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THE ECONOMIC BASIS OF INTEREST RATES

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1. INTRODUCTION

In a discussion I had with Professor J. J. McCutcheon and Dr W. F. Scott, the authors of the new book on compound interest, *An Introduction to the Mathematics of Finance* (Heinemann 1986) it was suggested that a chapter describing the theoretical economic background to the determination of interest rates might be of interest to actuarial students. I therefore drafted the substance of this note as such a chapter. In the event it was felt that the subject fitted uneasily with the mainly technical and numerical approach to compound interest appropriate for students at that particular stage of the examinations, and that the more theoretical approach in this note would in any case be of wider actuarial interest. It is therefore presented as this note in the *Journal*. I am grateful to Professor McCutcheon and Dr Scott for agreeing to this, and to the discussions I had with them while the note was being drafted.

The actuarial study of compound interest is mainly concerned with the calculation of interest rates or net present values in conditions of certainty, where the terms of the particular contract, loan or security under discussion are determined in advance. It is assumed that these conditions are given, and students of compound interest are not generally required to consider those factors in the economy which influence the level of interest rates from time to time. This is a complex subject in economics, on which there are, at an advanced level, conflicting views, discussed for example in Bliss (1975) and in Dougherty (1980). However, there is general agreement that the starting point for discussion is the model first propounded by Irving Fisher (1930), and fully developed for example by Hirshleifer (1970). As will be seen, this is a static equilibrium model, and assumes perfect foresight and hence certainty of outcome. The world is in fact uncertain and changing, so the Fisherian model does not tell the whole story. To do so would be too lengthy for this note, and the interested reader may consult the references given at the end.

In order to present a sufficiently simple model of the economy it is necessary to make some very great simplifying assumptions. This is a normal scientific technique, widely used in economics. At a later stage we shall discuss some of the real life complications, and the reader will be able to add some himself. In the first place, however, we ask the reader to imagine a very simple economy, consisting only of *individuals* (or families) who receive incomes which they may spend on the consumption of goods and services, and *firms* (which may be individuals, partnerships, companies, etc.) that undertake the production of goods or the supply of services for profit. We shall assume that there is no government, no overseas sector, no taxes and no inflation, so that prices measured in money and 'in real terms' are the same. We shall also, in the first instance, assume that there are only two time periods, which we shall call 'this year' or 'now' and 'next year' or 'the future' or 'later'. We shall also assume that there is no uncertainty about the future, that firms know exactly what they can produce with a certain amount of capital equipment, and that individuals know exactly what their present and future incomes are, and what they want to consume. The model therefore does not allow for uncertainty about the future, nor for changes in conditions as individuals change their minds, or firms discover different ways of producing things. The consequences of such changes will be discussed briefly later.

2. THE INDIVIDUAL'S CONSUMPTION CHOICE

Consider one individual or family unit. His, her or their income for this year we shall call Y_0, and that for next year we shall call Y_1. His consumption this year we shall call C_0, and that for next year C_1. We shall show this year's figures, Y_0 and C_0, on the horizontal axis of the following graphs, and next year's, Y_1 and C_1, on the vertical axis.

Assume that the individual's income for both periods is fixed, so that $Y_0 = y_0$ and $Y_1 = y_1$. If there is no borrowing or lending, and no way in which goods or services can be saved or carried forward to next year's consumption, then the individual has no choice about his consumption. Necessarily he must consume this year's income this year, and next year's income next year, so that $C_0 = y_0$ and $C_1 = y_1$. This is the economy, for example, of many animals, who can eat now only what they catch or graze now, with no inter-temporal choice. But some, such as squirrels, may put aside food gathered now in order to be able to consume it at a later date.

This is shown in Figure 1. This year's income, Y_0, is equal to y_0, and next year's income Y_1, is equal to y_1. The individual cannot rearrange his income, so this year's consumption C_0, is necessarily also equal to y_0 and next year's consumption, C_1, is necessarily equal to next year's income, y_1.

Now assume that by borrowing or lending the individual can rearrange his income in some way between the two periods. If he has more income than he wants now, he can lend now in exchange for a higher future income which he can spend on consumption. If his present income is small, he may prefer to borrow now to consume now, repaying out of next year's income. How might he decide what to do?

First, we assume that he has a preference for higher consumption. If he could get both higher consumption now and higher consumption later he would prefer this to his present position. Thus if he is comparing two consumption possibilities, say $(C_0, C_1) = (c_{0,1}, c_{1,1})$ or $(c_{0,2}, c_{1,2})$ he will prefer the second pair to the first if consumption now is increased and consumption later is unchanged, $c_{0,2} > c_{0,1}$ and $c_{1,2} = c_{1,1}$; or if consumption later is increased, consumption now remaining unchanged, $c_{1,2} > c_{1,1}$ and $c_{0,2} = c_{0,1}$; or if he can increase consumption in both periods, $c_{0,2} > c_{0,1}$ and $c_{1,2} > c_{1,1}$.

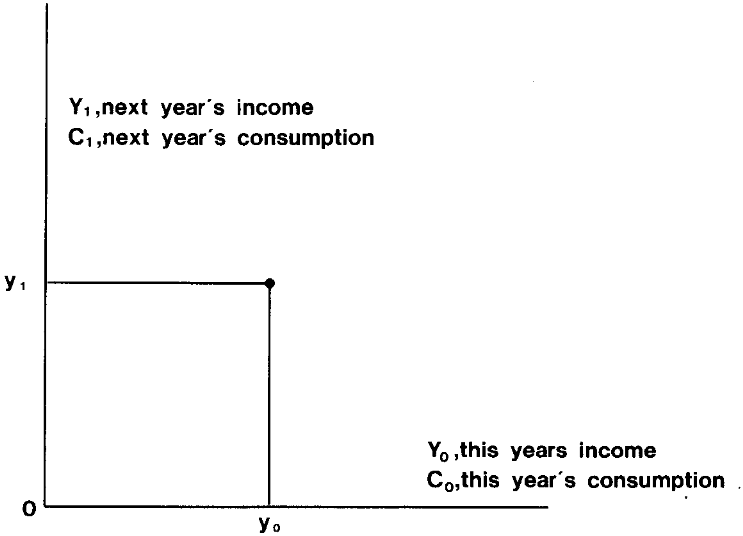

Figure 1. No borrowing or lending.
$C_0 = Y_0 = y_0$, $C_1 = Y_1 = y_1$

Figure 2 compares different consumption pairs. Point 2 $(c_{0,2}, c_{1,1})$ is preferred to point 1 $(c_{0,1}, c_{1,1})$ since $c_{0,2} > c_{0,1}$. Point 3 $(c_{0,1}, c_{1,3})$ is also preferred to point 1, since $c_{1,3} > c_{1,1}$ as is point 4 $(c_{0,4}, c_{1,4})$ since $c_{0,4} > c_{0,1}$ and $c_{1,4} > c_{1,1}$.

These preferences are assumed to hold for all (economically rational) individuals. But without further information about an individual's preferences, we cannot compare points 2, 3 and 4 with one another.

However, any individual will generally not be able to increase consumption now without giving up consumption later or vice versa. He therefore has to consider how much C_0 to give up to increase C_1, or how much C_1 to give up to increase C_0. We can imagine the perfectly knowledgeable individual being able to compare any pair (C_0, C_1) with any other pair, and to decide which of them he would prefer, or whether he would be indifferent between them. We can then imagine him plotting lines on the graphs of C_0 and C_1 joining all those points for which he has equal preference or between which he is indifferent. These lines we shall call *indifference curves*.

Specimen indifference curves L_1, L_2 and L_3 are shown in Figure 3. Each is the locus of consumption pairs that one particular individual is indifferent between. Thus he holds points 1 and 2, which both lie on L_1, equally preferable. Point 3 would be preferred to point 1 by everybody, since $c_{1,3} > c_{1,1}$. Point 4 is preferred to point 3 by everyone for a similar reason. This individual prefers any point on L_3 to any point on L_2, and in turn he prefers any point on L_2 to any point on L_1. Thus he prefers point 5 to either point 1 or point 2.

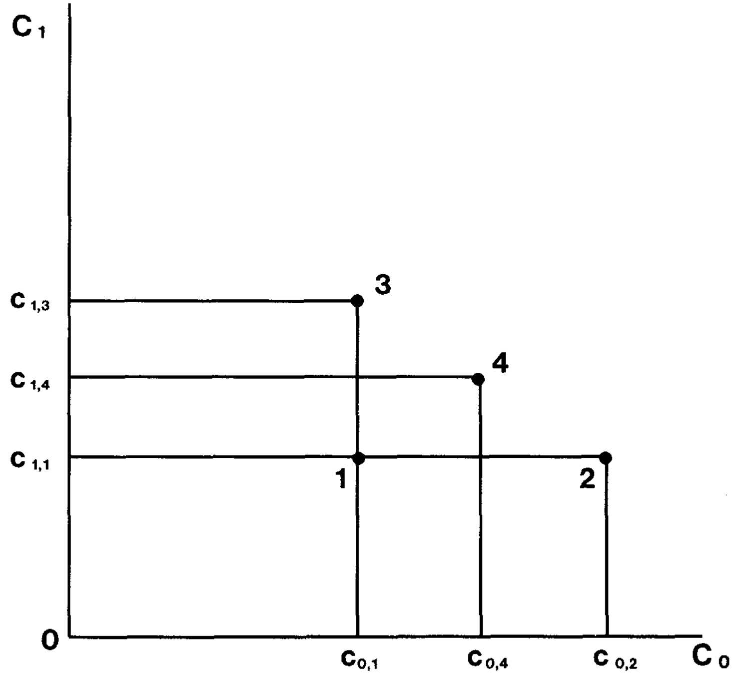

Figure 2. Consumption point 2 ($c_{0,2}, c_{1,1}$) is preferred to point 1 ($c_{0,1}, c_{1,1}$); point 3 ($c_{0,1}$, $c_{1,3}$) is preferred to point 1; point 4 ($c_{0,4}$, $c_{1,4}$) is preferred to point 1.

The curves can be thought of as contour lines on a hillside which rises towards the top right hand corner of the diagram, towards higher C_0 and C_1. However, we are not at this stage putting any numerical values on the 'heights' of the lines. Thus we do not compare the difference between L_2 and L_1 with the difference between L_3 and L_2. All we do is rank them, so that in some sense $L_3 > L_2 > L_1$, without numerical values being assigned.

It will be seen that the curves have been drawn downwards sloping. This follows necessarily from the logic of Figure 2, since along any horizontal line of constant C_1, a higher value of C_0 is preferred to a lower, and along any vertical line of constant C_0, a higher value of C_1 is preferred to a lower.

The lines have also been drawn continuous and smooth. This follows from the assumption that consumption is infinitely divisible. In reality, some consumption—a new car, a particular summer holiday—is only available in discrete lumps, but we assume continuity so that we can discuss the slope of the indifference curves. If we represent a particular indifference curve by the function $c_1 = f(c_0)$, we see that the (negative) slope of the indifference curve at any point is

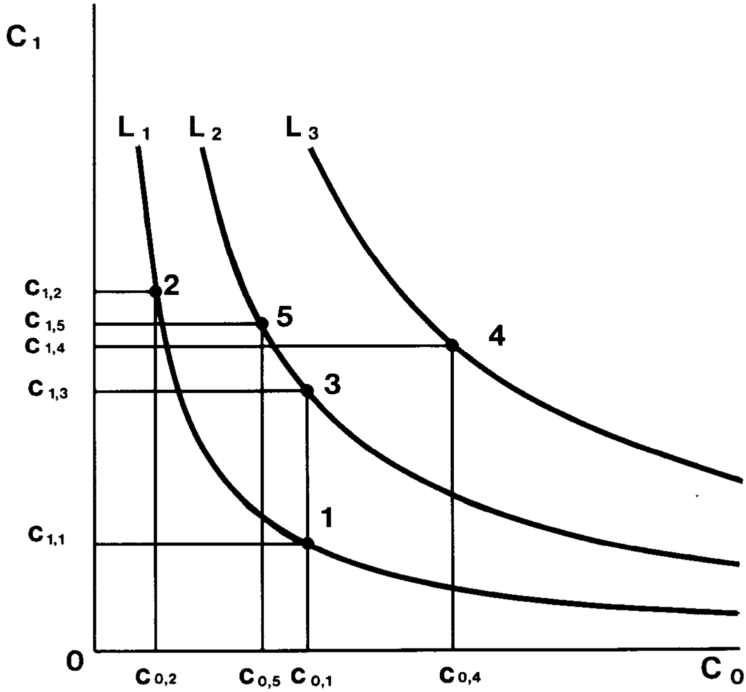

Figure 3. Indiffence curves, L_1, L_2, L_3 join points of equal preference for one individual.

given by the derivative, $f'(c_0)$. The positive value of this, $-f'(c_0)$ at any point can be thought of as the *marginal rate of time preference* of this individual between present and future consumption. Thus he is prepared to reduce his present consumption by 'δc_0' in order to increase his future consumption by 'δc_1' or vice versa. The value of this marginal rate of time preference depends on the values of C_0 and C_1 at any point.

It will also be seen that the curves have been drawn so that they are convex to the origin, so that the (absolute) slope is steep when C_0 is small and C_1 is big, and shallow when C_0 is big and C_1 is small. This is consistent with a principle of 'diminishing marginal utility', even though we are not giving specific values to the 'utility' of any particular consumption pair. But it seems reasonable for an individual who is currently in the position of having high C_0 and low C_1 to be willing to give up a lot of this year's superfluity to make provision for next year's famine; and vice versa, for an individual with low C_0 and high C_1 to be prepared to give up a large part of next year's plenty to avoid being on short commons today. Mathematically we assume that along any indifference curve, $c_1 = f(c_0)$, we have both $f'(c_0) < 0$ and $f''(c_0) > 0$.

Now let us consider the possibilities open to the individual if both borrowing and lending are possible. Assume, as before, that his incomes this year and next are initially fixed at $Y_0 = y_{0,1}$ and $Y_1 = y_{1,1}$. However, he may rearrange his income by lending, that is by giving up part of this year's income in exchange for income next year, at the rate of r next year for every 1 given up this year. Alternatively, he may borrow, increasing this year's income, but having to repay next year, at the same rate, so that for every 1 by which this year's income is increased, next year's is reduced by r. It can be seen that if we write $r = 1 + i$, then i is a rate of interest per period in the usual sense. It is more economical in the algebra that follows to use r.

We assume that the rates at which borrowing and lending take place are the same, and do not vary with the quantity borrowed or lent, nor with the status of the borrower or lender, and that there are no costs involved with the process of borrowing or lending. These practical considerations of course affect interest rates in real life in the expected ways.

It can easily be seen from Figure 4 that this individual has the opportunity to change his income pair to any point on the line M, whose equation is given by:

$$y_1 - y_{1,1} = -r\,(y_0 - y_{0,1})$$

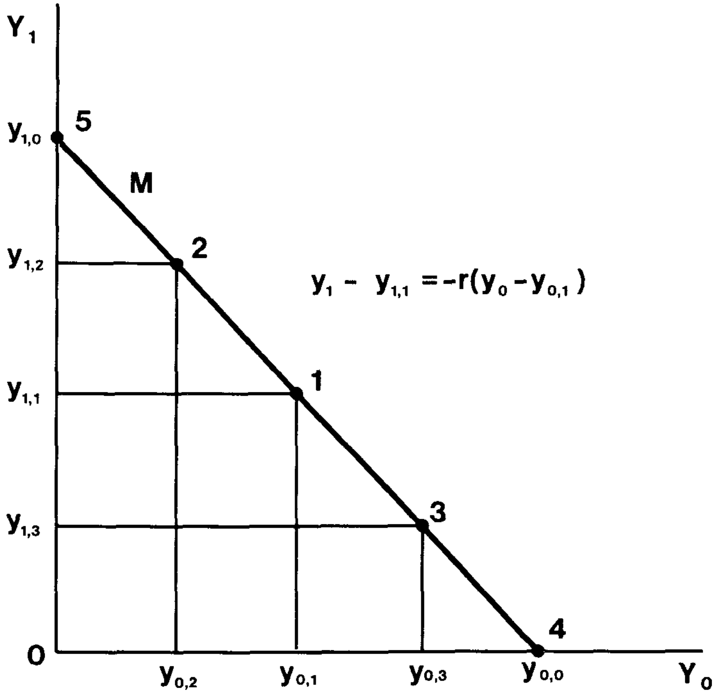

Figure 4. Income opportunity line, M, with possibility of borrowing or lending.

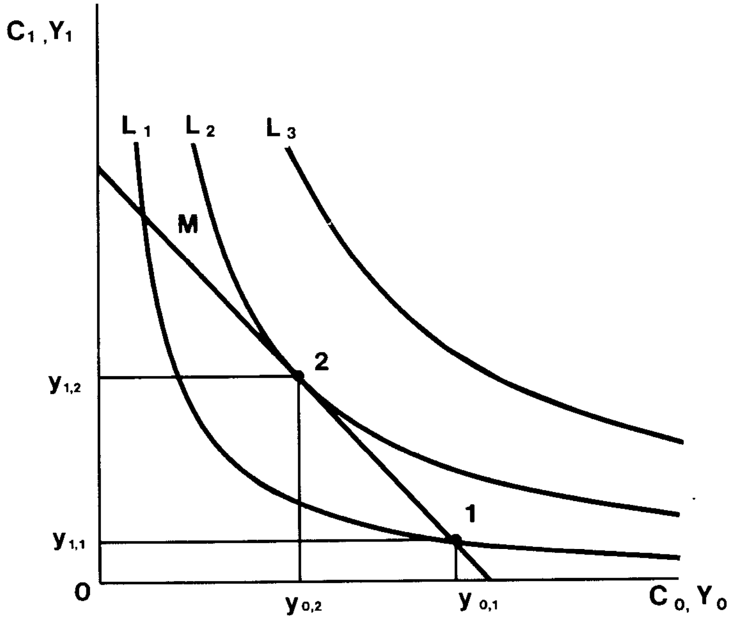

Figure 5. Individual who lends to go from point 1 ($y_{0,1}, y_{1,1}$) to point 2 ($y_{0,2}, y_{1,2}$).

Thus he may choose to lend an amount ($y_{0,1}-y_{0,2}$), bringing his income this year down to $y_{0,2}$ and putting next year's up to $y_{1,2}=y_{1,1}+r(y_{0,1}-y_{0,2})$ taking him to point 2. Alternatively he may borrow ($y_{0,3}-y_{0,1}$), increasing this year's income to $y_{0,3}$, but giving him only $y_{1,3}=y_{1,1}-r(y_{0,3}-y_{0,1})$ next year, shown by point 3.

He is limited only by the constraints that income in neither year can be negative. Thus at one extreme he may transfer all his income to this year, giving him $y_{0,0}=y_{0,1}+y_{1,1}/r$ this year and nothing next, shown as point 4. Or at the other extreme, he may reduce this year's income to 0, increasing next year's to $y_{1,0}=y_{1,1}+ry_{0,1}=ry_{0,0}$ shown as point 5.

Given this set of income opportunities what should the individual do to achieve his most preferred combination of consumption possibilities? Consider the individual in Figure 5, whose initial income pair is given by ($y_{0,1}, y_{1,1}$) at point 1, and some of whose indifference curves are represented by L_1, L_2 and L_3. He has a comparatively large current income, and a comparatively small future income. It is therefore worth his while lending some amount, so as to travel upwards along the line M until he reaches the available consumption position that he most prefers. This is seen to be at point 2, where the line M is tangential to the indifference curve L_2. This is a better position for him than point 1, or than any other point on line M. He would be happier still if he could reach a point on line

L_3, but with his present income combination this is unattainable. This individual's best policy is then to lend $(y_{0,1}-y_{0,2})$ leaving himself with $y_{0,2}$ to spend on consumption this year, and giving him $y_{1,2}=y_{1,1}+r(y_{0,1}-y_{0,2})$ to spend on consumption next year.

It can be seen that so long as the appropriate convexity conditions are fulfilled, so that no indifference curves have straight line sections or 're-entrants', then the optimum point for this individual is unique. At this point the slope of the consumption indifference curve, L_2, is equal to the slope of the income opportunity line, M, so that the 'rate of interest' (strictly $r=1+i$) is equal to the marginal rate of time preference between future and present consumption.

The possibilities for a different individual are portrayed in Figure 6. He has a relatively small immediate income, $y_{0,3}$ and a relatively large income in the future, $y_{1,3}$. He can therefore reach his most preferred consumption pattern by borrowing now an amount $(y_{0,4}-y_{0,3})$ allowing him only $y_{1,4}=y_{1,3}-r(y_{0,4}-y_{0,3})$ next year. However, this takes him to point 4 on his L_2 indifference curve, which is his most preferred position.

Although we have assumed that indifference curves are unique to each individual, it is clearly possible for two individuals to have the same indifference

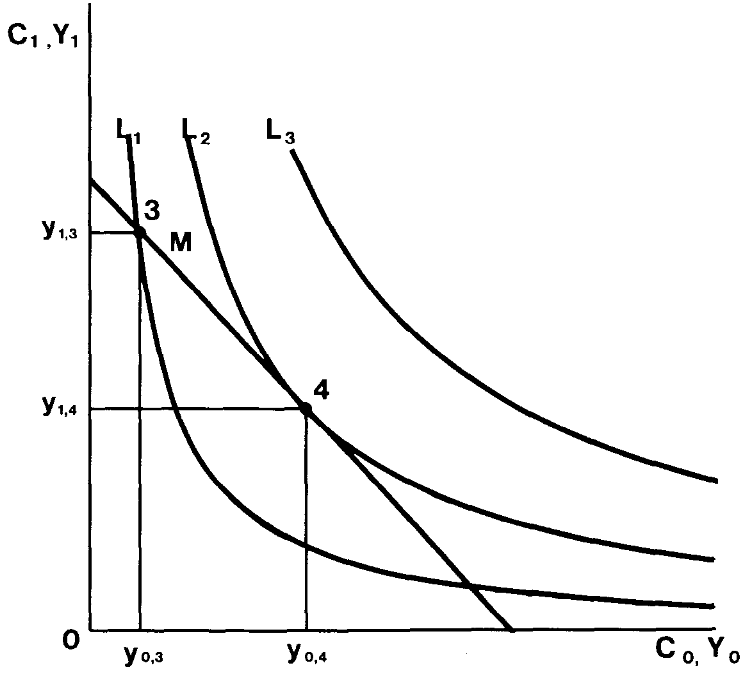

Figure 6. Individual who borrows to go from point 3 $(y_{0,3}\, y_{1,3})$ to point 4 $(y_{0,4}, y_{1,4})$.

curves, and yet for one to be a lender and the other to be a borrower. It depends on their initial income position. The lender in Figure 5 starts at point 1, where the slope of his indifference curve, L_1, is (absolutely) less than the slope on the line M. Conversely for the borrower who started at point 3; the slope of his indifference curve at that point is greater than that of line M. The marginal rate of time preference of an individual at a particular point can also be described as his *personal discount rate*. If his personal discount rate in present circumstances is lower than the market rate of interest, as in Figure 5, then he will be a lender. Conversely, if his personal discount rate in present circumstances is higher than the market rate of interest then he will tend to be a borrower.

We can see that these assumptions have some real life plausibility. The young couple, setting up house, but with prospects of an increasing or at least steady future income will borrow to purchase a house and furnish it. When they reach middle age, having set up a house and got their children off their hands, they may see the prospect of a diminishing income in retirement. They are therefore willing to save considerably out of their current higher income in order to secure adequate funds to purchase consumption in retirement. This inter-temporal rearrangement of incomes to match desired consumption patterns is facilitated by investment intermediaries such as building societies and pension funds.

We now return to our artificial two-period world: in equilibrium each individual will borrow or lend as appropriate until his marginal rate of time preference equals the market rate of interest. Since we are assuming a free market in which the interest rate is the same for all individuals, in equilibrium the marginal rate of time preference for all individuals must be equal.

3. FIRST DETERMINATION OF THE RATE OF INTEREST

The argument so far has told us how much each individual may wish to borrow or lend at a given rate of interest, but it does not tell us how the rate of interest is arrived at. To do this we shall in the first instances assume that all the borrowing and lending is done between individuals in order to adjust their predetermined incomes to suit their particular consumption preferences. Since every loan requires both a lender and a borrower, and we are assuming no transaction costs and no defaults, the total amount lent this year must exactly equal the total amount borrowed, and the total amount repaid next year must exactly equal the total amount received. Thus total income this year equals this year's total consumption, and next year's total consumption exactly equals next year's total income. At this stage we are assuming no transfer of production from one year to the next.

Our first statement of the determination of the rate of interest in equilibrium is therefore that the market rate of interest adjusts so that:

(i) the total amount borrowed equals the total amount lent, and

(ii) the marginal rate of time preference for each individual after borrowing or lending what he wishes exactly equals the market rate of interest.

This model, although it tells us the equilibrium results, does not explain how we actually get there. One explanation, due to Walras (1877), described also by Morishima (1977), involves the creation of an imaginary 'auctioneer'. This auctioneer announces a rate of interest, and asks for offers from lenders and bids from borrowers. Each individual knows his initial income, and instantaneously calculates from knowledge of his indifference curves how much he would be prepared to borrow or lend at the auctioneer's rate of interest. He can therefore offer funds for loan, or apply to borrow. The auctioneer instantaneously sums the total of offered loans and borrowings, and if these are not equal he adjusts his proposed rate of interest. If the offered lendings exceed the desired borrowings he assumed that his rate was too high, and tries a lower one. Contrariwise, if the offered lendings fall short of the desired borrowings he assumes his rate was too low and tries a higher one. He continues with this process of successive approximation until offered lendings exactly equal desired borrowings, whereupon he announces the now determined rate of interest. All this is presumed to happen instantaneously!

This fanciful explanation is not in practice so unrealistic. In reality one is not in a static position seeking an unknown rate of interest, but in a dynamic one, where the individuals, their incomes and their preferences are changing almost continuously. The market is therefore always seeking a new equilibrium position; but it is able to start from whatever the present position is. The place of the Walrasian auctioneer is taken by dealers in the money market, banks, building societies and discount houses, jobbers on the Stock Exchange etc., some of whom act as brokers (like the auctioneer) and others as principals (taking in deposits and granting loans themselves), but always in such a way as to keep the supply of loanable funds in equilibrium with the demand for borrowed funds, and hence determining a market rate of interest that changes frequently in response to changing conditions.

4. THE FIRM'S PRODUCTION CHOICE

So far we have assumed that the total amount produced in each period is fixed in advance, and that it must all be consumed in that period. The amount of income available to each individual was determined, not necessarily by what he produced, but in some way so that the total amount of income equalled the total value of production, and so that both equalled the total value of consumption. Now we have to consider the possibility that by rearranging the production process we can actually transfer production of consumables from one year to the next. In general this is described as capital investment. We have already referred to the squirrel who gathers food in the autumn and stores it for possible consumption in the winter. The individual can build and equip a house during his working lifetime which he can occupy in comfort during his retirement. These activities involve simply a transfer of consumption, without necessarily an increase in production.

Of more significance perhaps, the individual can spend time making tools now which will allow him to produce more next year. We can distinguish the activities of the individual as consumer from the individual as producer, and call all producers, whether they are individuals, partnerships, companies, corporations or whatever, *firms*. A firm may undertake the construction of factories, plant, machinery, railways, roads, offices, shops, etc. any of which allow the possibility of greater production of goods or greater provision of services in future years.

Many of the capital goods we have described last for many years, but again we must simplify, and assume that capital investment lasts for only one period. An individual or a firm may refrain from consuming now, but instead use his income to create a capital asset, which will give a return once and for all in the next and only future period. We assume that the return on any particular capital investment is known with certainty, and we also assume that capital investments are infinitely divisible. This allows us to deal in smooth curves, rather than stepwise ones. In reality at a practical level this is not true: one can't build half a bridge; but it is a good enough approximation if there are a number of separate possible projects, and none is large relative to the whole economy.

It is important to note that capital investment is a one-way process. One can invest some of this year's production to increase next year's; one cannot use next year's production to increase this year's. We therefore consider first an individual with income this year of $Y_0 = y_{0,1}$. He has the choice of consuming this income, or using it for capital investment to increase his next year's income. What next year's income would otherwise be is not at this stage relevant. What are the choices open to him?

He may, if he wishes, take all his income this year, leaving no excess income next year, represented by the point 1 $(y_{0,1}, 0)$ in Figure 7. Alternatively he may choose to invest in a capital project to give him a return next year. If he is sensible he will choose first to undertake whatever project will give him the best return next year. If he ranks all possible projects in order of their return, with the highest first, he can obtain a capital investment schedule where the returns will be high to begin with, falling off as the amount of investment increases. This is shown in reverse in Figure 7.

For example, if he invests $(y_{0,1} - y_{0,2})$ he can raise next year's income to $y_{1,2}$ taking him to point 2. If he invests a further $(y_{0,2} - y_{0,3})$ his income next year will increase by $(y_{1,3} - y_{1,2})$ to a total of $y_{1,3}$ taking him to point 3. But the return per unit invested for the second tranche of investment will be less than for the first, which is shown by the curve from points 2 to 3 being shallower than that from point 1 to point 2. At the extreme, he can invest the whole of $y_{0,1}$ giving him a return of $y_{1,4}$ taking him to point 4, but it can be seen that the marginal return of the last $y_{1,3}$ invested is much poorer than on the more preferred investment possibilities.

The whole curve P gives the locus of investment possibilities open to the individual. If it is expressed as $y_1 = g(y_0)$, then the negative of the derivative, $-g'(y_0)$ gives the *marginal return on capital invested*.

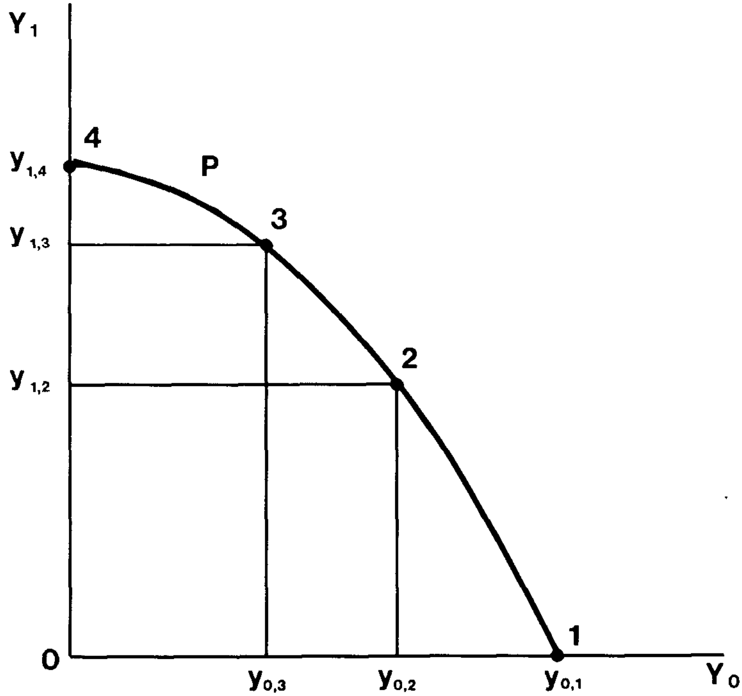

Figure 7. Individual's production possibilities.

How much of the individual's present income should he choose to devote to capital investment? Let us consider the answer first if borrowing or lending is not possible. Assume that the individual's initial position is given by point 1 in Figure 8, where he has income this year of $y_{0,1}$ and next year of $y_{1,1}$ if he does not invest. But if he engages in capital investment he can reduce this year's income and increase next year's, travelling up the curve *P*. Note that *P* does not extend to the right of point 1, because one cannot preempt next year's income.

If the individual wishes to attain his most preferred consumption pattern, he will move along curve *P* until he reaches his most preferred achievable consumption pattern. Let us assume that this is at point 2 in Figure 8, which he has reached by investing $(y_{0,1}-y_{0,2})$ giving him a return of $(y_{1,2}-y_{1,1})$ next year, so that his income pair is $(y_{0,2}, y_{1,2})$ which he can then consume. Point 2 is reached where the curve *P* is tangential to some indifference curve, say L_2, and at this point the slopes of the two curves are equal. At this optimum point, therefore, the marginal return on capital invested is exactly equal to the individual's marginal rate of time preference.

However, we have assumed that point 2 exists. It is possible that the position is

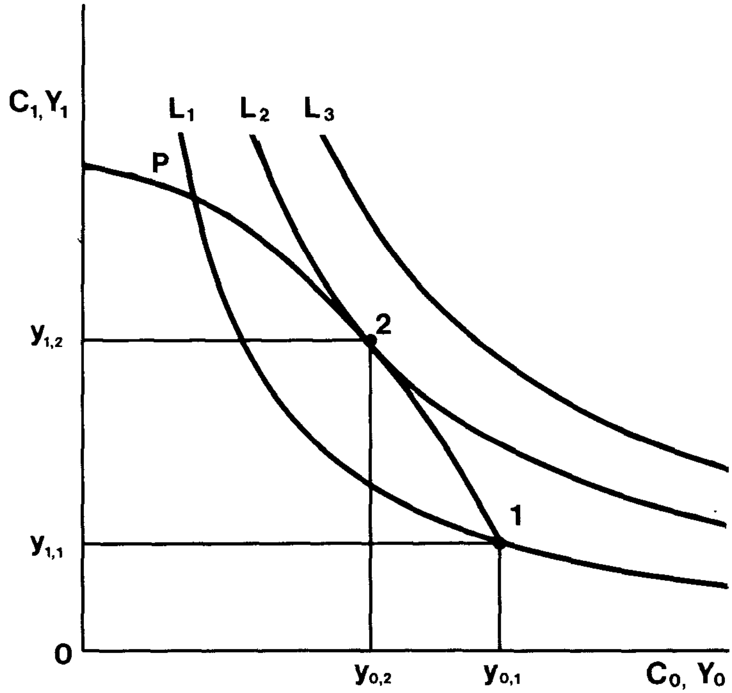

Figure 8. Individual's production choice with no borrowing or lending.

as shown in Figure 9, where point 1 on curve P is already on the most preferred indifference curve. If this is the case he will invest nothing, and consume simply according to his initial income pattern $(y_{0,1}, y_{1,1})$. In this case the curve L_2 that passes through point 1 must be steeper than the curve P and the individual's marginal rate of time preference at this point must be greater than the highest marginal return he could obtain on capital investment.

We have assumed in this section that the firm is owned by one individual, and we have seen how he would choose his capital investment to give himself the most preferred consumption pair. We have not yet allowed borrowing or lending. In the next section we see how borrowing or lending affects the situation.

5. THE PRODUCTION CHOICE WITH BORROWING OR LENDING PERMITTED

Let us now combine the results of the previous sections.

Consider again the individual with initial income pattern $(y_{0,1}, y_{1,1})$ represented by the point 1 in Figure 10. His production possibilities are represented by

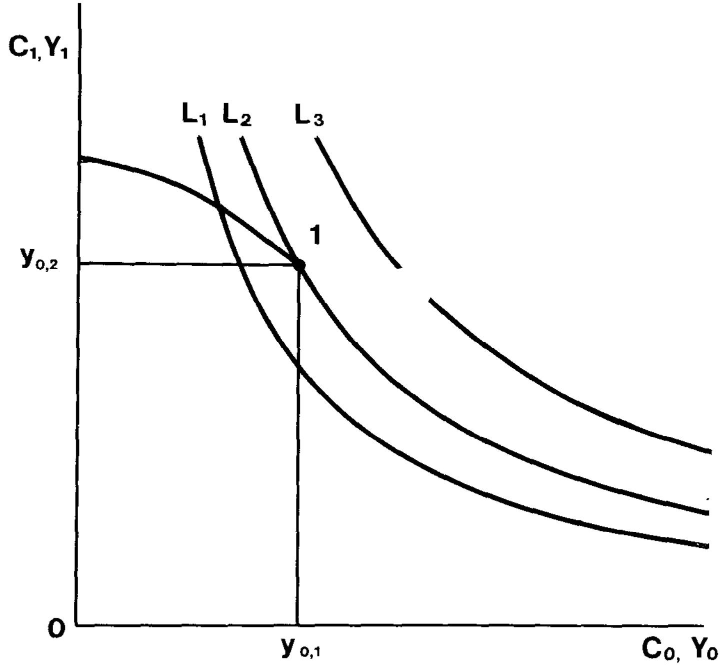

Figure 9. Individual who chooses not to invest.

the curve P. The individual can rearrange his income pattern to reach any point along P. But having got there, he could then rearrange his income pattern yet further by borrowing or lending, so as to move along any of the lines M_1, M_2, etc. each of which has a slope of $-r$, representing the market rate of interest (strictly $-(1+i)$).

It will be seen that the individual's best strategy is to move up curve P until he reaches the highest attainable of the M lines, in this case M_2, which is tangential to P at point 2. He can then slide along M_2 to his preferred position, in this case by lending at the market rate of interest to reach point 3, when he is on his highest attainable indifference curve, L_2. This individual therefore invests $(y_{0,1}-y_{0,2})$ to obtain a return of $(y_{1,2}-y_{1,1})$ and lends a further $(y_{0,2}-y_{0,3})$ to obtain a return of $(y_{1,3}-y_{1,2})=r(y_{0,2}-y_{0,3})$.

The individual in this case chose to lend further. A different individual might have chosen to borrow, taking him to a preferred point on M_2 which was below point 2. However, his production decision is the same. Whatever his preference for consumption, he can achieve his best consumption position by arranging his production at point 2, where the curve P is tangential to one of the interest lines M, so that the slope of P at point 2 is equal to $-r$. Thus the optimum production

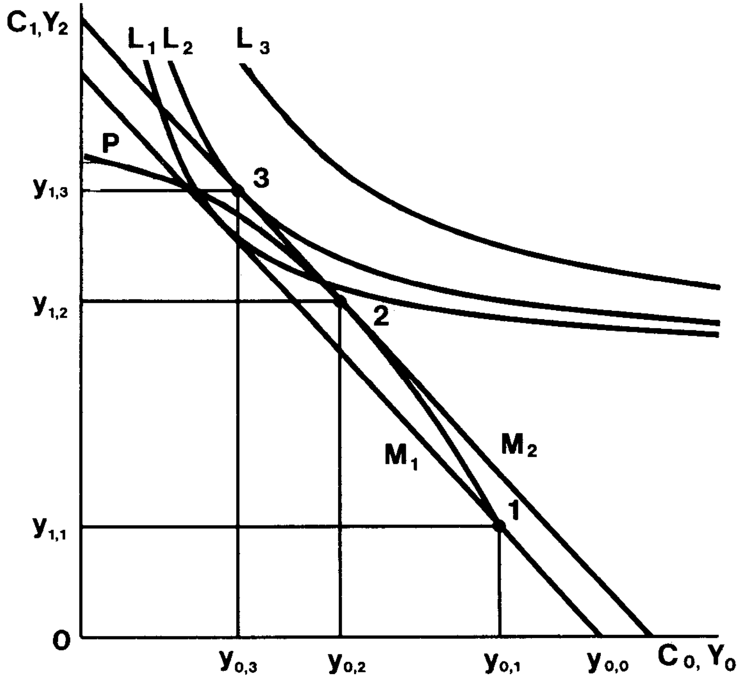

Figure 10. Individual's production choice with borrowing or lending.

position is achieved when the marginal return on capital invested is exactly equal to the market rate of interest.

This result is independent of the consumption preferences of the individuals who own the firm, and independent also of the initially given income for next year of $y_{1,1}$. We assume that if several individuals get together either as a partnership or a joint-stock company to form a firm, they share *pari passu* in the profits of the firm. Thus if Figure 10 represented the position of a firm which happened to have initial income of $(y_{0,1}, y_{1,1})$ and which engages in capital investment so as to transfer its outcome to $(y_{0,2}, y_{1,2})$ then the individuals with shares in the firm would share proportionately in the same way in $y_{0,2}$ distributed this year and $y_{1,2}$ distributed next.

If these conditions are fulfilled, then the correct objective for the management of the firm is to arrange its affairs to get onto the highest attainable M line, regardless of the consumption preferences of the individual shareholders. This objective is equivalent to pushing the point $(y_{0,0}, 0)$ as far to the right along the Y_0 axis as possible. It can be seen that $y_{0,0} = y_{0,2} + y_{1,2}/r$ which is the *present value* (in the usual compound interest sense) of the firm. This leads to the important separation theorem, which is valid in more general circumstances than we have described here:

> Provided that the shareholders of the firm rank *pari passu*, the management of the firm should attempt to maximize the net present value of the firm, regardless of the consumption preferences of the individual shareholders. The shareholders, by borrowing or lending (or trading their shares), can arrange their income to suit their consumption preferences regardless of the investment decisions taken by the firm.

In practice, the existence of taxation, which may apply at different rates to different shareholders, invalidates to some extent the *pari passu* assumption.

6. THE FIRM'S BORROWING OR LENDING DECISION

We assumed above that the individual or firm started with an initial income position given by $(y_{0,1}, y_{1,1})$. But a firm may have no initial endowment. It may only have a set of production opportunities which it could undertake if it could obtain the necessary finance. Assume that the firm starts from position $(0,0)$ in Figure 11. Its production possibility curve, P, has been moved so that the slope where P crosses the Y_1 axis is equal to the market rate of interest, $-r$. It is then worth the firm borrowing $y_{0,1}$ which it invests in projects that give a return next year of $y_{1,2}$. Next year it will have to repay $y_{1,1}=ry_{0,1}$, leaving it with a net profit of $y_{1,2}-y_{1,1}$. The present value of this profit is given by $y_{0,2}-y_{0,1}=(y_{1,2}-y_{1,1})/r$. The lines M_1 and M_2 each have slope $-r$, and represent points of equal present value.

If the firm were to borrow less than $y_{0,1}$ so that it could only engage in fewer production projects, then it would not be maximizing its present value. It would be missing out on projects that would provide a return greater than the cost of borrowing. Similarly, if the firm were to borrow more than $y_{0,1}$ and engage in further capital projects, its present value would also be diminished, since the marginal projects undertaken would yield less than the cost of capital.

A different situation is seen in Figure 12, where we assume that the firm starts with an amount of $y_{0,1}$ initially available this year. In order to maximize its present value it should move up its production curve, P, to point 2, spending $(y_{0,1}-y_{0,2})$ on capital projects, which next year will bring a return of $y_{1,2}$, and lending the remainder of its available cash, $y_{0,2}$, to bring a return next year of $ry_{0,2}$, giving it a total income next year of $y_{1,3}=y_{1,2}+ry_{0,2}$. The net present value of this is given by $y_{0,3}=y_{1,3}/r=y_{0,2}+y_{1,2}/r$.

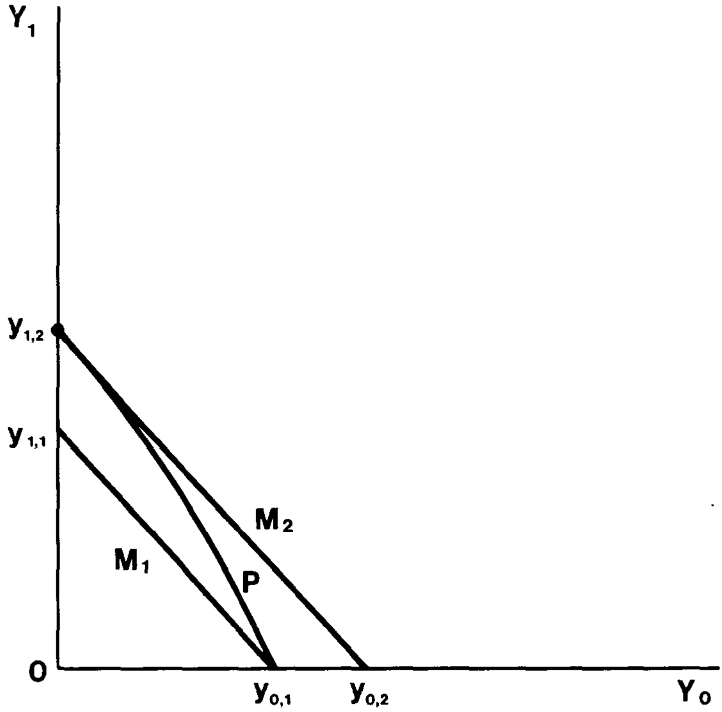

Figure 11. Firm's production choice with borrowing, starting at (0, 0).

We can thus state the optimum behaviour of the firm under our assumed conditions: a firm should undertake production opportunities which yield more than the market rate of interest, but should not undertake production opportunities that yield less. Its marginal rate of return on capital will therefore equal the market rate of interest. If it has insufficient funds it should borrow until its marginal cost of capital (in this case assumed to be the constant market rate of interest) is equal to the marginal return on investment. If it has surplus funds it should invest these at the market rate of interest.

7. SECOND DETERMINATION OF THE RATE OF INTEREST

We are now in a position to state all the conditions that must be fulfilled to determine an equilibrium market rate of interest. These are:

(i) the total amount borrowed, both by individuals and by firms, must equal the total amount lent, by individuals and by firms;

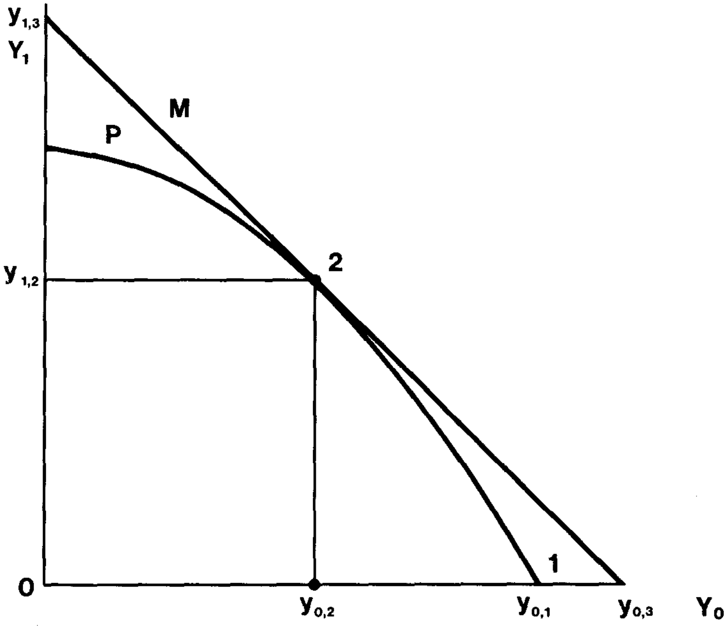

Figure 12. Firm's production choice with lending, starting at $(y_{0,1}, 0)$.

(ii) total income this year must equal the sum of total consumption by individuals and total capital investment by firms;
(iii) total income next year, including the proceeds of capital investment by firms, must equal the total consumption next year by individuals;
(iv) the marginal return on capital invested equals the market rate of interest, and all projects giving a higher return are carried out, with no projects giving a lower return being carried out;
(v) the marginal rate of time preference for each individual after borrowing or lending what he wishes equals the market rate of interest.

It can be seen that more conditions need to be satisfied in this case than in the first determination of the rate of interest. Our imaginary auctioneer has more to do to create equilibrium. But we can see that, if he pitches his rate of interest too high, there will be an excess of lending and a deficit of desired borrowing, as before; also that individuals will be inclined to postpone consumption to next year, because they wish to lend, and also that firms will be reluctant to invest, since only the most profitable capital investment would be undertaken. On two counts there will be a shortfall of expenditure this year, as compared with available production. A reduction in the auctioneer's rate of interest will stimulate capital investment and diminish the desire of individuals to postpone consumption till next year, i.e. reduce their savings. An equilibrium position is therefore, in principle, attainable. One possible equilibrium position, however, would involve no capital investment at all. This would occur if the general desire of individuals in the economy were such that the market rate of interest determined by the desire of individuals to borrow and their reluctance to lend was higher than the return obtainable on even the most profitable capital investment.

Thus we see that in a society which places a high value on immediate consumption interest rates are likely to be high and capital investment low, as compared with a society which places a high value on future consumption, in which interest rates are likely to be lower and capital investment higher. The time preference of individuals is one external factor; the available production possibilities are another. In a society where there are many available production possibilities, the result perhaps of new opportunities or new technological advances, both interest rates and capital investment may be high. In an economy where capital investment opportunities are less available, for any reasons, both interest rates and capital investment are likely to be lower.

8. THE EXTENSION TO MULTIPLE TIME PERIODS

In order to simplify the exposition, and so as to be able to draw graphs on a plane surface, we have restricted the discussion above to two time periods. The reader will be able to imagine how most of the same arguments apply when the number of time periods is increased, first to a finite number of discrete steps, $1, 2, \ldots, n$, and then to a continuous infinitely divisible time axis, t.

Instead of income 'now' and 'later', the individual has incomes $Y_1, Y_2, \ldots, Y_n$ in successive time periods, or a continuous income, represented by the function $Y(t)$ if time is treated as continuous. His indifference curves, which represent his choice of consumption patterns between different time periods, become hypersurfaces in an n-dimensional hyperspace, or become functionals of the function $Y(t)$. The market rate of interest is described not by straight lines, but by hyperplanes, or by a function determined by a discount function that varies with time, $v_1, v_2, \ldots, v_n$ or a continuous function $v(t)$. Production possibilities similarly become multi-dimensional, or functional.

In this formulation there is no requirement for the market rate of interest to be uniform over time. Individuals may be very willing to save for their own retirement, but not at all for their far-removed descendants. Some production possibilities may give good returns in the near future, with uncertainty thereafter; others may be slow to develop, but give satisfactory long-term returns.

For assessing capital investment projects, it can be shown that the correct criterion is to choose those projects that give a positive net present value, when all returns and costs are discounted at the (varying) market rates of interest. The uniform yield or internal rate of return is not in itself a valid criterion for comparing alternative investments.

9. THE INTRODUCTION OF UNCERTAINTY

We have so far assumed that the future is known with certainty. In reality it is uncertain and changeable. Individuals do not know with certainty their future incomes, nor probably what their future preferences may be. They do not know how long they may live. Capital investments also have uncertain returns. The firm may not know whether the machinery they have constructed will work in the way they had hoped; they may not know whether competition from other firms or other products will diminish the demand for their own; they may not judge consumers' preferences correctly; they may be influenced favourably or unfavourably, by the vagaries of the weather, by natural disasters, or by the accidents of political activity.

However, rather than abandon all investment decisions to a random and unfathomable chance, it is preferable to postulate probability distributions of alternative sets of outcomes. If this is done, we can make progress towards describing an individual's choices through the use of *utility functions*, which allow us to rank an individual's preferences for different probability distributions of events. We can then use the principles of *portfolio selection theory* to describe optimal investment strategies, both for the individual and for the firm. In essence this formalizes the simple concept that the spreading of risks is better than putting all one's eggs in one basket, but the details are beyond the scope of this note. The interested reader is referred to the references at the end.

10. OTHER PRACTICAL CONSIDERATIONS

We have assumed throughout that there is one market rate of interest, which applies equally to all borrowers and lenders. We also assumed that borrowers and lenders were in some way able to arrange their loans without any trouble or expense.

In practice rather few loans take place, at least formally, between individuals. Usually the individual lends to, or borrows from, an institution. The institution may itself be a firm, such as a company or a government; or it may be an investment intermediary, such as a bank, building society, or insurance company. The individual or the investment intermediary may supply capital to firms either on fixed contractual terms, such as fixed interest loans, debentures, loan stocks, mortgages, bonds, etc. or as 'equity' or 'risk-bearing' capital, in the form of ordinary shares or common stocks.

Investment intermediaries are in business themselves to provide a service, and cannot do this unless their charges adequately cover their costs. One way of imposing a charge is for the institution to lend at a rather higher interest rate than it pays on borrowings or deposits. Banks and building societies typically have a small margin between their lending rates to the most preferred borrowers, and the deposit rates to the most preferred depositors, at least where the loans are for the same time period. Since the administrative costs of arranging a loan or servicing a deposit are mainly a constant amount per loan, rather than depending on the amount involved, it is common for this fact to be reflected in interest rates. Small loans carry a high interest rate, and small deposits a low one.

We assumed above that all loans were repaid with certainty. In our assumption of a certain world with perfect knowledge this was practicable. In reality loans that have been granted are not always repaid. If a firm has borrowed for capital investment, it may be unable to repay if the capital investment proves substantially less profitable than was hoped for. If an individual has borrowed in the hope of making repayments out of future income, a change in his circumstances may make it impossible for him to repay. Other individuals or firms are more optimistic about their prospects than is perhaps justified, and yet others may even be less scrupulous about meeting their obligations. For whatever the cause, if there is a higher risk that the loan will not be paid on schedule, a charge for this risk may be made through a higher interest rate. In the United States companies are given ratings by a popular rating service, Moody's, which classifies corporate bonds (company loans) in various grades from the highest, AAA, downwards, depending on an assessment, partly based on objective facts about the capital structure of the company, partly on subjective assessments of its business outlook. Typically, the better the rating of the bond, the lower the redemption yield implied by the current market price. In most countries stocks or bonds issued by the government are considered to have the lowest probability of default, since it is assumed that the government can always raise taxes to pay interest charges and redemption amounts on such loans, and these usually carry the lowest redemption yields.

The risk of loss if a borrower cannot repay a loan may be reduced if he is able to provide some sort of asset as security or collateral. Thus an individual can usually arrange a mortgage on his house at a lower rate of interest than a personal loan backed by no physical security. A company debenture, which is secured against a specific property, may carry a slightly lower rate of interest than an unsecured loan stock, which may be secured only by a floating charge on all the assets of the company.

However, further discussion of these aspects of rates of interest would take us into the field of practical investments, for which the reader is referred, for example, to Van Horne (1984), or many other books.

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