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Proxy model: Out of sample validation

Yee Yung Yong, Deloitte
Christopher Hursey, Just Retirement Ltd

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Agenda

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- Introduction
- Scenario accuracy
- Distribution accuracy
- Practical implementation
- Market survey
- Conclusion
- Q&A

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Introduction

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Proxy model

- Proxy models approximate full models.
- Quick evaluation.
- Purposes: SCR calculation, capital projection, what-if investigations.
- Types of proxy model method: Replicating portfolio, curve fitting, radial basis function.
- Most popular approach: curve fitting.
- A proxy model (curve fitting) example with two risk drivers up to quadratic terms:

$$f(x_1, x_2) = a_0 + a_{11}x_1 + a_{12}x_2 + a_{21}{x_1}^2 + a_{22}{x_2}^2 + a_{23}x_1x_2$$

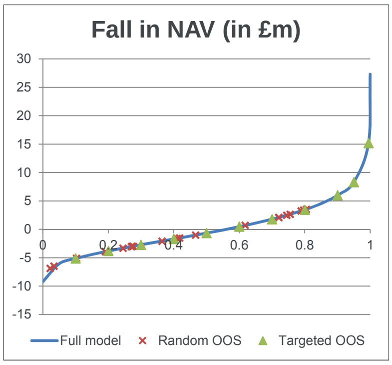
Aims of validation

- Good representation - depends on the use.
- SCR calculation: distribution accuracy.
  - Distribution of proxy values is same as distribution of actual values.
- Business purposes: scenario accuracy.
  - Individual proxy values are the same as individual actual values.
- Challenges:
  - Worst points unknown.
  - Run budget.

Level of validation should be specific to purpose.

Out of sample (OOS) testing

- Points not used for calibration. Impossible to test all.
- Two main questions:
  - How many points do we need?
  - How do we pick these points?
- Random OOS – Not enough coverage.
- Targeted OOS.



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Scenario accuracy

Least squares vs minimax

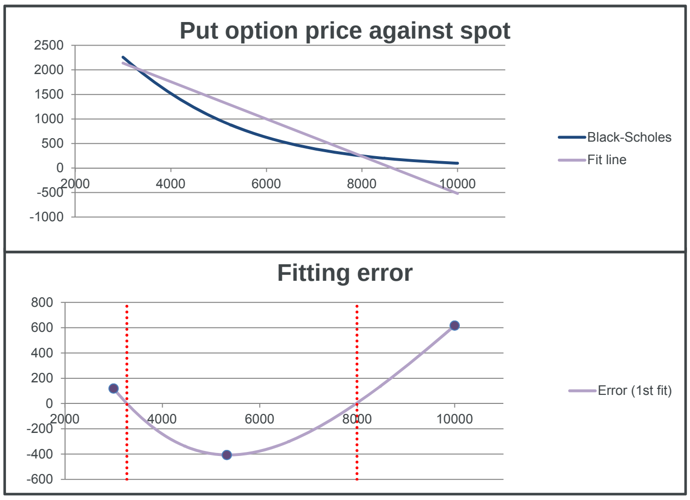
	Least squares	Minimax
Objective	Minimise root mean square error	Minimise maximum absolute error
Implementation	Easier to implement	Harder to implement

- Minimax fit is not too bad in least squares measure.
- But least square fit may not be a good minimax fit.

Minimax: Fitting in 1-D

- Suppose the proxy model is a polynomial of order n (i.e. n+1 basis function).
- Fitting error is a n+1 order polynomial.
- These n+1 zeroes partition the fitting range into n+2 intervals.

Minimax: Fitting in 1-D

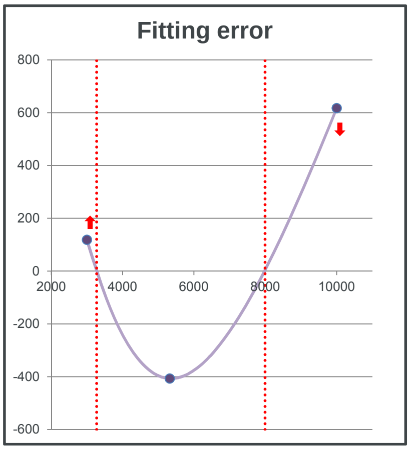


Achilles set

- Achilles set, Ω.
  - A set of points which gives maximum errors.
- Two Achilles subsets: Ω_- and Ω_+.
  - Ω_+ : Largest positive fitting error.
  - Ω_- : Largest negative fitting error.

Find minimax fit

- So how do we attain a minimax fit?
- Look at intervals where the absolute error is largest.
- Reduce error by adjusting the roots.
- But this increases errors in other areas.
- Adjustments are made until the maximum errors in each interval are equal.



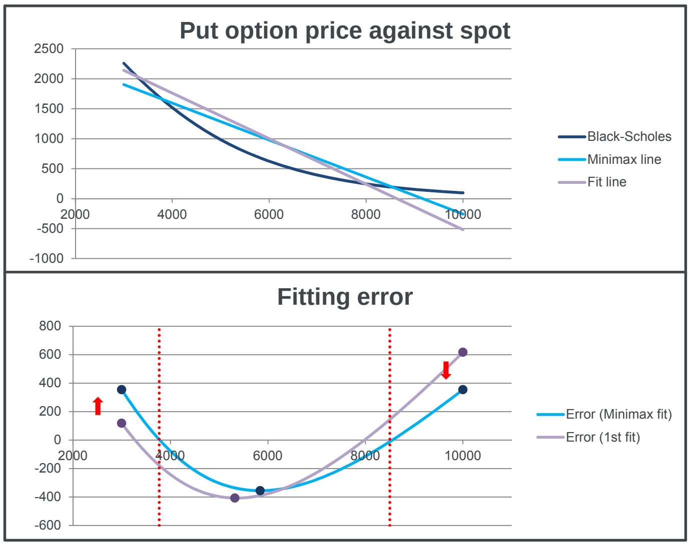
Find minimax fit

- Does a minimax fit exist?
- How many unknowns?
  - n internal points where the fitting errors are the worst.
  - n+1 coefficient β_j for the proxy model.
  - Minimax error m.

Find minimax fit

- Say ω_i is a point where it as the worst fitting error at an interval.
- So we have n+2 equations:
  - $Y(\omega_i) = \beta^T X(\omega_i) + (-1)^i m \,; 0 \le i \le n+1$
- Also for the internal points, these are turning points. Hence we have n equations:
  - $Y'(\omega_i) = \beta^T X'(\omega_i) \,; 1 \le i \le n$
- So we have 2n+2 equations.
- A solution is plausible.

Minimax fit



Sufficient conditions for minimax

- But how do we verify a claimed optimal minimax fit?
- ..when it meets the sufficient condition as set out below:
  - Non-empty Ω_+ and Ω_-.
  - There are two probability measures, $\mathbb{Q}_+$ on Ω_+ and $\mathbb{Q}_-$ on Ω_-, which give the same expectation of the basis function.

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Sufficient conditions for minimax

- Using the example,
  - $\Omega_+ = \{3000, 10000\}$
  - $\mathbb{Q}_+ = \{0.5940, 0.4060\}$
  - $\Omega_- = \{5842\}$
  - $\mathbb{Q}_- = \{1\}$
  - $X = \begin{pmatrix} 1 & 3000 \\ 1 & 5842 \\ 1 & 10000 \end{pmatrix}$
  - $\Rightarrow \mathbb{E}_+(X) = \mathbb{E}_-(X)$
  - Therefore optimal minimax fit achieved.

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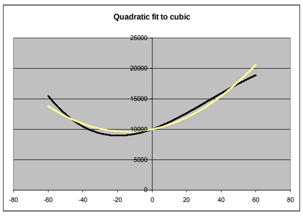
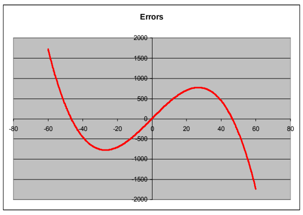
Least Squares

- Established theory based on Legendre polynomial roots for calibration.
- Turning points of error curve can be analytically estimated in advance.
- Constrained (non-Legendre) solutions can be derived.
- Maximum error usually found at extremes of domain and greater than minimax error.





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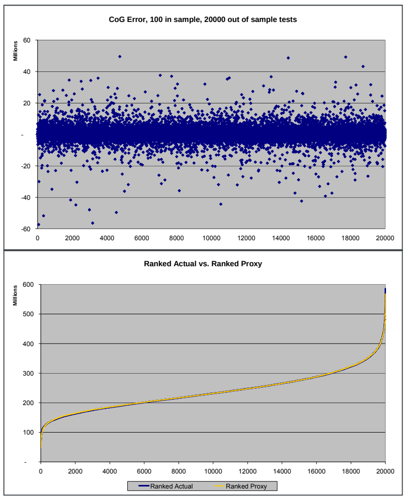
Distribution accuracy

Scenario accuracy vs. distribution accuracy

- Proxy models can be very inaccurate.
- Despite large scenario errors, distribution of values may be the same.
- Capital errors can be considerably less than scenario errors.
- SCR estimate may remain accurate despite scenario inaccuracy!



NB. These results are drawn from an artificial model to illustrate a concept

Distribution accuracy

- Scenario accuracy remains the gold standard, but…
- …If scenario accuracy is not achieved, validation fails?

Poor Scenario Accuracy does not invalidate the SCR!

- So how can we validate the SCR?

Scenario Accuracy	Distribution Accuracy
Hard to achieve	Easier to achieve
Easy to test	Hard to test
Hard to validate	Easier to validate?

Error distribution

- Want distributions of proxy values and actual values the same.
- Can infer conditions for error distribution:

Want	*E(Actual) = E(Proxy)*
But	*Actual = Proxy – Error*
Therefore	*E(Proxy – Error) = E(Proxy)*
Implies	***E(Error) = 0***

- Similarly, it can be shown:

Var(Error) = 2Cov(Proxy, Error)

Skew(Error) = 0

Implications

- Symmetrical errors desirable.
- If errors are non-zero and uncorrelated to proxy values then variance of proxy capital distribution is different from variance of actual capital distribution.
- Distribution accuracy relies on re-ranking scenario results.
- Successive scenario results must be close enough to allow re-ranking.
- The more extreme the percentile, the greater the number of scenarios required to smooth capital estimate volatility due to proxy errors.

Testing distribution accuracy

- Problem remains of picking sufficient points to cover whole risk distribution.
- Multi-dimensional risk distribution leads to a one-dimensional capital distribution.
- Start from capital distribution and work back to risk scenarios.
  - Rank the N proxy results from full proxy model run.
  - Take every n^{th} scenario to give every (100 x n/N) percentile scenario across whole distribution.
  - Run the selected scenarios through heavy model and test.

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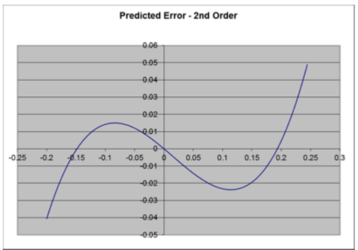
Practical implementation (Real World Example)

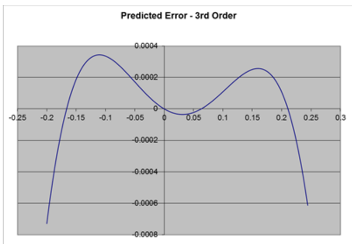
Key steps in process

1. Use risk factor distributions to set limits of domain, e.g. 1 in 2000.
2. Identify calibration nodes and OOS test points for scenario accuracy.
3. Perform heavy model calculations (1st drop).
4. Calibrate proxy model.
5. Test scenario accuracy.
6. Repeat steps 2 to 5 as required.
7. Run proxy model and identify OOS test points for distribution accuracy.
8. Perform heavy model calculations (final drop).
9. Test distribution accuracy.

Identifying fitting and test points

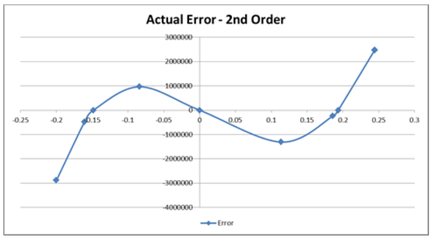
- For each risk factor identify 2nd and 3rd order nodes from predicted error curves.
- Total 12 points per risk.
  - 5 calibration
  - 7 validation
- Optional to include 1 in 200 as additional validation points.
- Analysis extended to risk interactions and multivariate non-linearity functions.

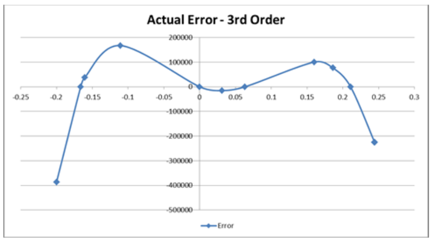




Calibrate and test scenario accuracy

- For each risk factor plot actual error curves.
- Compare to predicted error curves from previous step.
- Refine calibration as required.
  - 4th order
  - Minimax
- Select order of calibration for each risk component.

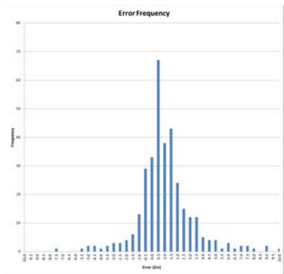


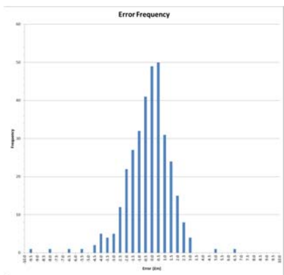


Additional Scenario testing

- Supplement targeted testing with random testing and real-world scenarios.
- 337 random (uncorrelated) scenarios, all assets & liabilities, all risks.

Metric	Assets	Liabilities
Average error	0.01%	(0.01%)
StdDev. Error	0.03%	0.02%
Max error	0.13%	0.09%
Min error	(0.11%)	(0.16%)
MSE	0.58%	0.46%
R^2	99.998%	99.999%





Testing distribution accuracy

- 50,000 Scenarios.
- Poor scenario accuracy by any measure, validation fails.
- Rank proxy results and read off every 0.5th percentile scenario.
- Evaluate test scenarios in heavy model.
- Perform statistical analysis on implied distributions.

50,000 Scenarios

Scenario	Actual	Proxy	Error
1	- 141,665,741	- 255,338,487	- 113,672,746
2	- 141,120,356	- 198,928,837	- 57,808,482
3	- 159,664,503	- 187,052,466	- 27,387,963
|	|	|	|
\V	\V	\V	\V
49998	376,298,206	346,215,023	- 30,083,183
49999	331,774,782	361,937,328	30,162,547
50000	424,210,220	386,176,744	- 38,033,476

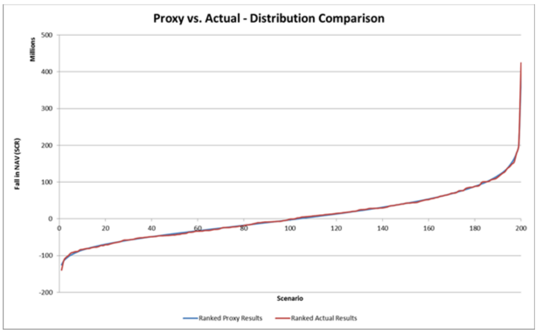
200 Scenarios

Scenario	Actual	Proxy	Error
250	- 116,877,674	- 124,792,951	- 7,915,277
500	- 110,942,144	- 113,696,369	- 2,754,224
750	- 110,836,964	- 107,627,093	3,209,871
|	|	|	|
\V	\V	\V	\V
49500	164,049,043	178,141,616	14,092,573
49750	212,021,731	201,283,494	- 10,738,237
50000	424,210,220	386,176,744	- 38,033,476

NB. These results are drawn from an artificial model to illustrate a concept

Initial Distribution Results

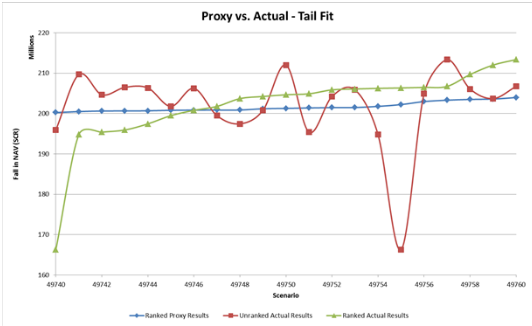
- Good distribution accuracy confirmed.
- 50,000 actual values not available in reality.
- Statistical analysis possible on the two capital distributions.
- How can we show SCR is valid despite scenario inaccuracy?



	200 Proxy	200 Actual	50,000 Proxy	50,000 Actual
Implied SCR	201.3	212.0	201.2	204.2

Final distribution results

- 20 scenarios around 99.5th percentile tested.
- Use ranked results to refine 200 'actual' at the tail.
- 'Actual' SCR estimate improved.
- Proxy SCR validated!



	200 Proxy	200 Actual (Refined)	50,000 Proxy	50,000 Actual
Implied SCR	201.3	204.7	201.2	204.2

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Market survey

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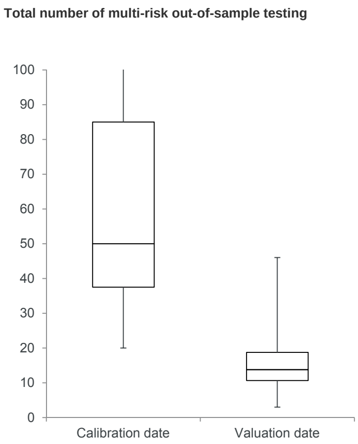
Market survey

- 9 participants for the survey.
- Calibrates and tests ahead of year-end then roll forward.
- Annual calibrations with some having more regular small refinements.

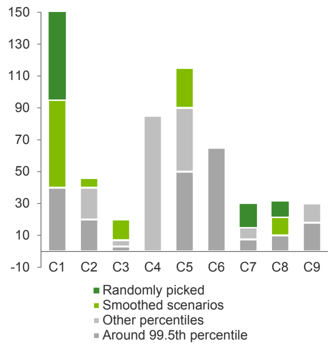
Market survey



- More effort at calibration date than valuation date due to time constraints.
- The median:
  - 50 tests at calibration date.
  - 10 tests at valuation date.
- The maximum number of tests performed at the calibration date is 1000.

Market survey

At calibration date



- Observations:
  - Majority around 99.5th percentile.
  - Random OOS not common.
  - One: 650 random scenarios.

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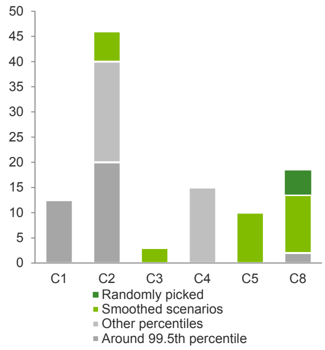
Market survey

At valuation date



- Observations:
  - 1st : smoothed scenarios.
  - 2nd: around 99.5th percentile.

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Conclusion

Conclusion

- Minimax fit has bounded least square errors.
- Can validate SCR without scenario accuracy.
- Efficiency of calibration and validation is vital.
- Out of sample scenarios:
  - Achilles set.
  - Smoothed scenarios and scenarios around 99.5^{th} percentiles.
  - Real-world scenarios.
  - Scenarios across whole capital distribution.
- Targeted out of sample scenarios are a powerful validation tool.

Questions and Comments

For further information please email:

yyong@deloitte.co.uk

chris.hursey@justretirement.com

References:

- Smith, A. Minimax proxy models
- The Proxy Model Working Party. *Heavy models, Light models and Proxy models*

Expressions of individual views by members of the Institute and Faculty of Actuaries and its staff are encouraged.

The views expressed in this presentation are those of the presenter.