Institute and Faculty of Actuaries

Ruin Theory Starter Kit #1

The Classical Model with Mixed Exponential Claims

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Abstract

The aim of the paper is to provide general insurance actuaries a starting point for evaluating and engaging with ruin theory in a practical manner. Towards this goal, the paper undertakes three tasks. First, it sets out the basic mathematical principles behind the classical model with mixed exponential claim distributions. In particular, this part contributes to the reference literature in that the discussion is extraordinarily detailed. Second, it highlights risk management questions that could be contemplated already by this classical model. A simple but novel modification of the model is discussed to allow dividend considerations to influence the dynamics. Third, suggestions of recent enhancements of the model are made for the practitioners to further consider.

For better appreciation of the concepts, a demonstration spreadsheet accompanies this paper.

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August 2013

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1   Introduction

The ground-breaking ideas of Filip Lundberg's 1903 thesis echo to our present day. On the one hand, the collective risk model is certainly one of the most important tools for general insurance actuaries. On the other hand, the idea of estimating ruin probabilities has had a mixed journey. Every year, the dozens of academic papers published in this area attest to the high level of interest in this topic in actuarial science research. Equally loud is the silence from practitioners on this subject in publicly available literature.

Research has improved the original Lundberg model a great deal. For example, rather than simple exponential severity distributions, the much more flexible mixed exponential distributions can now be used. In this year, the 110th anniversary of the thesis, a joint practitioner-academia study has been initiated – and it is still underway as this paper is being written.[3] The study's focus is novel: it aims to test how well ruin theory models could respond to modern risk management questions in general insurance companies, using realistic data. *Can Ruin Theory be easily adapted into useful tools for the general insurance actuary or risk manager, or will it remain only a quaint curiosity for our actuarial students?*

This present paper aims to introduce or refresh the topic to practitioners in a high-level manner, so that the readers would be equipped to evaluate the applicability of ruin theory concepts in their individual contexts. It outlines the mixed exponential model, highlights how it can potentially be used by the actuary, and provides some suggestions for them to further investigate the topic. Key fundamental mathematical concepts are clarified for the reader for further engagement with published papers in the area. The paper is intended neither as a survey paper nor as a detailed textbook. Such reference material does exist (see, for example, (Asmussen & Albrecher, 2010)), but they would not be helpful for providing a quick overview of the topic. More detailed results from the collaboration study will be discussed in a September 2013 seminar in Liverpool and naturally in the GIRO conference, 2013.

On first reading of the paper, we recommend that Sections 2.2 and 2.3 be skipped. These two sections contain details of the fundamental mathematical ideas used in ruin theory. However, we recommend that the reader return to them for more detailed study and fluent application of the theory.

Additionally, to help the reader get better acquainted with the ideas in this paper, we have put together a demo spreadsheet. This is introduced in Section 3 below.

The authors would like to thank the following colleagues for their helpful comments during the drafting of the paper:
- The members – both supervisors and students – of the aforementioned practitioner-academia study;
- Hansjoerg Albrecher, Andrew Couper, Ronnie Loeffen, Stephen Postlewhite and Enrique Thomann

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[3] The study involves the collaboration between Aspen's actuarial R&D and the Institute for Financial and Actuarial Mathematics at the University of Liverpool. The Aspen's side is led by the second author of this paper. Eight MSc. students (*) and their five supervisors / teachers (+) from the University of Liverpool are involved: Yunzhou Chen (*), Jiajia Cui (+), Suhang Dai (*), Jacob Hamanenga (*), Olivier Menoukeu-Pamen (+), Apostolos Papaioannou (+), Haoyu Qian (*), David Siska (+), Vasiliki Traga (*), Meng Wang (*), Xiaochi Wu (*), Jie Zhou (*) as well as the first author (+).

2   The Mixed Exponential Model and Probability of Ultimate Ruin

Consider that an insurance company starts with an initial capital, *u*, receives premiums at a constant rate *c* and pays claims arriving according to a compound Poisson process

$$S(t) = \sum_{k=1}^{N(t)} X_k,$$

where:
- $N(t)$ denote the number of claims up to time *t* with i.i.d. claim interarrival times, τ_k
- X_k represent the i.i.d. claim amounts with density f_X and finite mean

The classical collective risk model of Lundberg[4] can then be defined by
$$U(t) = u + ct - S(t).$$
One of the main goals is the analysis of the probability of ruin, where ruin is attained when *U(t)* is less than zero for some *t*. The probability of ruin is denoted by
$$\Psi(u) = \mathrm{P}(U(t) < 0, \text{ for some } t | U(0) = u).$$

Some basic assumptions for this classical model are:
- the time between claims is exponentially distributed with parameter *λ* (equivalently, for all positive *t*, $N(t)$ is Poisson distribution with parameter *λt*)
- the independence between the claim interarrival times and the claim severities,
- and the net profit condition, meaning that on average the company receives more in premiums than it pays out in claims (i.e. $c > \lambda \mathrm{E}(X)$)

One of the building blocks of the ruin theory models is the Lundberg equation
$$\mathrm{E}(e^{sX})\mathrm{E}(e^{-cs\tau}) = 1,$$
where т and *X* have the same distributions with any τ_k and X_k, respectively. The number and nature of solutions of the Lundberg equation is well analysed in the literature.

2.1   Equations for the ruin probability

The ruin theory literature abounds in method of calculating, approximating or asymptotically analysing the ruin probability. The methods vary from probability arguments, complex analysis, Wiener-Hopf factorisation to analysis of solutions of integro-differential equations (IDEs). We present here this last method, which for a large class of distributions permits algorithmic ways to derive exact solutions of differential equations, and thereby allowing explicit forms for $\Psi(u)$.

The starting place is observing the following relationship[5], obtained by conditioning on the time and the size of the first claim
$$\Psi(u) = \mathrm{E}\big(\Psi(u + cT_1 - X_1)\big).$$
Here, T_1 and X_1 are random variables and represent the time and the size, respectively, of the first claim. Using the assumption of independence, we have
$$\Psi(u) = \int_0^\infty \lambda e^{-\lambda t} \int_0^\infty \Psi(u + ct - x) f_X(x) dx \, dt.$$
Through integration by parts we then have the well-known IDE for the ruin probability
$$-c\Psi'(u) + \lambda\Psi(u) = \lambda \int_0^\infty \Psi(u - x) f_X(x) dx.$$
It is convenient to rewrite the differential part of the equation in operator form

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[4] The Swedish mathematician Harald Cramér is often jointly acknowledged for this model, for recognition of his publicising and developing of Flip Lundberg's ideas, which were only initially accessible by Swedish speakers.

[5] This type of relationship has deep links with *renewal theory*, which is outside the scope of this paper.

$$\left(-c\frac{d}{du} + \lambda\right)\Psi(u) = \lambda \int_0^\infty \Psi(u-x)f_X(x)dx.$$

The RHS can also be written in two parts, making use of the fact that $\Psi(v) = 1$ whenever v is negative:

$$-c\Psi'(u) + \lambda\Psi(u) = \lambda \int_0^u \Psi(u-x)f_X(x)dx + \lambda \int_u^\infty f_X(x)dx \qquad (1)$$

Before we carry on, it is useful to note the following properties of Ψ under this model:

- As u tends to infinity, $\Psi(u)$ tends to 0. Considering the simple purpose and form of the model, this is a reasonable property.
- The ruin probabilities should be a monotonically decreasing function of initial capital. Again, this is a characteristic proportionate to the assumptions of the model.
- Integrating equation (1), assuming the infinite capital point above, performing a swap in the order of integration to the first term, and recognising that $E(X) = \int_0^\infty \left(1 - F_X(u)\right)du = \int_0^\infty \int_u^\infty f_X(x)dx\,du$, we arrive at $\Psi(0) = \frac{\lambda E(X)}{c}$. This property can no doubt be scrutinised further, but we suggest that it is not a grossly unreasonable step for broadbrushed modelling.

The key mathematical result of this section is the following.

> When X is a mixture of n exponential distributions, then the probability of ultimate ruin has the form
> $$\Psi(u) = \sum_{i=1}^n C_i e^{-r_i u},$$
> for some constants C_i, provided that r_i are the positive solutions of the Lundberg equation, which is an (n+1)th order polynomial equation, with zero as one of its roots. The constants C_i's are determined by n initial conditions for $\Psi(0), \Psi'(0), \Psi''(0), \ldots, \Psi^{(n-1)}(0)$, derived from equation (1).

On first reading of this paper, the reader may want to skip the rest of this section and proceed to the considering risk management questions in Section 3. However, the material of this section contains fundamental mathematical ideas in ruin theory, and we strongly recommend its familiarity for fluent and serious use of ruin theory in practical applications.

We now present two methods of solving the above equation for the simple case when X is exponentially distributed. This then gives a springboard to the case when X is mixed exponentially distributed. The two methods are (1) through Laplace transforms and (2) through turning the IDE into an ordinary differential equation (ODE) that we can then solve. One would expect at least the elementary aspects of these methods covered in core undergraduate courses in mathematics or engineering. Good introductory references exist online and in print.

2.2 Exponential claims

Assume that the claim amounts are exponentially distributed with rate β (i.e. mean $1/\beta$), so that, for positive x,

$$f_X(x) = \beta e^{-\beta x}.$$

Then one can solve the IDE (1) via one of the two methods, as mentioned above.

2.2.1 Laplace transformation

Denote the Laplace transform of a function f by

$$\hat{f}(s) = \int_0^\infty e^{-st}f(t)dt.$$

For example, for the exponential function, $f(x) = e^{ax}$, the Laplace transform has the simple form of $\hat{f}(s) = \int_0^\infty e^{-sx}e^{ax}dx = 1/(s-a).$

The Laplace transform is an important tool for solving differential equations, used widely in the physical applied sciences, as well as, of course, in ruin theory. In probability theory, the Laplace transforms of densities are (almost!) the moment generating functions. The key steps are

- the differential equation is Laplace-transformed, term-by-term into an algebraic equation
- solve the algebraic equation
- perform inverse Laplace transform on the solution of the algebraic equation to obtain the solution for the original differential equation

The following properties of the Laplace transform will be helpful for us.

- The transform preserves linearity, which will be useful in particular when we consider the mixed exponential case later
- The transform of the convolution is just the product of the transforms: this is particularly handy for dealing with the RHS of equation (1)
- The transform of the derivative is simply, $\widehat{f}'(s) = s\widehat{f}(s) - f(0)$, an important property that helps create simple algebraic equations
- For our purposes, there is a one-to-one correspondence between functions and their Laplace transforms: this is clearly critical to perform the inverse Laplace transform after solving the algebraic equation

The Laplace transform of the IDE (1) then gives

$$-c\left(s\widehat{\Psi}(s) - \Psi(0)\right) + \lambda\widehat{\Psi}(s) = \lambda\widehat{\Psi}(s)\widehat{f_X}(s) + \lambda\frac{1 - \widehat{f_X}(s)}{s}, \qquad (2)$$

with solution

$$\widehat{\Psi}(s) = \frac{-c\Psi(0) + \lambda\dfrac{1 - \widehat{f_X}(s)}{s}}{-cs + \lambda\left(1 - \widehat{f_X}(s)\right)}.$$

When the claim amounts are exponentials, this reduces to

$$\widehat{\Psi}(s) = \frac{-c\Psi(0) + \lambda\dfrac{1}{\beta + s}}{-cs + \lambda\dfrac{s}{\beta + s}} = \frac{\Psi(0)}{s + \left(\beta - \dfrac{\lambda}{c}\right)}.$$

The second equality makes use of the assumption that $\Psi(0) = \frac{\lambda}{\beta c}$ (see Section 2.1 above).

By simple observation that the RHS is just the Laplace transform of an exponential function, we see that

$$\Psi(u) = \Psi(0)e^{-Ru},$$

where $R = \beta - \frac{\lambda}{c}$ is nothing else but the positive solution of the Lundberg equation.

2.2.2 Ordinary differential equation

Recall that, under quite general differentiability and continuity conditions,

$$\frac{d}{du}\int_{a(u)}^{b(u)} g(u,x)dx = g\left(u, b(u)\right)b'(u) - g\left(u, a(u)\right)a'(u) + \int_{a(u)}^{b(u)} \frac{\partial}{\partial u}g(u,x)dx. \quad (3)$$

The reader will also remember that the solution of the *m*th order homogeneous ODE with constant coefficients

$$\sum_{j=0}^{m} a_j y^{(j)} = 0$$

depends on the roots of the characteristic equation

$$\sum_{j=0}^{m} a_j s^j = 0 \qquad (4)$$

In the simplest cases, when equation (4) has distinct roots, $\{\tilde{r}_1, \dots, \tilde{r}_m\}$ then all the solutions to the ODE are of the form

$$y(x) = \sum_{j=1}^{m} C_j e^{\check{r}_i x},$$

for some constants $\{C_1, \ldots, C_m\}$. Solutions for cases when there are repeated roots and complex roots are also available. Note that to help make clear Ψ is a decreasing function of u, we usually re-define the constants in the exponent, so that the resulting form of Ψ is a combination of $e^{-\check{r}_i x}$, where $\check{r}_i$ are positive constants.

Applying the operator $\left(\frac{d}{du} + \beta\right)$ to the RHS of IDE (1), using rule (3) once on each of the two terms, integrating once by-parts on the first term, and the properties of exponentials, one derives an ODE with constant coefficients

$$\left(\frac{d}{du} + \beta\right)\left(-c\frac{d}{du} + \lambda\right)\Psi(u) = \lambda\beta\Psi(u)$$

which can be solved via characteristic polynomials

$$(s + \beta)(-cs + \lambda) - \lambda\beta = 0. \tag{5}$$

Note that this is equivalent to

$$\frac{\beta}{(s + \beta)}\frac{\lambda}{(-cs + \lambda)} = 1,$$

which is just the Lundberg equation. Factoring out the quadratic equation

$$s(-cs + (\lambda - c\beta)) = 0,$$

one can write the exact ruin probability in this special case,

$$\Psi(u) = C_1 + C_2 e^{-\left(\beta - \frac{\lambda}{c}\right)u}.$$

Here the boundary condition $\Psi(\infty) = 0$ gives us $C_1 = 0$, leaving us with $C_2 = \Psi(0)$. Thus we arrive once again at the celebrated form of ruin probability

$$\Psi(u) = \Psi(0)e^{-Ru}.$$

2.3   Mixed Exponential Claims

When $f_X(x) = \sum_{i=1}^{n} A_i \beta_i e^{-\beta_i x}$ for positive x, then one can show that

$$\Psi(u) = \sum_{i=1}^{n} C_i e^{-r_i u},$$

for some constants C_i, provided that r_i are the distinct positive solutions of the Lundberg equation

$$\left(\sum_{i=1}^{n} A_i \frac{\beta_i}{\beta_i + s}\right)\left(\frac{\lambda}{-cs + \lambda}\right) = 1. \tag{6}$$

are distinct.

As in the simple exponential case, this can be derived either by using Laplace transform, or by solving the ODE derived from the IDE (1).

For Laplace transform, we have, from equation (2)

$$\widehat{\Psi}(s) = \frac{-c\Psi(0) + \lambda\dfrac{1 - \widehat{f}_X(s)}{s}}{-cs + \lambda\left(1 - \widehat{f}_X(s)\right)} = \frac{-c\Psi(0) + \lambda\dfrac{1 - \widehat{f}_X(s)}{s}}{-cs + \lambda\left(1 - \sum_{i=1}^{n} A_i \dfrac{\beta_i}{\beta_i + s}\right)}.$$

As $\widehat{\Psi}(s)$ is a rational function, and observing that the denominator is zero whenever s is some r_i (by definition), using the theory of partial fractions, and assuming distinct roots to (6), we can write

$$\widehat{\Psi}(s) = \frac{C_0}{s} + \sum_{i=1}^{n} \frac{C_i}{r_i + s},$$

where each term – apart from the first one – can be easily Laplace-invert back into an exponential. The first term inverts to the constant function for $x > 0$. This then gives $C_0 = 0$ to satisfy $\Psi(\infty) = 0$.

If, on the other hand, we start with the IDE (1), then applying the operators $\left(\frac{d}{du} + \beta_1\right)$, $\left(\frac{d}{du} + \beta_2\right)$, ..., $\left(\frac{d}{du} + \beta_n\right)$ successively, and using similar techniques as with the simple exponential case, we obtain

$$\prod_{i=1}^{n}\left(\frac{d}{du} + \beta_i\right)\left(-c\frac{d}{du} + \lambda\right)\Psi(u) = \lambda\sum_{i=1}^{n} A_i\beta_i \prod_{i\neq j}\left(\frac{d}{du} + \beta_j\right)\Psi(u). \qquad (7)$$

The characteristic equation has a similar form:

$$\prod_{i=1}^{n}(s + \beta_i)(-cs + \lambda) = \lambda\sum_{i=1}^{n} A_i\beta_i \prod_{i\neq j}(s + \beta_j)$$

which by inspection is just a rearrangement of the Lundberg equation (6). The result then follows when the roots to (6) are distinct.

At least two interesting questions arise. First, what if we have repeated roots for (6)? Solutions do exist using the theory of characteristic equations, giving rise to terms that are polynomial multiples of the exponentials.

Second, how should we determine the constants C_j's? First, constants associated with increasing terms should be set to zero, to make sure that $\Psi(u)$ tends to zero as u tends to infinity. Then, other constants will need to refer to the boundary conditions. In the simple exponential case, the constant was determined by the initial condition $\Psi(0) = \frac{\lambda E(X)}{c}$. For this more complex case, we would require further initial conditions based on derivatives of Ψ at 0. These can be derived through the IDE (1) to get

$$\Psi'(0) = \frac{\lambda}{c}(\Psi(0) - 1);$$

through differentiating (1) once to get

$$\Psi''(0) = \lambda\Psi'(0) - \lambda f_X(0)(\Psi(0) - 1);$$

and so on.

3 Application of the Model and Risk Management Questions

The environment in which general insurance companies currently operate is challenging in at least two aspects. First, investment incomes are squeezed by unprecedented low levels of interest rates. Second, for some classes of business, premium rates are relatively low due to abundance of industry capacity. Risk management is therefore relied upon not only for monitoring of risks, but also for evaluating management action options that would take the organisation to a more optimal state on the risk-return plane.

As organisations face unique sets of such risk management challenges, it is outside the scope of a general ruin theory starter kit to discuss risk management questions in detail. This section therefore aims to describe the kind of questions, which ruin theory models would potentially be deployed as part of a wider set of tools. It lists example questions and describes one of them in detail. In helping the reader to appreciate more concretely the concepts, a demo spreadsheet accompanies it: this section also describes this spreadsheet

The mathematical starting point is the classical model with mixed exponential claims (which in this section is simply referred to as "the classical model"), as described in Section 2 – in particular, its key result as described in Section 2.1. Where the model is found wanting, Section 4 provides suggestions of recent ruin theory results that the reader could examine for further refinements of the classical model.

First, we consider a particular risk management question.

3.1   Example question: how much capital do we need?

A simple question is that of the level of capital requirement. How much capital, *u*, should a company hold, given its business plans and strategies?

Tackling this question would involve consideration of regulatory requirements, as well as the company's specific risk appetite. Internal capital models would likely be key. There would also be wider and more general consideration of the commercial environment, such as the rating agencies' assessments, expense of raising capital and the return on capital expected by the market.

The use of the classical model to consider this problem can at first sight seem obvious. On premium and claim frequency and severity assumptions, we can test various levels of *u* to the model, to derive $\Psi(u)$, the probability of ultimate ruin. One could then select the minimum level of initial capital, $u^{\#}$, such that $\Psi(u^{*})$ satisfies the risk appetite of the company. This minimum exists because the classical model gives rise to monotonically decreasing relationship between Ψ and *u*.

However, there are at least three problems we need to deal with.
- i)   A wide variety of items should also be modelled on top of premiums: e.g. planned expenses including brokerage; reinsurance premiums and associated overrider commissions; investment income.
- ii)   As well as the individual claims, there could be other significant risks that need to be considered: e.g. emergence of latent claims; asset defaults; operational risk losses.
- iii) Return for the capital the investors provide should also be considered.

3.1.1 Incorporation of items at constant rates

The classical model's parameters can be quite easily modified to incorporate most of the above. Those in point (i) can, for a broadbrushed analysis, be considered to pose insignificant stochastic variations, and be incorporated into the parameter *c*. For example, if the plan premium income were 120 p.a. and the plan expense ratio were 25%, then *c* can be taken to be 90 rather than 120.

3.1.2 Incorporation of stochastic items

Those in point (ii) can be incorporated into the mixed exponential distribution. The mixed exponential distribution should be flexible enough to allow mixture of more exponential distributions for extra risk types. (We recall that the mixed exponential distributions can approximate *any* distribution arbitrarily closely.[6]) For instance, suppose the insurance claims can be represented by a mixture of three exponentials with means 3.0, 5.0 and 8.0, with weights, 2/9, 3/9 and 4/9, respectively. Suppose also that the claim arrival rate is 9.0 p.a. Now, if our asset portfolio has a default rate of 1.0 p.a., and severity upon default is an exponential distribution with mean 15.0, then, the asset portfolio defaults can be incorporated, by setting λ to be 10.0 p.a. and the severity be a mixture of four exponentials with means 3.0, 5.0, 8.0 and 15.0, with weights 20%, 30%, 40% and 10%.

3.1.3 Incorporation of dividends

Compensations for the capital provided (corresponding to point (iii) above) often crystallises in the form of regular and discretionary dividends and share buy-backs when excess capital is accumulated. As a first step, we could make the assumption that a regular form of dividends is paid out until ruin. This regular dividend can be calibrated as a percentage of the initial capital, *u*, and incorporated into *c*. So, continuing the example above, if *u* is 170.0, and the dividend percentage is 7.0%, then the dividends would be 11.9 p.a., and *c* could be set to 78.1 (from subtracting 11.9 from 90.0).

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[6] Allowing negative weights in the definition of the mixed exponentials, the general combinations of exponential distributions are shown to be dense in the collection of all distributions on the positive reals in (Botta & Harris, 1986). Convergence is pointwise on the positive reals.

This is clearly a vast simplification of the usual dividend dynamics. However, maintenance of regular dividends are often taken by the market as an indication of a company's health: the assumption that the inability to pay out dividends is linked with negative surplus is not as simplistic as appears on the surface. Moreover, since the model contemplates a very long time horizon, smoothing the return of excess capital may potentially be tolerated for obtaining indicative answers.

Dwelling on point (iii) a little more, we should consider this dividend rate to be real: i.e. after allowing for inflation. The parameters of the classical model apply at all positive *t*, as long as we are not in ruin. This is clearly unrealistic, considering inflation would likely drive up the monetary parameters of the model. However, if we interpret the parameterisation to apply post-inflation, then the dynamics become more realistic, representing an operation at equilibrium in real terms.

Certainly, there is a key assumption that the commercial environment remains constant (in real terms) and that inflation of all items (premiums, claims, expenses, etc.) are the same. The calibration of this model, therefore, should allow for these aspects. One may, for the example of a growing portfolio, select parameters that represent target portfolios rather than the current ones. The usual actuarial practice of sensitivity analysis of results should also be helpful here.

3.1.4 Optimal level of initial capital

The suggested incorporation of dividends into the parameter *c* creates a negative relationship between *c* and *u*. The higher the initial capital, *u*, is, the lower the rate of the positive and constant cashflow, *c*. Rather than a straightforward decreasing relationship between *u* and $\Psi(u)$, we now have a more dynamic relationship, with the possibility of finding an optimal – and finite – capital level u^* that gives the lowest $\Psi(u)$. The chart of $\Psi(u)$ vs *u* below demonstrates this for a particular setup, for the example situation suggested above:

- with a dividend rate of 7% of the initial capital, we obtain a U-shape curve, with $u^* = 165$, with $\Psi(u) = 6.7\%$

- when no dividend is modelled, we obtain a curve that monotonically decreases to 0
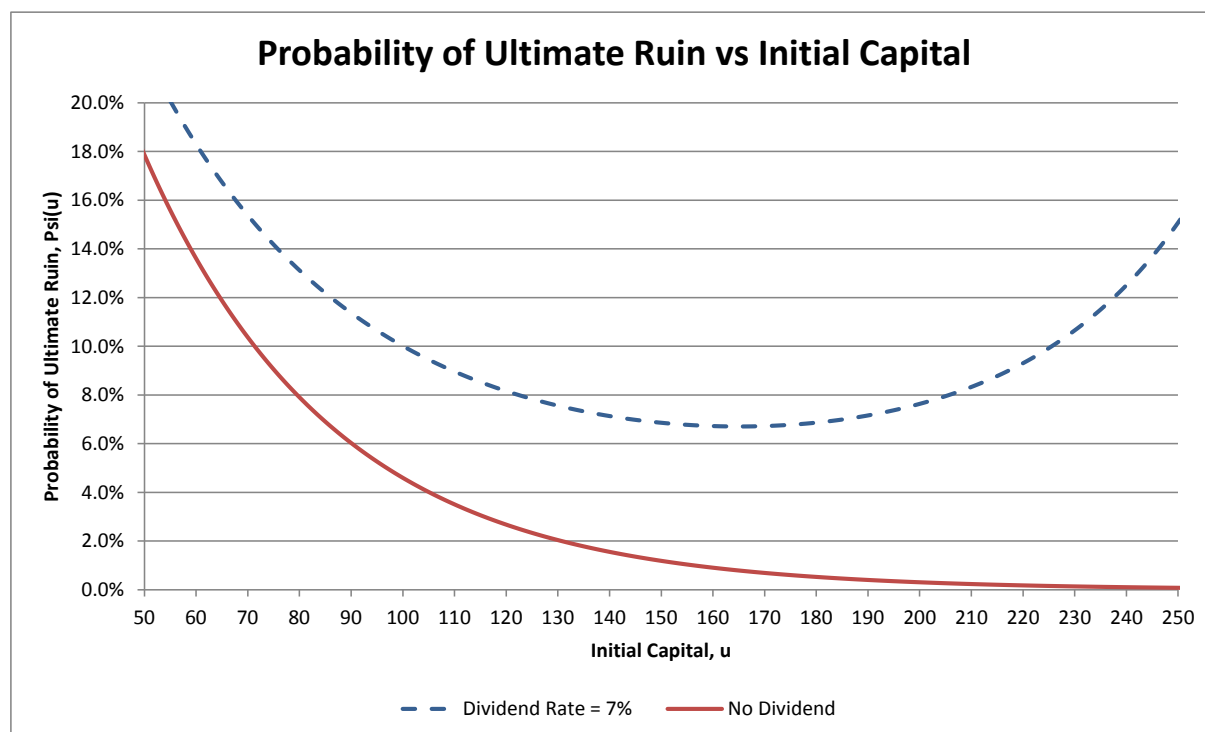
The optimal capital value, u^*, can then be tested against other requirements (e.g. regulatory, internal risk appetite) to assess suitability and inform debates. It can be interpreted as an indicative optimal point at which the company could provide the maximum long-run survival probability. Although $\Psi(u)$ is instrumental in obtaining the optimal point u^*, the quantity $\Psi(u^*)$ would likely only play a minor role in

management discussions, especially since many other forms of capital requirements are calibrated to a much shorter time horizon (e.g. one year for Solvency II).

We now introduce the demo spreadsheet.

3.2 Demo spreadsheet

It should be noted that the purpose of this spreadsheet is only for demonstration purposes, to help the authors enhance communication of mathematical ideas in this paper. The reader should perform the appropriate checks and tests themselves before using the spreadsheet or the formulas therein for any other purposes.

The spreadsheet uses the solution of the IDE when claims are mixtures of four exponential distributions to calculate the probability of ultimate ruin. It assumes the characteristic equation of the ODE (7) has four distinct and negative roots. Additionally, it allows incorporation of a constant dividend stream that depends on the initial capital level *u*.

The inputs are:
- Constant premium rate (p.a.)
- Constant expenses (as % of premium)
- Constant dividend rate (as % of initial capital)
- Initial capital, *u*
- Claim arrival rate, λ
- Claim severity weights and means, $A_1, ..., A_4, \beta_1, ..., \beta_4$

The output is:
- Probability of ultimate ruin, $\Psi(u)$

The orange cells indicate input items, and the one green cell indicates the output. Intermediate calculations and checks can be seen also. The description of the different items makes use of the notation in this paper. The reader will also be interested in using the Excel SOLVER capability to test optimal parameters.

3.3 Other example risk management questions

In light of the above discussions, here are a few other interesting questions the reader might want to try out with the model.
- For organic expansion or acquiring certain new business, would it be better to purchase reinsurance or raise capital? At what level of volatility is reinsurance a preferred option? If reinsurance is required or desired, what kind of reinsurance may be optimal?
- For a fixed amount of capital, *u*, what is the optimal level of exposures?
- How much more premium should we charge to cover a new peril or for an emerging risk?
- How much more capital should we hold to go into more risky corporate bonds?
- Should we invest $Xm in an I.T. system to reduce the chances of various operational risks?

4 Further Investigation Ideas for the Reader

The 110 years since Lundberg's thesis have seen developments in many directions. We list a few that the interested reader might wish to investigate further as a next step.
- Distribution of time to ruin
- Distribution of the amount of negative surplus at ruin
- Distribution of maximum attained surplus before ruin
- Dividend barriers and optimal dividend policies
- Allowance for temporary negative net asset values

- Perturbed models through Brownian motion

We note also that more flexible claim inter-arrival times (using mixed exponential distributions) can be incorporated using similar techniques as those detailed in Section 2.

Apart from the 2010 book, (Asmussen & Albrecher, 2010), the *ASTIN Bulletin* and the *Insurance: Mathematics and Economics* ("IME") are two journals that frequently publish research papers on the subject. The ASTIN Colloquia and IME Congresses are held regularly – the interested readers in this part of the world may wish attend the latter's 2015 congress in Liverpool, U.K., to gain more in-depth insight for themselves into the latest advances in Ruin Theory.

5   Conclusions

Contemporary risk management questions demand a diverse toolkit. We trust this note has raised practitioner interest in Ruin Theory for high-level assessments of the risk-return impacts of specific management actions. The key is being able to derive and solve differential equations and we have demonstrated two common ways of doing so: using characteristic equations, and using Laplace Transforms. Their solutions would provide straightforward formulas for spreadsheet implementations, giving rise to quick quantitative tools for the actuaries and risk managers.

Interest from the academic community is showing few signs of fatigue. Industry engagement with academia would be worthwhile in further enhancing the models.

6   Bibliography

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