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FURTHER REMARKS ON THE RELATIONSHIP BETWEEN THE VALUES OF LIFE ANNUITIES AT DIFFERENT RATES OF INTEREST, INCLUDING A DESCRIPTION OF A METHOD OF FIRST-DIFFERENCE INTERPOLATION AND A REFERENCE TO ANNUITIES-CERTAIN

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THE present note has its origin in an earlier contribution to the *Journal* (Vol. LXIII, pp. 70–81). The initial object in view was an improvement in the method discussed in 1932, but in the course of the investigation there came to light certain other matters which it is thought may be worth placing on record.

It is convenient to group the approximate methods to be discussed under three separate headings.

I. PROPOSED MODIFICATION OF THE METHOD DESCRIBED IN *J.I.A.* VOL. LXIII

The central idea underlying this method was that, given annuity-values and increasing annuity-values at a single rate of interest (preferably a central rate), it was possible to obtain approximate annuity-values at other rates of interest by means of the formula

$$\log_e \frac{a'}{a} = -\frac{k}{1+k/\mathrm{R}},$$

where a and a' are at rates i and $i+h$ respectively and $k = hv\,(\mathrm{I}a)/a$.

An essential feature was that the quantity R should be capable of being considered, with fair approximation, a constant except for variations in i or in the type of annuity, i.e. single-life, joint-life, or last-survivor.

The examples worked out in support of the formula showed that there was some justification for the assumption made about R, but consideration of the errors indicated that the following factors, all of which were ignored, had also an influence on R:

(1) The value of i' $(=i+h)$.
(2) The age or ages involved in the annuity.
(3) The mortality table employed.

As a first step in the search for possible improvement in the formula some specimen values of $1/\mathrm{R}$ were calculated on the basis of the A 1924–29 ultimate table (this table was not available in 1932). The values shown in Table 1 are in respect of single lives.

The figures given in brackets in Table 1 may be termed central values based upon the mean of the values for $h = -\cdot 01$ and $h = +\cdot 01$. They could have been obtained directly from the expression

$$\frac{1}{2}\left[\frac{1}{\log_e (a/a')} + \frac{1}{\log_e (a/a'')}\right],$$

where a'' is at rate $i-h$. The reason for the final column of Table 1 will appear immediately.

Table 1

$i = \cdot 03$

Age	Values of $\frac{1}{\log_e (a/a')} - \frac{1}{k} = \frac{1}{R}$					$\frac{\cdot 4v\,(\mathrm{I}a)}{a^2}$
	$h = -\cdot 01$		$+\cdot 01$	$+\cdot 02$	$+\cdot 03$	
10	·310	(·315)	·320	·323	·326	·322
20	·299	(·304)	·308	·312	·314	·311
30	·291	(·295)	·298	·302	·305	·301
40	·291	(·293)	·295	·299	·301	·294
50	·301	(·302)	·303	·303	·305	·293
60	·317	(·318)	·318	·321	·322	·305
70	·350	(·353)	·355	·350	·355	·337

$i = \cdot 04$

Age	Values of $\frac{1}{\log_e (a/a')} - \frac{1}{k} = \frac{1}{R}$					$\frac{\cdot 4v(\mathrm{I}a)}{a^2}$
	$h = -\cdot 02$	$-\cdot 01$		$+\cdot 01$	$+\cdot 02$	
10	·342	·347	(·350)	·353	·356	·344
20	·326	·331	(·335)	·338	·340	·332
30	·315	·320	(·322)	·324	·327	·320
40	·309	·313	(·316)	·319	·318	·310
50	·313	·314	(·314)	·314	·318	·306
60	·330	·333	(·333)	·333	·332	·314
70	·361	·362	(·362)	·362	·363	·343

If now we examine the variations in $1/R$ we see that, reading from left to right, there is a clearly marked tendency for the value to increase although at older ages the movement is but slight. These lateral movements correspond to changes in i'. Vertical movements corresponding to changes in the age are capable of fair representation by the values in the last column; in other words, $1/R$ is roughly proportionate to $v\,(\mathrm{I}a)/a^2$. This is consistent with the remarks made in *J.I.A.* Vol. LXIII, p. 77 (fifth paragraph), because $v\,(\mathrm{I}a)/a^2$ is approximately proportionate to $\Delta\,(1/a)$.

It must be remembered that our primary concern is to ensure a close correspondence between actual and assumed values at the *younger* ages. As the age increases, appreciable errors in $1/R$ have less and less effect on the results of the approximation. Bearing this in mind, it was decided to try the effect of replacing $1/R$ by an expression of the form

$$C\,(1+h) \times \frac{v\,(\mathrm{I}a)}{a^2} = \frac{Ck\,(1+h)}{ha},$$

where C could, it was hoped, be assumed to vary only with i.

The basic formula thus became

$$\log_e \frac{a'}{a} = -\frac{k}{1 + Ck^2\,(1+h)/ha},$$

which for purposes of calculation could be more conveniently written as

$$\log_{10} a' = \log_{10} a - \frac{\cdot 43429}{1/k + Ck\,(1+h)/ha}. \qquad (1)$$

Experiment, based upon the A 1924–29 ultimate table, suggested that the following would be suitable values for C:

at 2% interest, ·392,
at 3% interest, ·396,
at 4% interest, ·404,
at 5% interest, ·416.

The formula has also been tested with a base rate of interest of 6%, C being taken as ·422. In this case, better results are obtained by replacing $(1+h)$ in the formula by $(1+\frac{1}{2}h)$.

Furthermore, e_x can be used as a base with very fair results, C being taken as ·390 and $(1+h)$ replaced by $(1+\frac{3}{2}h)$.

Fairly extensive trials (including the recalculation by the modified formula of the examples given in 1932) suggest that the following claims may be made in respect of C:

(1) The same values which are suitable for single-life annuities are also satisfactory for joint-life annuities and are reasonably satisfactory for last-survivor annuities.

(2) As between different mortality tables there is no occasion for appreciable adjustment. Identical values of C appear to suit the more modern tables

Table 2

(A) $a=a_x$, $i=\cdot 03$, $C=\cdot 396$

x	$h=-\cdot 01$	$+\cdot 01$	$+\cdot 02$	$+\cdot 03$
10	+·010	+·003	+·005	+·010
30	+·003	+·002	+·003	+·006
50	−·003	−·002	−·004	−·006
70	·000	−·002	−·001	−·002

(B) $a=a_x$, $i=\cdot 04$, $C=\cdot 404$

x	$h=-\cdot 02$	$-\cdot 01$	$+\cdot 01$	$+\cdot 02$
10	−·004	−·003	−·001	−·001
30	+·006	·000	+·001	+·004
50	−·009	−·001	·000	−·001
70	−·002	·000	·000	−·001

(C) $a=a_{xx}$, $i=\cdot 04$, $C=\cdot 404$

x	$h=-\cdot 015$	$-\cdot 01$	$+\cdot 01$	$+\cdot 02$
10	−·001	−·001	−·001	+·002
30	+·001	+·001	+·001	+·003
50	−·003	−·001	−·002	−·002
70	−·001	·000	·000	·000

(D) $a=a_{\overline{xx}}$, $i=\cdot 04$, $C=\cdot 404$

x	$h=-\cdot 015$	$-\cdot 01$	$+\cdot 01$	$+\cdot 02$
10	+·011	+·001	+·001	+·002
30	+·019	+·006	+·004	+·013
50	+·001	·000	+·002	+·003
70	−·004	−·001	·000	−·003

(e.g. A 1924–29, $a(f)$, and E.C.R.D.), while values higher by only ·004 fit the older tables (e.g. $O^{[af]}$, H^M, and Carlisle).

(3) The use of the 'C' formula leads to greater accuracy and range than the '1/R' formula. By the revised formula the total errors for the 1932 examples are cut down by more than 60%.

Some examples of errors in applying formula (1) to the A 1924–29 ultimate table are given in Table 2.

While the method is naturally at its best when i is a central rate like ·03 or ·04, the results are also good for other values of i. Generally speaking, for close approximation the formula may be considered to have an effective range of from 2% below to 3% above the base rate of interest.

Another application of the formula is the approximate calculation of increasing annuity-values from either two or three annuity-values at different rates of interest, the latter process enabling us to dispense with the value of C. Taking the familiar example where we require (Ia) from a, a' and a'', the two latter annuities being at rates $i+h$ and $i-h$ respectively, the appropriate formula is

$$\frac{1}{k}=\frac{a}{hv\,(Ia)}=\frac{\cdot 43429}{2}\left[\frac{1-h}{\log_{10}(a/a')}-\frac{1+h}{\log_{10}(a/a'')}\right]. \qquad (2)$$

To give an indication of the degree of accuracy, some values of $(Ia)_x$ by H^M (Text Book) $4\frac{1}{2}\%$ are shown in Table 3, h being taken as ·015—a fairly wide interval.

Table 3

x	$(Ia)_x$	
	True	Approximate
10	313·93	314·21
20	270·67	270·84
30	225·44	225·52
50	120·05	120·11
70	34·50	34·53

A general formula corresponding to (2), where the annuity-values are at rates of interest i, $i+h$ and $i+rh$, may readily be obtained.

Probably these formulae for (Ia) are in general more accurate—particularly at the younger ages—than any existing formulae based on only three annuity-values.

II. METHOD OF FIRST-DIFFERENCE INTERPOLATION

It is assumed in this section that values of (Ia) are not available.

So far as the writer is aware, there is no record of any reliable method of interpolation (based on only two annuity-values) which does not depend upon some such device as an equivalent term-certain or reference to a different mortality table or to substituted equal ages—the last-mentioned for joint-life annuities. It may therefore be of interest to see whether annuity-values can be made to provide their own source of interpolation, without recourse to external aids.

The annuity-value itself, when i is variable, does not respond well to first-difference interpolation. On various occasions log a and $1/a$ have been suggested as substitutes; they undoubtedly give better results than a, although usually not up to the standard of accuracy of the devices already mentioned.

Let us consider formula (1) given on p. 75 of *J.I.A.* Vol. LXIII. If we stop at the term involving k^2, this becomes

$$\log_e a' = \log_e a - k + c_2 k^2, \text{ approximately,} \tag{3}$$

where c_2 (for a single-life annuity) $= N_x \times \Sigma S_x/(S_x)^2 - \frac{1}{2}$.

The corresponding formula, if we had operated on $1/a$ instead of $\log_e a$, is

$$\frac{1}{a'} = \frac{1}{a}[1 + k - (c_2 - \tfrac{1}{2})\, k^2], \text{ approximately.} \tag{4}$$

It may be proved that c_2 is always positive, and in practice it is found to have a maximum value of $\frac{1}{2}$, which it attains at the last age in the mortality table for which a_x has a value. Consequently $-(c_2 - \frac{1}{2})$ is also positive, falling to zero at the end of the table.

The fact that the respective coefficients of k^2 in the above two equations are of the *same* sign, bearing in mind that the second equation deals with a *reciprocal*, gives good ground for assuming that separate first-difference interpolations based on $\log a$ and $1/a$ respectively will yield results which bracket the true value.

By a system of weights applied to the results of each interpolation it should be possible to get close to the true value. There is, however, a more direct process than this. If we operate on $(1/a)^p$, where p is any unknown quantity, we arrive at a third equation analogous to the two already given. This is

$$\left(\frac{1}{a'}\right)^p = \left(\frac{1}{a}\right)^p [1 + pk - p\,(c_2 - \tfrac{1}{2}p)\, k^2], \text{ approximately.} \tag{5}$$

Now put $p = 2c_2$, with the result that the term involving k^2 vanishes.

Consequently we may conclude that, with a suitable value for p, $(1/a)^p$ should form an excellent medium for first-difference interpolation.

In the 1932 note, c_2 was given (in the form of $1/R$) an average value for each value of i. It is essential to preserve simplicity for a practical method of interpolation, and we clearly cannot cater for any variations in c_2 (and consequently in p) otherwise than with i. Fortunately, such other variations are only of minor effect, as the previous work has shown.

A simple expression for p giving good results for practical rates of interest is $\cdot 45 + 5i$, which is approximately consistent with our previous knowledge of the movement in c_2 resulting from variations in i. In this expression i, for interpolation purposes, is best taken as the average of the three rates of interest involved.

Examples of errors arising from this method of interpolation are given in Table 4.

Table 4

A 1924–29 ultimate, $a = a_x$

x	2% value from 3% and 4% ($p = \cdot 6$)	4% value from 2% and 6% ($p = \cdot 65$)	5% value from 4% and 6% ($p = \cdot 7$)	6% value from 4% and 5% ($p = \cdot 7$)
20	$-\cdot 011$	$+\cdot 014$	$+\cdot 004$	$-\cdot 007$
40	$+\cdot 005$	$-\cdot 014$	$-\cdot 005$	$+\cdot 006$
60	$-\cdot 003$	$+\cdot 002$	$-\cdot 001$	$\cdot 000$

There may be some slight variation in accuracy as between different mortality tables, and we should expect $a_{\overline{xy}}$ to respond rather less readily to the method than a_x or a_{xy}.

Formula (5) supplies another method of calculating (Ia) from only two annuity-values, because (with an appropriate value for p) we obtain the approximate relationship

$$k=\frac{hv\,(\mathrm{I}a)}{a}=\frac{1}{p}\left[\left(\frac{a}{a'}\right)^p-1\right]. \tag{6}$$

Table 5 shows some results by formula (6) on the basis of $\mathrm{H^M}$ (Text Book) 3%, with $p=\cdot 45+5i=\cdot 6$, and $h=\cdot 01$.

Table 5

x	$(\mathrm{I}a)_x$	
	True	Approximate
20	402·38	400·36
40	232·89	232·35
60	85·22	85·01

The practical value of the method of interpolation just described is, of course, intimately bound up with the comparative lack of variation in the values of c_2. It does not appear that the same characteristic applies to a sufficient extent in respect of actuarial functions other than the ordinary annuity-values. On the other hand, it is believed that, with any actuarial function of the form $\sum_1^n v^t f(t)$, separate first-difference interpolations based on logarithm and reciprocal respectively (i being the variable) will give results bracketing the true values.

A problem which may be worth investigating is whether, and with what limitations as to type of function, the feature just referred to is true of second-difference interpolations.

III. SOME APPLICATIONS TO ANNUITIES-CERTAIN

Life annuities and annuities-certain have much in common, and it is not surprising that the work in the foregoing section should have some bearing on the latter type of annuity. In the Institute Text-Book on Compound Interest (revised edition, 1931) attention has been drawn to the fact that $1/a_{\overline{n}|}$ is a more suitable medium for interpolation than $a_{\overline{n}|}$. A good illustration of this may be found in the comparative success of the well-known formula on p. 170 of the Text-Book, which is derived from the expansion of $1/a_{\overline{n}|}$.

By analogy with life annuities we may expect interpolation based on $(1/a_{\overline{n}|})^p$ to be even more accurate than when $1/a_{\overline{n}|}$ is operated on. In the case of annuities-certain, an investigation into the values of c_2* shows that a single

* The exact value of c_2 is $\dfrac{a_{\overline{n}|}\times\sum_1^n \frac{1}{2}t\,(t+1)\,v^t}{\left[\sum_1^n tv^t\right]^2}-\dfrac{1}{2}$,

but it is quicker to calculate it approximately from $\dfrac{1}{2}\left[\dfrac{1}{\log_e(a_{\overline{n}|}/a'_{\overline{n}|})}+\dfrac{1}{\log_e(a_{\overline{n}|}/a''_{\overline{n}|})}\right]$, (cf. expression on p. 447).

average value of p, namely, the convenient one of $\cdot 5$, will serve well for interpolation purposes provided that $n/a_{\overline{n|}} < 3$. This covers the great majority of cases likely to arise in practice, but if $n/a_{\overline{n|}} > 3$ it is usually better to operate on $1/a_{\overline{n|}}$.

As an example we may consider the effect of operating on $(1/a)^{\frac{1}{2}}$ instead of $1/a$ in Chap. VIII of the Text-Book, Arts. 8 and 9. The formula corresponding to (5) in Art. 8 would be

$$i = i' + 2i' \frac{(a'/a)^{\frac{1}{2}} - 1}{1 - nv'^{n+1}/a'}, \tag{7}$$

and the numerical result for the example in Art. 9 is $i = \cdot 028447$.

We thus obtain in one step an answer practically as accurate as that resulting from two steps in the Text-Book.

Similar improvement in accuracy should result when the method is one of direct first-difference interpolation, as discussed in Chap. VIII, Art. 16.

The compound interest function A—representing the present value of a debenture or other security—is another function which may with advantage be dealt with on the same lines as $a_{\overline{n|}}$. That is to say, whether for the purpose of obtaining a short expansion or for direct interpolation, $(1/A)^{\frac{1}{2}}$ will normally be a better medium on which to operate than A or $1/A$.

Finally, it may be worth studying from the angle of this note the old problem of determining, without the aid of interest tables, the rate of interest in $a_{\overline{n|}}$.

If in formula (6) we put $a' = a_{\overline{n|}}$ and $a = n$, we shall have $h = i$ and

$$\frac{hv\,(\mathrm{I}a)}{a} = i \times \left[\frac{v\,(\mathrm{I}a)_{\overline{n|}}}{a_{\overline{n|}}}\right]_{i=0} = \frac{n+1}{2}\, i.$$

Hence, putting $p = \frac{1}{2}$, we obtain the approximate formula

$$i = \frac{4}{n+1}\left\{\left(\frac{n}{a_{\overline{n|}}}\right)^{\frac{1}{2}} - 1\right\}. \tag{8}$$

Formula (8) is found to give rough values of i which (provided $n/a_{\overline{n|}} < 4\cdot 5$) do not differ from the true values by more than about 3% thereof.

A considerably more accurate formula may be obtained, though largely by empirical means. Reverting this time to formula (5) and again putting $a' = a_{\overline{n|}}$ and $a = n$, we shall find that the term involving k^2 vanishes when

$$p = \frac{n \times \sum_{1}^{n} t\,(t+1)}{\left[\sum_{1}^{n} t\right]^2} - 1,$$

which simplifies to

$$p = \frac{n+5}{3\,(n+1)}.$$

This may be taken as a first approximation to p. Now let us assume that

$$i = \frac{2}{(n+1)\,p}\left[\left(\frac{n}{a_{\overline{n|}}}\right)^{p} - 1\right], \tag{9}$$

where

$$p = \frac{n+5}{3\,(n+1)} + m.$$

Inverse calculation from formula (9) of various values of p shows that over a considerable range m is closely proportionate to $(n+1)i$ and can be represented with good approximation by $\frac{(n+1)i}{30}$. Thus

$$p=\frac{n+5}{3(n+1)}+\frac{(n+1)i}{30}.$$

It is not, however, convenient to have i in the expression for p, so we replace the rate of interest by its approximate equivalent in terms of formula (8), leading to

$$p=\frac{n+5}{3(n+1)}+\frac{2}{15}\left\{\left(\frac{n}{a_{\overline{n}|}}\right)^{\frac{1}{2}}-1\right\}=\frac{1}{5}+\frac{4}{3}\left[\frac{1}{n+1}+\frac{1}{10}\left(\frac{n}{a_{\overline{n}|}}\right)^{\frac{1}{2}}\right],$$

the latter expression being slightly simpler for calculation.

A further small addition to p of ·001 for every complete 20 years in n makes for greater all-round accuracy.

Adopting the expression for p just given, including the small adjustment referred to, Table 6 gives some values of $100i$ by formula (9).

Table 6

n	2%	4%	6%	8%	10%
100	2·001	4·002	—	—	—
80	2·001	4·000	—	—	—
60	2·000	3·998	5·998	—	—
40	2·000	3·998	5·997	8·001	10·006
20	2·000	4·000	6·001	8·002	10·003
10	1·999	4·000	6·000	8·001	10·004

The formula is a simple one to operate because, having determined $1/(n+1)$ and $\log(n/a_{\overline{n}|})$ in order to find p, we use the same quantities again in calculating i. Close results are obtained for values of $n/a_{\overline{n}|}$ up to about 4·5 and, within this range, errors in excess of one-halfpenny per cent in the rate of interest are likely to be rare.

If $n/a_{\overline{n}|}>4{\cdot}5$, close approximations to i can be obtained from the formula

$$i=\frac{1-(1+1/a_{\overline{n}|})^{-n}}{a_{\overline{n}|}}. \qquad (10)$$

Formulae (8) and (9) may have other applications, e.g. to the function θ described in *J.I.A.* Vol. LX, pp. 343–5.

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