Application of EVT to Risk Capital Estimation

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Generalized Pareto Distribution

- The most widely used EVT models are the models for threshold exceedances using the GPD.
- The GPD is a two parameter distribution with df

$$G_{\xi,\beta}(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi} & \xi \neq 0\,, \\ 1-\exp(-x/\beta) & \xi = 0\,, \end{cases}$$

  where $\beta > 0$, and the support is $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\beta/\xi$ when $\xi < 0$.
- This subsumes:

$\xi > 0$	Pareto (reparametrized version)
$\xi = 0$	exponential
$\xi < 0$	Pareto type II.

- Moments. For $\xi > 0$ distribution is heavy tailed. $E\left(X^k\right)$ does not exist for $k \geq 1/\xi$.

The Role of the GPD

- The GPD is a natural limiting model for excess losses over high thresholds.
- The excess distribution for a (high) threshold u is given by

$$F_u(x) = P\left(X - u \leq x \mid X > u\right) = \frac{F(x+u) - F(u)}{1 - F(u)} \,,$$

  for $0 \leq x < x_F - u$ where $x_F \leq \infty$ is the right endpoint of F.
- The mean excess function of a rv X is

$$e(u) = E(X - u \mid X > u).$$

  It is the mean of the excess distribution above the threshold u expressed as a function of u.

Asymptotics of Excess Distribution

Theorem. [Balkema and de Haan, 1974, Pickands, 1975]
We can find a function $\beta(u)$ such that

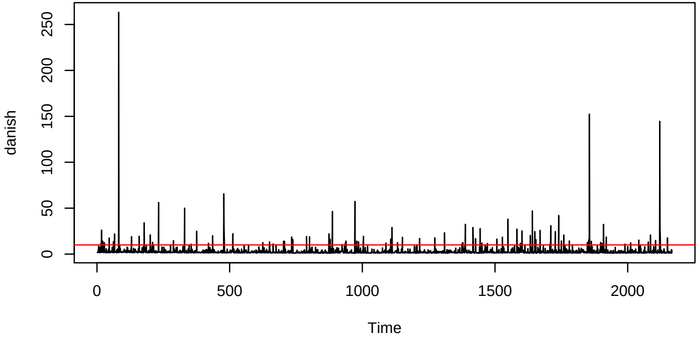
$$\lim_{u \to x_F} \sup_{0 \leq x < x_F - u} \left| F_u(x) - G_{\xi, \beta(u)}(x) \right| = 0 \,,$$

if and only if $F \in \text{MDA}\,(H_\xi)$, $\xi \in \mathbb{R}$.

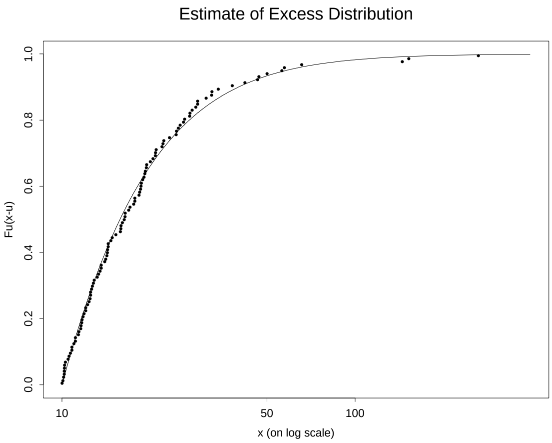
- Thus there is a class of probability distributions MDA(H_ξ) whose excess distributions converge to generalized Pareto with shape parameter ξ.
- All the common continuous distributions used in risk management or insurance mathematics are in MDA(H_ξ) for some value of ξ.

Danish Fire Loss Data

The Danish data consist of 2167 losses exceeding one million Danish Krone from the years 1980 to 1990. A threshold at 10M gives 109 exceedances. ξ and β estimated by fitting GPD to the excess amounts, usually by maximum likelihood.



Estimating Excess df



Excesses Over Higher Thresholds

- If we assume $F_u(x) = G_{\xi,\beta}(x)$ we can infer a model for the excess distribution over any higher threshold. We have that $F_v(x) = G_{\xi,\beta+\xi(v-u)}(x)$ for $v \geq u$.
- The excess distribution over v remains GPD with the same ξ parameter but a scaling that grows linearly with v. Provided $\xi < 1$ the mean excess function is given by

$$e(v) = \frac{\beta + \xi(v-u)}{1-\xi} = \frac{\xi v}{1-\xi} + \frac{\beta - \xi u}{1-\xi}, \qquad (1)$$

  where $u \leq v < \infty$ if $0 \leq \xi < 1$ and $u \leq v \leq u - \beta/\xi$ if $\xi < 0$.
- The linearity of the mean excess function in v is commonly used as a diagnostic for data admitting a GPD model for the excess distribution. It forms the basis for a simple graphical method for deciding on an appropriate threshold as follows.

Using Mean Excess Plot to Set a Threshold

- For positive valued loss data $X_1, \dots, X_n$ we define the sample mean excess function to be an empirical estimator of the mean excess function given by

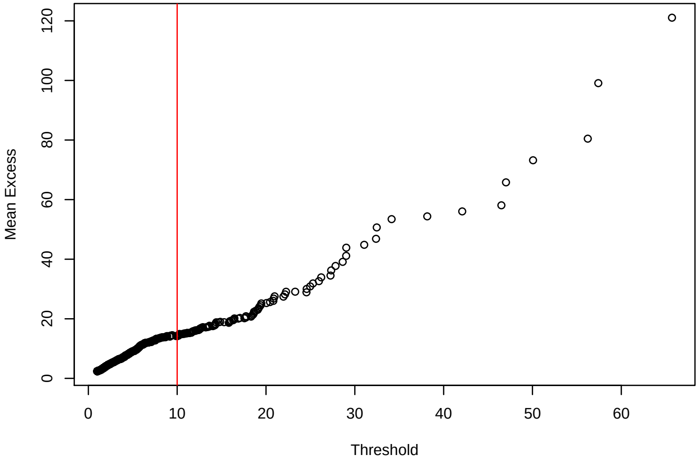
$$e_n(v) = \frac{\sum_{i=1}^{n} (X_i - v) 1_{\{X_i > v\}}}{\sum_{i=1}^{n} 1_{\{X_i > v\}}}.$$

- To view this function we generally construct the mean excess plot

$$\{(X_{i,n}, e_n(X_{i,n})) \ : \ 2 \leq i \leq n\},$$

  where $X_{i,n}$ denotes the ith order statistic. If the data support a GPD model over a high threshold we would expect this plot to become linear in view of (1).

Mean Excess Plot for Danish Data



Modelling Tails of Loss Distributions

Under our assumption that $F_u = G_{\xi,\beta}$ for some u, ξ and β we have, for $x \geq u$,

$$\begin{aligned}\overline{F}(x) &= P(X > u)P\left(X > x \mid X > u\right) \\ &= \overline{F}(u)P\left(X - u > x - u \mid X > u\right) \\ &= \overline{F}(u)\overline{F}_u(x - u) \\ &= \overline{F}(u)\left(1 + \xi\frac{x - u}{\beta}\right)^{-1/\xi}, \end{aligned} \tag{2}$$

which, if we know $F(u)$, gives us a formula for tail probabilities. This formula may be used to derive formulas for risk measures like VaR and expected shortfall.

Calculating VaR and Expected Shortfall

For $\alpha \geq F(u)$ we have that VaR is equal to

$$\mathrm{VaR}_\alpha = q_\alpha(F) = u + \frac{\beta}{\xi}\left(\left(\frac{1-\alpha}{\overline{F}(u)}\right)^{-\xi} - 1\right). \quad (3)$$

Assuming that $\xi < 1$ the associated expected shortfall can be calculated easily to be

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_x(F)dx = \frac{\mathrm{VaR}_\alpha}{1-\xi} + \frac{\beta - \xi u}{1-\xi}. \quad (4)$$

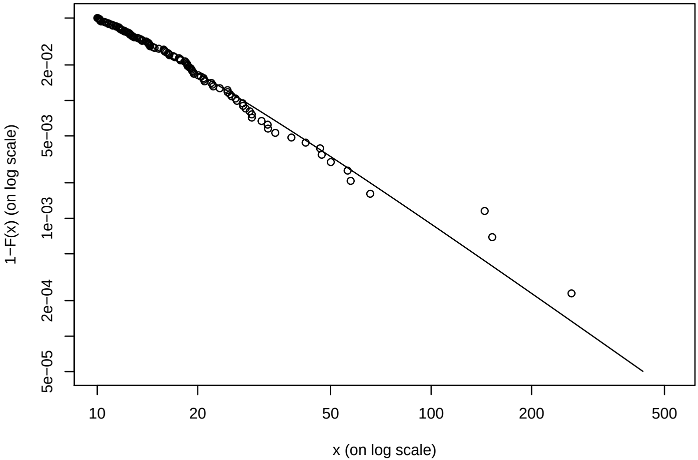
Estimating Tails and Risk Measures

- Tail probabilities, VaRs and expected shortfalls are all given by formulas of the form $g(\xi, \beta, \overline{F}(u))$. We estimate these quantities by replacing ξ and β by their estimates and replacing $\overline{F}(u)$ ny the simple empirical estimator N_u/n.
- For tail probabilities we the estimator of [Smith, 1987]

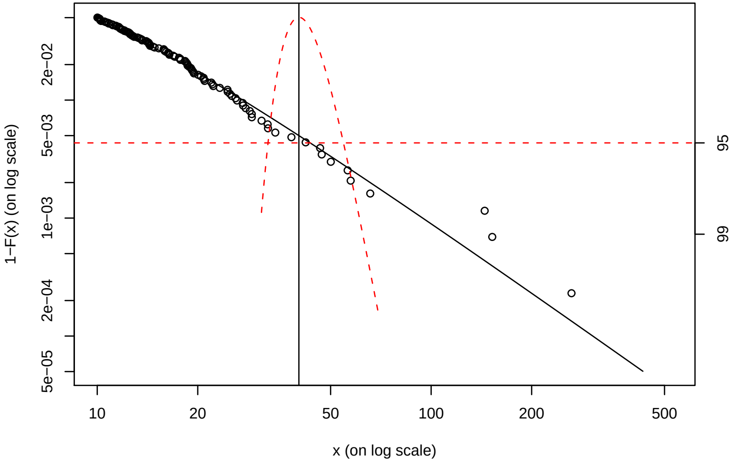
$$\widehat{\overline{F}}(x) = \frac{N_u}{n}\left(1 + \widehat{\xi}\,\frac{x-u}{\widehat{\beta}}\right)^{-1/\widehat{\xi}}, \qquad (5)$$

  which is valid for $x \geq u$. For $\alpha \geq 1 - N_u/n$ we obtain analogous point estimators of VaR_α and ES_α.
- Asymmetric confidence intervals can be constructed using profile likelihood method. [McNeil et al., 2005]

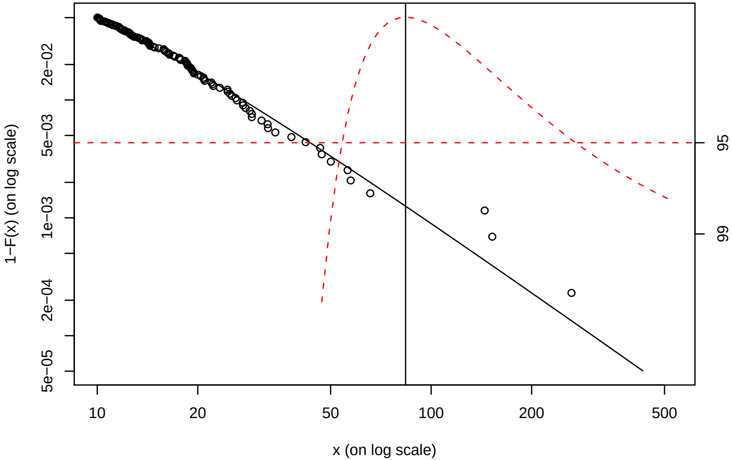
Estimating Tail of Underlying df



Estimate of 99.5% Quantile (VaR)



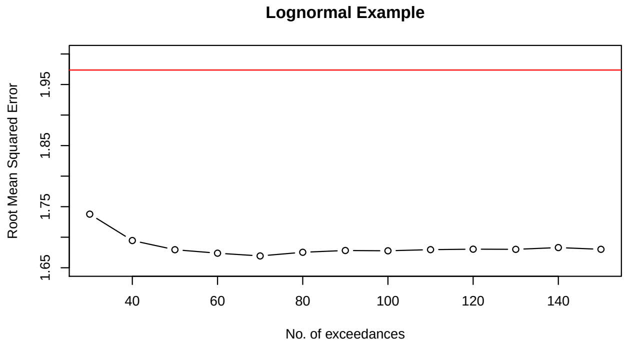
Estimate of 99.5% ES (or cVaR)



Accuracy of EVT Estimates?

Consider problem of estimating 0.995 quantile based on 1000 data.

- How does accuracy of estimate change with threshold?
- How does it compare with empirical quantile estimation?



Accuracy of the Confidence Interval?

- How good is the coverage of the estimated 95% confidence interval? Does it contain the true value 95% of the time?
- The following simulation results are for estimates of the 0.995 quantile based on samples of size 1000 and a threshold at the 90th percentile. 1000 replications.

Distribution	below	within	above
Student t	3.1%	92.7 %	4.2%
Lognormal	2.1%	94.2 %	3.7%

- Intervals very slightly too narrow (neglect error in estimating $\bar{F}(u)$).

For Further Reading

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