DYNAMIC DYNAMIC-PROGRAMMING SOLUTIONS FOR THE PORTFOLIO OF RISKY ASSETS

MARK S TENNEY

1998 GENERAL INSURANCE CONVENTION
AND
ASTIN COLLOQUIUM

GLASGOW, SCOTLAND: 7-10 OCTOBER 1998

Dynamic Dynamic-Programming Solutions for the Portfolio of Risky Assets

Mark S. Tenney *

April 28, 1995

Abstract

Dynamic programming solutions for optimal portfolios in which the solution for the portfolio vector of risky assets is constant were solved by Merton in continuous time and by Hakansson and others in discrete time. There is no case with a closed form solution where this vector of risky asset holdings changes dynamically. This paper derives such solutions for the first time, and is thus a dynamic dynamic-programming solution as opposed to a static dynamic-programming solution for this vector. The solution is valid when there is a set of basis assets whose excess expected return is linear in the state vector, whose variance-covariance matrix is time-dependent and for which the interest rate is a quadratic function of the state vector.

Classification codes from Journal of Economic Literature: C61, G11.

Key words: dynamic programming, non-linear quadratic problem, linear risk premium, vector auto regression, multivariate quadratic interest rate model.

We consider the optimal portfolio problem for the case where the investment opportunity set is stochastic. We allow the state variables to follow a vector Gaussian process with linear dynamics among the state variables, and with time-dependent coefficients. We assume that interest rates are quadratic in the state variables. We allow the expected asset returns to be linear in the state vector, and the variance matrix to be time-dependent. We solve the case of power utility with no consumption during the horizon. We allow both the dimension of the state vector and the number of risky assets to be unrestricted. We solve cases with both complete and incomplete markets.

After this solution, we extend our solution to cover cases where the assets follow more general processes, but a transformation can be made to a basis that satisfies our assumptions. We show that this can happen quite generally with derivative securities, if the risk premia coefficients across assets are linear in the state vector. We give a specific treatment of the time-dependent case of Beaglehole and Tenney's Multivariate Quadratic Interest Rate Model [1], [2]. The solution for the prices of bonds and the state price for this case of BT's model was indicated to be possible in the footnotes of their paper. It was independently worked out by Eterovic [6] and by Tenney [15], who also constructed a general equilibrium economy for this model. Jamshidian [9] considerably simplified some of the integrals in the time-dependent case for evaluating derivative prices.

Merton [11], [12] developed a framework for solving continous time optimal consumption and portfolio problems. He solved a variety of problems which we summarize below. In discrete time, similar work was developed by Hakansson [7], actually considerably earlier than Merton and Samuelson, and by Samuelson [14], although not for the three more exotic cases solved by Merton and discussed below. Since then considerable progress has been made in solving his problem with the addition of non-negativity constraints on wealth and consumption to the constant investment problems considered by him, see Cox and Huang [3] and Karatzas et. al. [10]. The methodology of Cox and Huang can be applied to additional problems once one solves for the joint density of asset prices and the value of the portfolio of a lognormal investor with on consumption. The Cox Huang methodology requires complete markets. Merton [13] has developed an alternative approach, where one solves for the state price for this economy. So far, this more powerful methodology has been limited by the cases where the above joint distribution can be solved, which heretofore have been quite limited. The results of this paper, together with the results of Tenney [15] allow the determination of a related joint density, one that includes the above quantities and also the state vector in our different information structure.

Cox, Ingersoll and Ross [4], [5] solved a dynamic programming problem with no risk free borrowing or lending in equilibrium, and in which a single stochastic factor influenced asset returns proportionately. In this case, the optimal holdings of the risky assets were constant, and the utility function was logarithmic. In addition, to the cases discussed previously, the case of power utility with one risky asset and no consumption during the horizon when the interest rate follows a one factor normal process with constant drift was solved by Ingersoll [8].

Merton solved the general HARA utility with constant coefficients for an unrestricted set of n assets with lognormal returns in a finite horizon in Merton [12] which is reproduced on page 139 of Merton [13], he then solves for the additional complication of constant non-capital income. He then solves for a set of complications in the assumptions on the environment, when there is only one equity asset. The first set concern Poisson processes, and we shall not address those type of processes here.

He then considers three variations on the case where the equity asset follows a lognormal process, except that the expected return is linear in a state variable. In two of the examples, he allows for two specific forms of time-dependency in the coefficients. In each case he solves for the case of exponential utility in an infinite horizon. His three cases have quite interesting interpretations. The first is where there is a long-run normal price to which market prices are converging on a proportional basis, so that the absolute level of prices can still fluctuate arbitrarily. The second case is where the expected return follows a mean reverting process. The third case is where an investor estimates the expected return based on a time-series of observations.

We can summarize this area of work by stating that Merton developed a general framework for continuous time dynamic programming, but that closed form solutions were difficult to obtain. The solved problems were for the case that the investment opportunity set was not stochastic or contained only limited stochastic elements. Considerable work has been done since then on handling non-negativity constraints in the unconstrained problems with solutions, but little work has been done on extending the problems that can be solved to non-trivial dynamic environments. This paper provides solutions for the optimal portfolio problem for significant dynamic environments.

1 Economic Framework

We assume a variation of the information structure in the Cox, Ingersoll and Ross general equilibrium model [4]. In our formulation, we allow for the case that the primary assets form a dynamically complete market by themselves as well as the case that they do not dynamically complete the market. This information structure together with our assumptions on asset returns are considered in Tenney [15], who shows that there is a general equilibrium economy in which these assumptions are realized. For our information structure, we assume that there is a $k \times 1$ vector of state variables, that is governed by $dY = \mu^Y(Y,t)dt + S(t)dw$, where $\mu^Y(Y,t) = b(t) + A(t)Y$ where S is $k \times p$ and a function of time t only, and dw is a $p \times 1$ vector of independent Wiener processes. There is an $n \times 1$ vector of asset prices $P(Y,t)$ [1], which evolve according to $dP = I_P\alpha dt + I_P Gdw$, where G is $n \times p$, and I_P is a $n \times n$ diagonal matrix with diagonal elements P_i. We do not require that $n + k = p$ as do Cox, Ingersoll and Ross. We have the equation for wealth, W $dW = W[a'(\alpha - r) + r]dt + Wa'Gdw$, where a is the $n \times 1$ vector of asset holdings, and $a_0 = 1 - a'1$ is the level of short term lending, or if negative borrowing. We allow a to be positive or negative to reflect short selling. We assume no transaction costs. The investor maximizes $J(W,Y,t) = E[U(W,T)]$, where $U(W,T) = W^\gamma/\gamma$.

---

[1] We shall sometimes use the CIR notation of η for P

2   Solving the Dynamic Programming Equation with Power Utility

We now solve the dynamic programming problem for power utility with no consumption during the horizon for the case of an interest rate quadratic in the state vector, Y, for expected returns linear in Y, with time-dependent coefficients, and for the variance-covariance matrix of returns being only time-dependent. We actually solve a more general case than this, which is stated in the theorem, by specifying that certain coefficient functions be time-dependent, linear in Y with time-dependent coefficients or quadratic in Y with time-dependent coefficients. After the theorem, we discuss the relation of this more abstract set of assumptions to the more intuitive set discussed above.

We define a small piece of notation for convenience. If V is a quadratic form, $V = V^0 + Y'V^1 + Y'V^2Y$, then we define $L_0[V] = V^0$, $L_1[V] = V^1$, and $L_2[V] = V^2$.

Theorem 1 *Suppose an investor has no consumption over a period, and has utility of final wealth at T, $U = W^\gamma/\gamma$, then the solution for the indirect utility function J is $J(W,Y,t) = f(Y,t)W^\gamma/\gamma$, where $f(Y,t) = B^{qn}(Y,t,T|D^{qn}, b^{qn}, A^{qn}, S^{qn}, R^{2qn}, R^{1qn}, R^{0qn})$ which is a function defined in Theorem 4 where $S^{qn} = SS'$ is time-dependent only,*

$$D^{qn} = \frac{1}{2}\gamma(\gamma - 1)a^{1'}GG'a^1 + \gamma a^{1'}GS' \tag{1}$$

and D^{qn} is time-dependent only

$$V = \gamma[a^{0'}(\alpha - r) + r] + \frac{1}{2}(\gamma)(\gamma - 1)a^{0'}GG'a^0 \tag{2}$$

and V is quadratic in Y, with time-dependent coefficients,

$$\mu^{qn} = \gamma(\alpha - r)'a^1 + \gamma(\gamma - 1)a^{0'}GG'a^1 + \mu'_Y + \gamma(a^0)'GS' \tag{3}$$

and μ^{qn} is linear in Y with time-dependent coefficients, and for $i = 0,1,2$, we define $R^{iqn} = -L_i[V]$, where the operator L_i acting on a quadratic form was defined previously, $b^{qn} = L_0[\mu^{qn}]$, $A^{qn} = L_1[\mu^{qn}]$ The optimal portfolio weights are

$$a(Y,t) = a^0(Y,t) + 2a^1(t)\Gamma(t)Y \tag{4}$$

The level of short term lending, or borrowing is the scalar

$$a_0(Y,t) = 1 - 1'a = 1 - 1'a^0(t) - 21'a^1(t)\Gamma(t)Y \tag{5}$$

where the $n \times 1$ vector a^0 is

$$a^0(Y,t) = (GG')^{-1}(\alpha(Y,t) - r(Y,t)1)/(1 - \gamma) \tag{6}$$

and the $n \times 1$ vector a^1 is

$$a^1(t) = G^{-1'}S'/(1 - \gamma) \tag{7}$$

Proof.

The Bellman equation is

$$J_t + J_W(a'(\alpha-r)+r)W + \mu_Y' J_Y + \tfrac{1}{2}W^2 J_{WW} a'GG'a + W a'GS' J_{WY} + \tfrac{1}{2}\mathrm{tr}\,(SS'J_{YY}) = 0$$

subject to the boundary condition

$$J(W,Y,T) = W^\gamma/\gamma \tag{8}$$

Assume a solution for J of the form

$$J(W,Y,t) = f(Y,t)W^\gamma \frac{1}{\gamma} \tag{9}$$

We remind ourselves that a'1 need not be equal to 1, since we can have short term borrowing or lending. The Bellman equation becomes

$$f_t + [\gamma[a'(\alpha-r)+r] + \tfrac{1}{2}(\gamma)(\gamma-1)a'GG'a]f + [\mu_Y' + \gamma a'GS']f_Y + \tfrac{1}{2}\mathrm{tr}\,(SS'f_{YY}) = 0$$

The equation for the optimal portfolio weights is

$$\alpha - r1 + GG'a \cdot \frac{WJ_{WW}}{J_W} + GS'\frac{J_{WY}}{J_W} = 0 \tag{10}$$

This simplifies to the following under our assumed solution for J

$$\alpha = r(Y,t)1 + GG'a^*(1-\gamma) - GS'\frac{f_Y}{f} \tag{11}$$

We can solve for a, we drop the asterisk,

$$a = (GG')^{-1}(\alpha - r(Y,t)1)/(1-\gamma) + G^{-1'}S'\frac{f_Y}{f}/(1-\gamma) \tag{12}$$

We can write this as

$$a^0 = (GG')^{-1}(\alpha - r(Y,t)1)/(1-\gamma) \tag{13}$$

$$a^1 = G^{-1'}S'/(1-\gamma) \tag{14}$$

$$a = a^0 + a^1 f_Y/f \tag{15}$$

We substitute this into the equation for f, to obtain the following equation for f that depends only on α, r, G and S, $\mu^{Y'}$ all of which are given functions.

$$f_t + [\gamma[(a^0+a^1f_Y/f)'(\alpha-r)+r] + \tfrac{1}{2}(\gamma)(\gamma-1)(a^0+a^1f_Y/f)'GG'(a^0+a^1f_Y/f)]f + [\mu_Y' + \gamma[(a^0 + a^1 f_Y/f)'GS']f_Y + \tfrac{1}{2}\mathrm{tr}\,(SS'f_{YY}) = 0$$

We can group terms to obtain

$$f_t + [\gamma[a^{0'}(\alpha - r) + r] + \tfrac{1}{2}(\gamma)(\gamma - 1)a^{0'}GG'a^0]f + [\gamma(\alpha - r)'a^1 + \gamma(\gamma - 1)a^{0'}GG'a^1 +$$
$$\mu'_Y + \gamma(a^0)'GS']f_Y + f'_Y[\tfrac{1}{2}\gamma(\gamma - 1)a^{1'}GG'a^1 + \gamma a^{1'}GS']f_Y/f + \tfrac{1}{2}\text{tr}\ (SS'f_{YY}) = 0$$

We define S^{qn}, D^{qn}, μ^{qn} and V as in the theorem, and they are the coefficients of f in the equation. We further define R^{iqn}, b^{qn} and A^{qn} as in the theorem. This equation is of the form of the quadratic non-linear problem, Theorem 4, if S^{qn} and D^{qn} are time-dependent only, μ^{qn} is linear in Y with time-dependent coefficients and V is quadratic in Y with time-dependent coefficients. The solution for f is, from Theorem 4,

$$f(Y, t) = B^{qn}(Y, t, T | D^{qn}, b^{qn}, A^{qn}, S^{qn}, R^{2qn}, R^{1qn}, R^{0qn}) \tag{16}$$

In this case

$$f_Y = 2\Gamma Y f \tag{17}$$

So the portfolio weights solution is

$$a = a^0 + 2a^1\Gamma Y \ ^\cdot \tag{18}$$

We can solve for the weight of short term lending as

$$a_0 = 1 - 1'a = 1 - 1'a^0 - 21'a^1\Gamma Y \tag{19}$$

QED.

The conditions on a solution are expressed in terms of S^{qn}, D^{qn}, V, and μ^{qn}, which are not customary quantities. We can consider some non-exhaustive conditions that lead to these quantities necessary fulfilling the conditions imposed by the theorem. The theorem is satisfied, if the following conditions are met: (1)a^0 is linear in Y; (2) $G'a^1$ is independent of Y; (3)$a^{0'}G$ is linear in Y (4) r is quadratic in Y. These in turn are achieved if (1) a^0 is linear in Y; (2) G is independent of Y; (3) a^1 is independent of Y; (4) r is quadratic in Y. These conditions in turn are realized if G is time-dependent only, $\alpha - r$ is linear in Y, with time-dependent coefficients, and $\mu^{Y'}$ is linear in Y with time-dependent coefficients, and r is quadratic in Y with time-dependent coefficients.

We can summarize our results as follows. We have solved for the case that $\alpha - r$ is linear in Y, G is time-dependent only, S is time-dependent only, and r is quadratic in Y. For this case, we solved for f, and then obtain the closed form solution for the optimal weights a, which is linear in Y, and obtained the amount in the risk free asset, which is also linear in Y. The joint density of the state vector Y, portfolio wealth W, and the values of the asset prices for portfolios which need not be optimal has already been solved for in Tenney [15]. That result applies to portfolios and asset environments of the type solved for here.

3 Basis transformations

If we have a set of traded assets which does not solve the assumptions of Theorem 1, then we can look for transformations to a new set of assets that do solve those assumptions, but which do not give up any of the trading opportunities available with the first set. We call such transformations, basis transformations. In our analysis, we assume no arbitrage, which means that there is no portfolio that is riskless that earns a rate of return that is different than that of the risk free rate, r. We consider theorems in this section and its subsections that show that if security returns satisfy some simple relationships, which are motivated by equilibrium and no-arbitrage results, then the problem of finding a basis that satisfies the assumptiions of Theorem 1 is simplified.

Let $dx_i = dP_i/P_i$, i=1,...,n. Then we have

$$dx = \alpha dt + G dw \tag{20}$$

We write G in the form

$$G_{jk} = G_{jk}^0 + G_{jk}^{1m} Y^m \tag{21}$$

where G^0 depends on t only, and G^1 can depend on Y and t. In this equation and throughout the rest of the paper, we employ a summation convention in which repeated indices such as m in the previous equation are automatically summed over their range, in this case, n^Y, the dimension of Y.

A basis transformation consists of a set of portfolios of the retuns on the original assets that replicates the new returns. We form portfolios, indexed by i of the original assets, indexed by j=1,...,m. We assume that the new basis is non-singular, i.e. has no redundant assets, but the original basis can be singular. The weight of original asset j in portfolio i is given by a_{ij}. Let dx_j be the rate of return of the original asset j, and dy_i be the rate of return of portfolio i over dt. The return to this portfolio is then given by

$$dy_i = a_{ij} dx_j \tag{22}$$

We shall say that a set of processes dy spans G dw, or equivalently spans dx if using the set dy, each element of dx can be replicated, i.e. form a process that is perfectly correlated over dt, whether with coefficients that are a non-linear function of Y or t or not. If the coefficients a_{ij} of the transformation from dx to dy are functions of t only, and if dy itself spans dx, then we call the transformation from dx to dy a non-stochastic basis transformation. For the transformed basis, we have

$$\alpha_i^* = a_{ij} \alpha_j \tag{23}$$

and

$$G_{ik}^* = a_{ij} G_{jk} \tag{24}$$

We assume that $\sum_j a_{ij} = 1$ for each i=1,...,m.

Theorem 2 *If $\alpha_j - r = \Lambda_j^0 + \Lambda_k^1 G_{jk}^{1m} Y_m$, for $j=1,...,n$, and Λ^0 is linear in Y, and Λ^1 and G^1 can depend on Y and t, then if there is a non-stochastic basis transformation from the original assets dx, to a basis dy, given by a_{ij}, such that $G^* = a_{ij}G_{jk}^0$ is independent of Y, then $\alpha^* - r = a_{ij}\Lambda_j^0$) is linear in Y, and if r is quadratic in Y, we can solve the optimal portfolio optimization problem for power utility in the new basis. For an optimal portfolio $b_i, = 1,...,m$ in the new basis, the holdings in the original basis are $c_j = b_i a_{ij}$.*

Proof

We can write in component form

$$dx_j = \alpha_j dt + G_{jk}dw_k \tag{25}$$

We can write

$$dy_i = a_{ij}\alpha_j dt + a_{ij}G_{jk}dw_k \tag{26}$$

and substituting for G

$$dy_l = a_{ij}\alpha_j dt + a_{ij}(G_{jk}^0 + G_{jk}^m Y^m)dw_k \tag{27}$$

One chooses a_{ij}, such that

$$a_{ij}G_{jk}^m = 0 \tag{28}$$

for each i and m, but such that

$$a_{ij}G_{jk}^0 \neq 0 \tag{29}$$

and such that the resulting set of assets spans the space of Gdw with a resulting variance-covariance matrix in the new basis that is not singular. The variance covariance matrix is given by $\Sigma_{im} = a_{ij}G_{jk}^0 G_{kl}^{0'} a_{lm}'$. We can simply require that $|\Sigma| \neq 0$ in order to insure non-singularity.

For our new basis, we have

$$dy_i = (r + a_{ij}\Lambda_j^0)dt + a_{ij}G_{jk}^0 dw_k \tag{30}$$

since $a_{ij}G_{jk}^{1m}Y^m\Lambda_k = 0$ by virtue of $a_{ij}G_{jk}^{1m}Y^m = 0$ for all k, which follow from Equation 28 We thus have a new basis with $\alpha_i^* - r = a_{ij}\Lambda_j^0$ that is linear in Y, and $G_{ik}^* = a_{ij}G_{jk}^0$. The solutions for the a_{ij} are time-dependent if the G matrix has time-dependent coefficients. However, if the G coefficients are independent of time, then we can choose a coefficients that are as well. In either case, we obtain a basis which satisfies the assumptions of the problem for which we have a closed form solution.

QED.

Thus, if the G_{jk}^0 are not systematically related to the G_{jk}^{1m}, and the initial set of assets is sufficiently large, with no systematic relationship among the G_{jk}^{1m}, and if G_{jk}^{1m} are polynomials in Y with time-dependent coefficients, then we can choose a set that satisfies our requirements.

3.1 Beaglehole and Tenney's Multivariate Quadratic Interest Rate Model

When the asset returns are generated by a no-arbitrage pricing model, or equilibrium pricing model, then we can consider a special basis transformation. We now consider the application of this approach to Beaglehole and Tenney's Multivariate Quadratic Interest Rate Model [1], which we shall refer to as BT. In Tenney [15] it is shown that this model obtains in a general equilibrium economy, and that in that economy the risk premia are linear in Y. That paper also obtains the solution for a zero coupon bond price B with time-dependent coefficients. That solution is of the form $B(Y, t, T) = e^{Y'\Gamma Y + \gamma'Y + \zeta}$, where Γ, γ and ζ are time-dependent functions dependent on t and T, but not Y. This is the same form as obtained in the original solution by BT. BT had indicated in their footnotes that a solution for time-dependent b and A was possible but did not derive it. An independent solution of the time-dependent case of BT's model was obtained by Eterovic [6].

There are two main issues to applying our solution methodology to a set of bond prices governed by this model, together with a set of equity assets. The first is that the transformation that changes G from linear in Y to time-dependent only, also simultaneously reduces the expected return on zero coupon bonds from non-linear in Y to linear in Y, if the risk premia coefficients are themselves linear in Y. The second is that we can look for a transformation based on all possible zero coupon bonds that might exist, not merely those that are actually traded. We can then construct a basis for the theoretical zero coupon bonds used to solve the optimal portfolio problem with any traded securities, including coupon bonds, futures and options and other instruments.

Theorem 3 *If out of all possible zero coupon bonds that could exist, there is a basis for a transformation of the type in Theorem 2 for these zero coupon bonds together with the available equity assets, that preserves the space spanned by the original equities plus all possible zero coupon bonds, then under that transformation, the optimal portfolio problem can be solved if the risk premia coefficients for bond pricing are linear in the state vector Y. Furthermore, if the set of traded bonds plus equities spans the space of the equities and all possible bonds, then the plan can be implemented with traded assets only.*

Proof.

Under no arbitrage, the return of the ith bond, or other derivative, is given by

$$\frac{dB^i}{B^i} = (r + \lambda'S'\frac{B_Y^i}{B^i})dt + dY'\frac{B_Y^i}{B^i} \tag{31}$$

where λ' is independent of i, and is a common risk premia coefficient across assets.

This in turn becomes

$$\frac{dB^i}{B^i} = (r + \lambda'S'\frac{B^i_Y}{B^i})dt + dw'S'\frac{B^i_Y}{B^i} \tag{32}$$

In the BT model, we have $B^i_Y/B^i = 2\Gamma^i Y + \gamma^i$ for zero coupon bonds, where Γ^i and γ^i depend on t, the maturity date, the process parameters and the risk premia time-dependent coefficients. We thus have

$$dB^i/B^i = (r + \lambda'S'(2\Gamma^i Y + \gamma^i))dt + dw'S'(2\Gamma^i Y + \gamma^i) \tag{33}$$

We choose a_{ij}, such that $a_{ij}\lambda'S'\Gamma^j = 0$, which follows by our assumption that a transformation of the previous form is possible. Thus we obtain

$$dy_i = (r + \lambda'S'a_{ij}\gamma^j)dt + dw'S'a_{ij}\gamma^j \tag{34}$$

where the dy_i include bonds and equities. If now λ' is linear in Y, then we obtain the linear return, time-dependent variance form that we need. This linear form of the risk premium has been shown to occur in the multivariate quadratic production economy, which is a general equilibrium economy in which the multivariate quadratic interest rate model is realized, see Tenney [15]. Because the Γ^i and γ^i form a continuum of quantities, the requirement of a basis transformation is a weaker condition than for a fixed set of zero coupon bonds, such as those traded.

However, we do not need a continuum of traded assets to exist. Whatever starting basis we arrive at by picking from the continuum, we only need to have a set of traded securities that are a basis for that set. This latter basis, need not even be a basis of zero coupon bonds, but can be any set of derivative contracts including coupon bonds, and exchange traded futures and options.

QED.

The approach applied to the BT model can be generalized for general derivative contracts, both within their model or within another derivative pricing model that is based on no arbitrage. We note that the result obtained was a consequence of the risk premium being linear in Y and proportional to the elasticity vector, B^i_Y/B^i, Y follwing a vector autoregression diffusion with $S(t)$ being time-dependent only, the risk premia coefficient being linear in Y, and the elasticity vector being linear in Y. However, as indicated in the comments after Theorem 1, we can apply the basis transformation, even when the G matrix is non-linear. We see that the cancellation of terms in Y, that occurs in the transformed G term, also occur in the transformed α term, even for non-linear in Y terms. As a consequence, if we can make this transformation to simplify G to a time-dependent coefficient, then we will simultaneously eliminate the Y dependence in α^* due to the Y dependence in the elasticity. This leaves the Y dependence in λ. If this function is linear in Y, then we obtain a basis that is tractable for our methodology. This includes a very wide variety of models.

Furthermore, as in the theorem, we do not require that all possible derivatives be traded, only that the traded set be a basis for the derivatives used to make the transformation to the new basis. In this way, in making this transformation, we can consider a continuum of derivatives to choose from to make the transformation, if the traded basis spans the space of Gdw. Because of this, the condition on the transformation requires only finding a finite set of points in this continuum that satisfy the necessary conditions of the transformation.

4   Conclusion

What we have done is to greatly expand the application of dynamic programming in portfolio problems. We have expanded what can be solved for from problems in which very little is changing, to ones in which the state vector can follow a vector autoregressive markov diffusion with time-dependent coefficients, in which the expected return vector in some basis is linear in Y, and the variance-covariance matrix is time-dependent.

We have shown that such a basis can be constructed for derivatives priced under no-arbitrage when the risk premia coefficient is linear in Y, and Y follows a vector autoregression diffusion. Such a risk premium occurs in the general equilibrium economy developed by Tenney for problems with the information structure assumed here. The most notable application is to the multivariate quadratic interest rate model of Beaglehole and Tenney.

Furthermore, this approach works for empirically based expected return and variance-covariance functions for equity returns which are linear in the state variable for the expected return and quadratic in the variance-covariance matrix. This framework is therefore sufficient for solving the optimal portfolio problem with power utility for fairly general and realistic assumptions on stochastic returns and variance-covariances for equities and a stochastic interest rate model with a closed form solution, namely BT, for bonds that is of arbitrary dimension and has a number of desirable properties.

The problems solved here are examples of the original goal of dynamic programming applied to portfolio problems, namely developing a dynamic optimal strategy for non-trivial dynamic securities markets where the interest rate was stochastic and bond and equity returns had stochastically varying expected returns and variance-covariance matrices. Furthermore the solution methods are sufficiently robust that derivative contracts can be added to the mix, and the optimal strategies obtained making use of those contracts.

A   Non-Linear Quadratic Problem

The following problem was solved in Tenney [15].

Theorem 4 (Non-linear Quadratic Problem) *We consider the non-linear differential equation,*

$$\frac{1}{2}B_{yy} \cdot S(t) + B_y' D B_y / B + (b(t) + A(t)y)' B_y + B_t - r(y,t)B = 0 \qquad (35)$$

where $r(y,t) = R_0(t) + R_1'(t)y + y'R_2(t)y$, *and* D *and* R_2 *are symmetric. The solution, subject to the boundary condition* $B(y,T) = 1$ *is given by*

$$B^{qn}(y,t,T|D,b,A,S,R_0,R_1,R_2) = e^{y'\Gamma(t)y + \gamma'y - \int_t^T \eta(t')dt'} \qquad (36)$$

where Γ *is symmetric, and where*

$$2\Gamma'(S + 2D)\Gamma + A'\Gamma + \Gamma A + \dot{\Gamma}_t - R_2 = 0 \qquad (37)$$

$$\gamma'(2S + 4D)\Gamma + \gamma'A + 2b'\Gamma + \gamma_t' - R_1' = 0 \qquad (38)$$

$$\Gamma \cdot S + \frac{1}{2}\gamma'(S + 2D)\gamma + b'\gamma + \eta - R_0 = 0 \qquad (39)$$

This theorem can be verified by direct substitution of the proposed solution in the partial differential equation for f, and then observing that this equation is trivially satisfied as a consequence of the 3 final equations listed in the theorem.

References

[1] David Beaglehole and Mark Tenney. General solutions to some interest rate contingent claims pricing equations. *Journal of Fixed Income*, September 1991.

[2] David Beaglehole and Mark Tenney. Corrections and additions to,"a nonlinear equilibrium model of the term structure of interest rates". *Journal of Financial Economics*, 32(3):345–354, December 1992.

[3] John Cox and Chi-fu Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49:33–83, 1989.

[4] John Cox, Jonathon Ingersoll, and Stephen A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384, March 1985.

[5] John Cox, Jonathon Ingersoll, and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[6] Adrian Luis Eterovic. *Essays on multifactor models of the term structure of interest rates.* PhD dissertation, Harvard University, Department of Economics, May 1994.

[7] Nils H. Hakansson. Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica*, 38(5):587–607, September 1970.

[8] Jonathan Ingersoll. *Theory of Financial Decision Making.* Rowman and Littlefield, Totowa, New Jersey, 1987.

[9] Farshid Jamshidian. Bond, futures, and option evaluation in the quadratic interest rate model. Technical report, Fuji International Finance PLC, 7-11 Finsbury Circus, London EC2M 7NT, March 1993.

[10] Ioannis Karatzas, John P. Lehoczky, and Steven Sethi, Suresh P.and Shreve. Explicit solution of a general consumption/investment problem. *Mathematics of Operation Research*, 11(2):261–294, May 1986.

[11] Robert C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, (51):247–257, August 1969.

[12] Robert C. Merton. Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, (31):373–413, 1971.

[13] Robert C. Merton. *Continuous-Time Finance.* Basil Blackwell, Cambridge, Massachusetts 02142, 1990.

[14] Paul A. Samuelson. Lifetime portfolio selection by dynamic programming. *Review of Economics and Statistics*, (51):239–246, August 1969.

[15] Mark S. Tenney. Methodology for solvable general equilibrium economies. Technical report, Alexandria, Virginia, 1994.