performance for the case study

Strategy 1

Percentile	0	1	2	3	4	5	6	7	8	9	10
1%	0%	-17%	-18%	-21%	-21%	-21%	-19%	-18%	-18%	-16%	-14%
5%	0%	-11%	-12%	-14%	-14%	-13%	-13%	-12%	-9%	-8%	-6%
25%	0%	-3%	-3%	-2%	-1%	1%	2%	6%	7%	10%	13%
50%	0%	3%	6%	9%	12%	15%	18%	21%	25%	29%	34%
75%	0%	9%	17%	23%	28%	33%	40%	46%	53%	62%	66%
95%	0%	23%	37%	49%	61%	74%	84%	97%	106%	124%	138%
99%	0%	34%	54%	73%	93%	105%	130%	149%	162%	197%	227%
Expectation	0%	4%	8%	12%	16%	20%	25%	30%	35%	41%	47%

Strategy 2

Percentile	0	1	2	3	4	5	6	7	8	9	10
1%	0%	-13%	-12%	-11%	-8%	-4%	-1%	4%	8%	13%	17%
5%	0%	-10%	-10%	-9%	-7%	-3%	-1%	5%	10%	13%	16%
25%	0%	-4%	-3%	-3%	-1%	2%	4%	7%	11%	14%	19%
50%	0%	2%	5%	6%	8%	10%	10%	14%	14%	17%	20%
75%	0%	9%	15%	21%	25%	27%	34%	36%	39%	50%	47%
95%	0%	24%	38%	51%	64%	84%	93%	112%	116%	142%	155%
99%	0%	39%	61%	83%	114%	129%	160%	190%	200%	256%	294%
Expectation	0%	4%	8%	12%	16%	20%	25%	30%	35%	40%	47%

Strategy 3

Percentile	0	1	2	3	4	5	6	7	8	9	10
1%	0%	-16%	-15%	-16%	-12%	-9%	-5%	-1%	3%	7%	11%
5%	0%	-11%	-11%	-12%	-11%	-9%	-6%	0%	4%	7%	11%
25%	0%	-3%	-3%	-3%	-2%	1%	2%	6%	9%	9%	12%
50%	0%	3%	5%	8%	10%	13%	15%	18%	16%	19%	20%
75%	0%	9%	17%	23%	27%	32%	39%	45%	48%	61%	64%
95%	0%	23%	38%	51%	63%	77%	88%	102%	111%	131%	149%
99%	0%	35%	55%	75%	97%	111%	135%	156%	169%	205%	238%
Expectation	0%	4%	8%	12%	16%	20%	25%	30%	35%	41%	47%

Strategy 4

Percentile	0	1	2	3	4	5	6	7	8	9	10
1%	0%	-15%	-15%	-16%	-13%	-9%	-6%	-1%	3%	7%	11%
5%	0%	-11%	-11%	-11%	-11%	-9%	-7%	0%	4%	6%	11%
25%	0%	-3%	-3%	-2%	-1%	1%	2%	6%	8%	9%	12%
50%	0%	3%	6%	8%	11%	13%	16%	18%	17%	19%	19%
75%	0%	9%	17%	23%	28%	32%	39%	46%	51%	61%	64%
95%	0%	22%	37%	49%	61%	75%	86%	99%	108%	129%	148%
99%	0%	33%	52%	70%	91%	104%	126%	145%	157%	187%	217%
Expectation	0%	4%	8%	12%	16%	20%	25%	30%	35%	41%	47%

Strategy 5

Percentile	0	1	2	3	4	5	6	7	8	9	10
1%	0%	-15%	-15%	-16%	-13%	-10%	-6%	-2%	2%	6%	10%
5%	0%	-10%	-10%	-11%	-11%	-9%	-7%	0%	4%	5%	10%
25%	0%	-2%	-2%	-1%	-1%	2%	2%	6%	8%	8%	12%
50%	0%	3%	6%	10%	12%	15%	18%	19%	19%	20%	18%
75%	0%	9%	17%	23%	28%	33%	40%	49%	55%	61%	64%
95%	0%	20%	35%	46%	58%	71%	82%	94%	102%	125%	145%
99%	0%	29%	47%	63%	80%	92%	110%	126%	136%	157%	182%
Expectation	0%	4%	8%	12%	16%	20%	25%	30%	36%	41%	47%

Strategy 1

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	0%	-16%	-20%	-20%	-21%	-21%	-20%	-19%	-18%	-16%	-14%
5th	0%	-12%	-14%	-14%	-14%	-13%	-13%	-11%	-10%	-8%	-6%
25th	0%	-4%	-3%	-2%	0%	2%	4%	5%	8%	10%	13%
50th	0%	3%	6%	9%	13%	16%	20%	24%	28%	32%	36%
75th	0%	11%	17%	24%	30%	36%	42%	49%	57%	64%	71%
95th	0%	23%	38%	51%	63%	78%	91%	107%	123%	136%	152%
99th	0%	34%	55%	77%	98%	119%	141%	161%	191%	216%	240%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%

Strategy 2

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	T=8y	t=9y	t=10y
1st	0%	-14%	-13%	-11%	-9%	-4%	-1%	4%	7%	13%	19%
5th	0%	-11%	-11%	-9%	-6%	-4%	0%	4%	9%	13%	18%
25th	0%	-4%	-3%	-4%	0%	-1%	4%	7%	10%	13%	19%
50th	0%	3%	4%	8%	10%	11%	13%	17%	19%	21%	24%
75th	0%	10%	16%	19%	26%	31%	39%	43%	56%	47%	55%
95th	0%	25%	43%	55%	69%	90%	111%	119%	144%	157%	154%
99th	0%	37%	65%	92%	120%	149%	183%	202%	244%	281%	305%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%

Strategy 3

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	T=8y	t=9y	t=10y
1st	0%	-15%	-17%	-15%	-14%	-9%	-6%	-1%	1%	8%	13%
5th	0%	-11%	-13%	-12%	-11%	-9%	-5%	-1%	5%	7%	9%
25th	0%	-4%	-3%	-3%	0%	-1%	3%	4%	9%	8%	15%
50th	0%	3%	6%	9%	12%	14%	17%	20%	22%	24%	27%
75th	0%	10%	17%	24%	30%	36%	42%	49%	57%	61%	70%
95th	0%	24%	39%	52%	66%	80%	96%	112%	130%	144%	156%
99th	0%	35%	56%	80%	101%	124%	145%	169%	199%	226%	249%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%

Strategy 4

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	0%	-15%	-16%	-15%	-14%	-9%	-6%	-2%	1%	8%	12%
5th	0%	-11%	-13%	-12%	-11%	-9%	-5%	-1%	4%	7%	9%
25th	0%	-3%	-3%	-3%	0%	0%	3%	5%	8%	8%	15%
50th	0%	3%	6%	9%	12%	15%	18%	21%	23%	25%	27%
75th	0%	10%	17%	24%	30%	37%	43%	50%	59%	63%	72%
95th	0%	23%	37%	50%	65%	77%	92%	108%	128%	139%	156%
99th	0%	33%	53%	74%	93%	115%	132%	157%	183%	207%	227%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%

Strategy 5

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	t=8y	t=9y	t=10y
1st	0%	-14%	-16%	-15%	-14%	-10%	-6%	-2%	1%	7%	12%
5th	0%	-10%	-12%	-11%	-10%	-9%	-6%	-1%	4%	7%	9%
25th	0%	-3%	-2%	-1%	1%	1%	3%	5%	7%	8%	16%
50th	0%	3%	7%	10%	13%	16%	19%	22%	24%	27%	25%
75th	0%	10%	17%	25%	31%	38%	45%	51%	62%	67%	74%
95th	0%	21%	34%	47%	62%	70%	85%	102%	123%	128%	154%
99th	0%	30%	46%	66%	81%	98%	110%	135%	156%	173%	190%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%

Strategy 6

Percentile	t=0	t=1y	t=2y	t=3y	t=4y	t=5y	t=6y	t=7y	T=8y	t=9y	t=10y
1st	0%	-17%	-19%	-17%	-17%	-11%	-8%	-4%	-2%	5%	10%
5th	0%	-13%	-15%	-14%	-13%	-11%	-7%	-4%	2%	4%	6%
25th	0%	-5%	-4%	-4%	-2%	-3%	1%	2%	6%	5%	12%
50th	0%	3%	5%	8%	11%	13%	16%	19%	21%	22%	26%
75th	0%	11%	18%	24%	30%	36%	43%	50%	58%	62%	71%
95th	0%	26%	42%	56%	70%	86%	102%	118%	138%	153%	165%
99th	0%	38%	61%	86%	109%	135%	158%	182%	215%	246%	270%
Mean	0%	4%	8%	13%	17%	22%	27%	32%	38%	44%	50%