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32nd ANNUAL GIRO CONVENTION

The Imperial Hotel, Blackpool

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Information and Entropy
From Black Holes to Black Scholes

David Sanders

Entropy and Information
From Black Holes to Black- Scholes

Agenda:

- Introduction to Black Holes
- Introduction to Probability Theory and the Inversion Problem
- A solution
- Introduction to Risk Measures
- Generalisation of Black Scholes with Risk Measures
- Work in Progress

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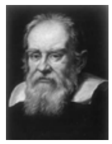
A Thought Experiment – after Einstein

Take two stones of different weight
Drop them from the Tower
Which hits the ground first!

Answer – Galileo
In a vacuum both at the same time



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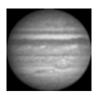
Principle of Equivalence



- This works no matter what the weight
- Even for a photon
- Now throw the stones (and photon)
  - They will follow the same trajectory
- We know the speed of light is finite
  - Jupiters moons
- Therefore (from Newton) there is a mass from which light can't escape
- Thus Newton and Galilieo predict Black Holes!

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Don't believe all you read in papers

- Newton Predicts light bends as it passes a large body
- Einstein (1911) did the same
- Einstein bends twice as much
- Eddington (1919) proved him right
  By ignoring all photographic plates that proved him wrong!
- What has this to do with actuarial science?

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Hawking Radiation

The radiation from a Sshwarzschild black hole is black body ratiation with temperature:

$$T = \frac{\hbar\, c^3}{8\pi G M k}$$

where $\hbar$ is the Reduced Planck Constant, *c* is the speed of light, *k* is the Bolltzmann Constant, *G* is the Gravitational constant, and *M* is the mass of the black hole.

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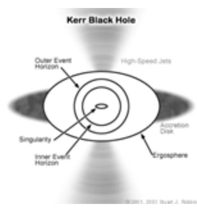
From Black Holes to Black- Scholes

What does this say



In the most perfect mathematical physical concept all the Information is contained in the Entropy of the body!

INFORMATION = ENTROPY

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Entropy and Information
From Black Holes to Black- Scholes

In reserving in actuarial science we have two main objectives

1. **The completion of an undeveloped claims triangle to its ultimate position = the inversion problem**
2. **The measurement of uncertainty or risk in that estimation**

   THE SOLUTIONS OF BOTH THESE PROBLEMS INVOLVE INFORMATION AND ENTROPY

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Probability Theory

- Probability is nothing but common sense reduced to calculation – Laplace
- Probability was first stated as an explicit formal principle in the Ars Comjectandi of Jacob Bernouilli (1713). It was given an interesting title; the Principle of Insufficient Reason.
- Keynes renamed it the Principle of Indifference.

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Plausible Reasoning

- Policeman sees someone wearing a mask, crawling through a broken window in a jewellers with a bag of gemstones and watches
- Conclusion – he's a robber
- Plausible reasoning – not certainty
- Effected by experience
- Prior information and common sense

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Plausible Reasoning

- Plausible reasoning of outcome
- Qualitative correspondence with common sense
- Consistency
- "I know this defies the law of gravity, but, you see, I haven't studied law!" - Bugs Bunny

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Plausible Reasoning Examples

- Bending of light
- Sun travels around the earth (and I mean this!)
- The claims reserve required is £1 m
- My estimate for the Katrina Loss is $20bn
- My estimate for the Katrina Loss is $60bn

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Plausible Reasoning - Axioms

Rule 1: $(AB|C) = (A|BC)(B|C) = (B|AC)(A|C)$

Rule 2: $(A|B) + (a|B) = 1$

Rule 3: $(A+B|C) = (A|C) + (B|C) - (AB|C)$

Rule 4: If $\{A_1 \ldots A_n\}$ are mutually exclusive and exhaustive, and B does not favor any over any other, then $(A_i|B) = \frac{1}{n}$, i=1, 2, . . . n.

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Probability Theory

- We recognise that a probability assignment is a means of describing a particular state of knowledge .
- if the available evidence gives us a reason to consider a proposition A1 neither more or less likely than Proposition A2, then the only way is to ascribe equal probabilities; p(A1)=p(A2.)

$$p(A) = \frac{M}{N} = \frac{(Number of case favorable to A)}{(Total Number of equally possible cases)}$$

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Probability Theory – Binary Trial

Assume we have a binary trial. Then we have the

$$P(m|n,p) = \binom{n}{m} p^m (1-p)^{n-m}$$

For such a model and for a large number of trials, the observed frequency tends to p

$$P(p - \varepsilon < f < p + \varepsilon|p,n) \to 1$$

However, we don't know how large n must be!

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Probability Theory – Continuous case

For this we have the more detailed solution where f is considered as a continuous distribution and the probability that (f < m/n <f+df) becomes Guassian

$$P(df|n,p) \sim \left[\frac{n}{2\pi p(1-p)}\right]^{½} \exp\left[- \frac{n(f-p)^2}{2p(1-p)}\right] df$$

This is the de Moivre-Laplace Theorem

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Probability Theory – Inversion Problem

These results concern sampling distributions, and when the population numbers (N,M) are known,

The problem left by Bernoulli can be summaries as follows, These results concern sampling distributions, i.e. given p=M/N what is the probability that we shall see specific sample numbers (m.n). Bernoulli tried to solve when the sample was known, but not only is the total population is unknown, but its existence is a tentative hypothesis (eg what is the number of diseases)

This gives rise to the INVERSION PROBLEM

Given (M.N) and the **correctness of the whole conceptual model,** then it is likely that in many trial the observed frequency f will be close to the probability p. But can we make this in a theorem similar to the de Moivre-Lapalace
The binomial law gives a probability m given (M,N,n)
Can we find a formula for the probability of M given (m,.N,n)

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Probability Theory – Solution Bayes 1763

Given the data(m,n) the probability that M/N lies in the interval p<(M/N)<p+dp is

$$P(dp|m,n) = \frac{(n+1)!}{m!(n-m)!} p^{m} (1-p)^{n-m} dp$$

(A Beta distribution and not a binomial!)

But for large n the equation tends asymptopically to de Moivre

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Probability Theory – Solution Laplace 1774

Let E stand for some observable event and (C1,….,CN) the set of conceivable causes. Suppose we have a conceptual model with the "sampling distribution" or "direct" probabilities of E for each cause: (P(E|Ci) i= 1,2,…,N.
Then if initially the causes were considered equally likely, then having seen the event E, the different causes are indicated with probability proportional to P(E|Ci); i.e
With uniform prior probabilities, the posterior probabilities of Ci are

$$P(C_i|E) = \left[\sum_{j=1}^{N} P(E|C_j)\right]^{-1} P(E|C_i)$$

If Ci correspond to possible values of M in the Bernoulli Model then P(E|Ci) is binomial.

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Probability Theory – Solution Laplace's Generalisation

$$P(C_i|E,I) = \frac{P(E|C_i)P(C_i|I)}{\sum_j P(E|C_j)P(C_j|I)}$$

- **This did NOT solve Bernoulli's problem.**
- **His original motivation was that the Principle of Insufficient Reason is inapplicable in many real problems because we are unable to break things down into "equally possible" cases.**
- **Laplaces only useful results relied on P(Ci|I) =1/N**
- **PRINCIPLE OF INVERSE PROBABILITY**

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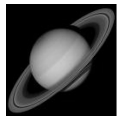
Example of Laplace Mass of Saturn

Proposition A might be the statement that the unknown mass MS of Saturn lies in a specified interval,
B the data from observatories about the mutual perturbations of Jupiter and Saturn,
C the common sense observations that MS cannot be so small that Saturn would lose its rings; or so large that Saturn would disrupt the solar system.



Laplace reported that, from the data available up to the end of the 18'th Century, Bayes' theorem estimates MS to be (1/3512) of the solar mass, and gives a probability of.99991, or odds of 11,000:1, that MS lies within 1% of that value. Another 150 years' accumulation of data has raised the estimate 0.63 percent.

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So where are we?

- Bernoulli defining probability as a representation of a particular state of knowledge, with the equations of probability representing the process of plausible reasoning for cases with not enough information i.e. deductive reasoning
- Laplace represents learning by experience
- However, since then probability has not been seen as describing a state of knowledge, but a formulistic number.
- To follow down the route started by Bernoulli, Bayes and Laplace we need to throw away the concept of probability = frequency of a random experiment
- Bayes theory has the concept of probability of an hypothesis.
- Mainline statistics is sampling theory

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So where are we?

- Maximum Likelihood = Bayes with Uniform Prior Distribution
- Evidence and Bayes representation
- Bayes is symmetric and can be used backwards to test strength of proposition

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Jaynes Definitions

In my terminology, a *probability* is something that we *assign*, in order to represent a state of knowledge, or that we *calculate* from previously assigned probabilities according to the rules of probability theory. A *frequency* is a factual property of the real world that we *measure* or *estimate*. In this terminology, the phrase "estimating a probability" is just as much a logical incongruity as "assigning a frequency".

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Complete ignorance

- How do we express "complete ignorance" of a continuously variable parameter
- Bayes and Laplace had used uniform prior densities…but these are not invariant under changes of parameter.
- This leads to Jeffreys Rule
- To express ignorance of a scale parameter x whose possible domain is 0<x<infinity, assign uniform prior distribution to its log P(dx|I)=dx/x
- It has been shown that dx/x is uniquely determined as the only scale parameter that is completely uninformative in that it leads us to the same conclusions about other parameters as if the parameter x had been removed from the model
- This set the stage for the for the generalisation of the Principle of Insufficient Reason to the Principle of Maximum Entropy.

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The Principle of Maximum Entropy

As promised no maths!

- The **principle of maximum entropy** is a method for analyzing the available information in order to determine a unique epistemic probability distribution
- Put in an actuarial way– we analyse the available information to get a series of outcomes.
- By PME we have NO confusion over best estimate and range
- BUT depends on information

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The Principle of Maximum Entropy - constraints

- The principle of maximum entropy is only useful when all of our information is of a class called **testable information**.
- A piece of information is testable if we can determine whether or not a given distribution is consistent with it.
- For example, the statements
  - "The expectation of the variable *x* is 2.87" ; and
  - "*p2+p3* > 0.9"

  are statements of testable information.

- Given testable information, the maximum entropy procedure consists of seeking the probability distribution which maximizes information entropy, subject to the constraints of the information. This constrained optimization problem is typically solved using the method of Lagrange Multipliers
- The maths now becomes difficult!

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The Principle of Maximum Entropy -Jaynes

The maximum entropy distribution ``is uniquely determined as the one which is maximally noncommittal with regard to missing information and that is ``agrees with what is known, but expresses `maximum uncertainty' with regard to all other matters, and thus leaves a maximum possible freedom for our final decision to be influenced by the subsequent sample data"

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Example – loaded dice

- In a normal dice the probability of each side is 1/6
- Each face is equally likely
- Principle of Insufficient Reason
- H(max) = $\log_e 6$ = 1.79176 (range 1.783 -1.792)
- Average no of spots =3.5
- BUT dice is loaded and average = 4.5

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Example – loaded dice

- Max entropy distribution (using Lagrangians) has

Freq(1) = 0.0543, Freq(2) = 0.0788
Freq(3) = 0.1142, Freq(4) = 0.1654
Freq(5) = 0.2398, Freq(6) =0.3475
Hmax=1.614
BUT
This is the best estimate of the probabilities!

Maximum Entropy – Actuarial Science

- Given a set of data and information (constraints) what is the best estimate and range of outcomes
- Best estimate given by PME
- Different actuaries may have different information and hence different best estimates
- Chain ladder MAY NOT GIVE best estimate as not derived from PME

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Entropy –what is it?

- Entropy is connecting with
  - Heat
  - Disorder
  - information theory
  - statistical mechanics
- Quantitative not qualitative
- Entropy is what the equation defines it to be
- There is no such thing as an "entropy", without an equation that defines it.

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Entropy –what is it?

- **The entropy is a measure of the probability of a particular result.**
- **The entropy is a measure of the disorder of a system.**

$$S = -k \cdot \sum [P_i \log(P_i)]$$

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Entropy –what is it?

- Entropy is also sometimes confused with *complexity*, the idea being that a more complex system must have a higher entropy.
- In fact, that is in all likelihood the opposite of reality. A system in a highly complex state is probably far from equilibrium and in a low entropy (improbable) state, where the equilibrium state would be simpler, less complex, and higher entropy
- Relative Entropy =Kullback-Leibler Information Criterion,

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Economic Entropy and Value

Value is a function of scarcity. Scarcity can be defined as a probability measure *P* in a certain probability space. It is generally agreed that value of products satisfies the following properties:

(a) The value of two products should be higher than the value of each of them.

(b) If two products are independent, that is, if the two products are not substitutes or partial substitutes of each other, then the total value of the two products will the sum of two products.

(c) The value of any product is non-negative.

The only mathematical functions that satisfy all the above properties are of the form

$$V(P) = -\log_b P$$

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Economic Entropy and Value

- In general, suppose a service or product, *X*, can perform different tasks, with probability of p1, p2, . pn, Then the value of this product is the average of the value of each task. That is

$$V(X) = \sum_{i=1}^{n} p_i(-\log_b p_i)$$

- For Shannon b=2 (0,1)
- In general b – no of producers
- Entropy is fairly fundamental to economics and actuarial science

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Fisher Information and Entropy

- Fisher Information measures spread and is inversely related to the ENTROPY
- B.R. Frieden, "Physics from Fisher Information: a Unification" (Cambridge Univ. Press, 1998)
- (Be careful – it's a good read but!)
- Entropy looks for minimum spread
- Measure of amount of uncertainty in a distribution

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Reading

- Pierre Simon de Laplace (1812) *Analytical Theory of Probability*
- Andrei Nikolajevich Kolmogorov (1933) *Foundations of the Theory of Probability*
- H. **Jeffreys**. Theory of **probability**. Oxford University Press, Oxford
- **Probability Theory: The Logic of Science.** E. T. Jaynes
- R. T. Cox, "Probability, Frequency, and Reasonable Expectation," *Am. Jour. Phys.,* 14, 1-13, (1946).
- R. T. Cox, *The Algebra of Probable Inference,* Johns Hopkins University Press, Baltimore, MD, (1961).

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Risk measures – an example of entropy at work

- Risk in mathematical terms become synonymous with standard deviation (also variance, semi-variance etc) of returns following Markowitz's seminal paper
- This has lead to people not understanding risk and hence misunderstanding many of the issues we face today

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Risk measures

- A risk measure is simply function that assigns a number to set of "risks"
- Examples
  - Number of contracts
  - Number of underwriters
  - Sum insured
  - Variance
  - Semi Variance
  - Value at Risk
  - Tail at Risk

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Risk measures

(i) Monotonicity
If $P(X \le Y) = 1$, then $\rho(X) \le \rho(Y)$.

(ii) Positive homogeneity
For all $\lambda \ge 0$, $\rho(\lambda X) = \lambda\rho(X)$.

(iii) Translation invariance
For all risks and real numbers, α, $\rho(X + \alpha) = \rho(X) + \alpha$.

(iv) Subadditivity
For all risks X, Y, $\rho(X + Y) \le \rho(X) + \rho(Y)$.

(v) Relevance
For all non-negative risks with $P(X > 0) > 0$, $\rho(X) > 0$.

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Coherent Risk Measures

- A risk measure is convex if it satisfies the three axioms of monotonicity, translation invariance, and subadditivity (although some authors include a weak homogeneity
  $$\pi(\lambda X + (1-\lambda)Y) \geq \lambda\pi(X) + (1-\lambda)\pi(Y),$$
- A risk measure is coherent if it satifies the four axioms of monotonicity, positive homogeneity, translation invariance, and subadditivity.

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Risk measures –some nice requirements

Risk, Entropy, and the Transformation 0f Distributions

by

R. Mark Reesor and Don L. McLeish

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Risk measures –some nice requirements

The exponential family, relative entropy, and distortion are methods of transforming probability distributions. We establish a link between those methods, focusing on the relation between relative entropy and distortion. Relative entropy is commonly used to price risky financial assets in incomplete markets, while distortion is widely used to price insurance risks and in risk management. The link between relative entropy and distortion provides some intuition behind distorted risk measures such as value-at-risk. Furthermore, distorted risk measures that have desirable properties, such as coherence, are easily generated via relative entropy.

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The Black Scholes Bit

- Black Scholes formula is driven by the implied volatility
- This requires continuous independent random shocks
- BUT
- Can also be derived by Martingales and MPE
- Generalised Black Scholes equation
- See
  - Simple Entropic Derivation of a Generalized Black-Scholes Option Pricing Model by Michael J. Stutzer (Entropy)
  - Gerber, H.; Shiu, E. Option pricing by Esscher Transforms. T*ransactions of the Society of Actuarie*s **1994**, *46*, 99-140.

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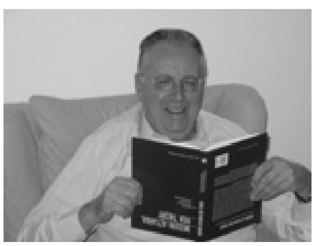
Work in Progress- Being reviewed by Hans Gerber



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