The Actuarial Profession
making financial sense of the future

The Model Underlying Solvency II

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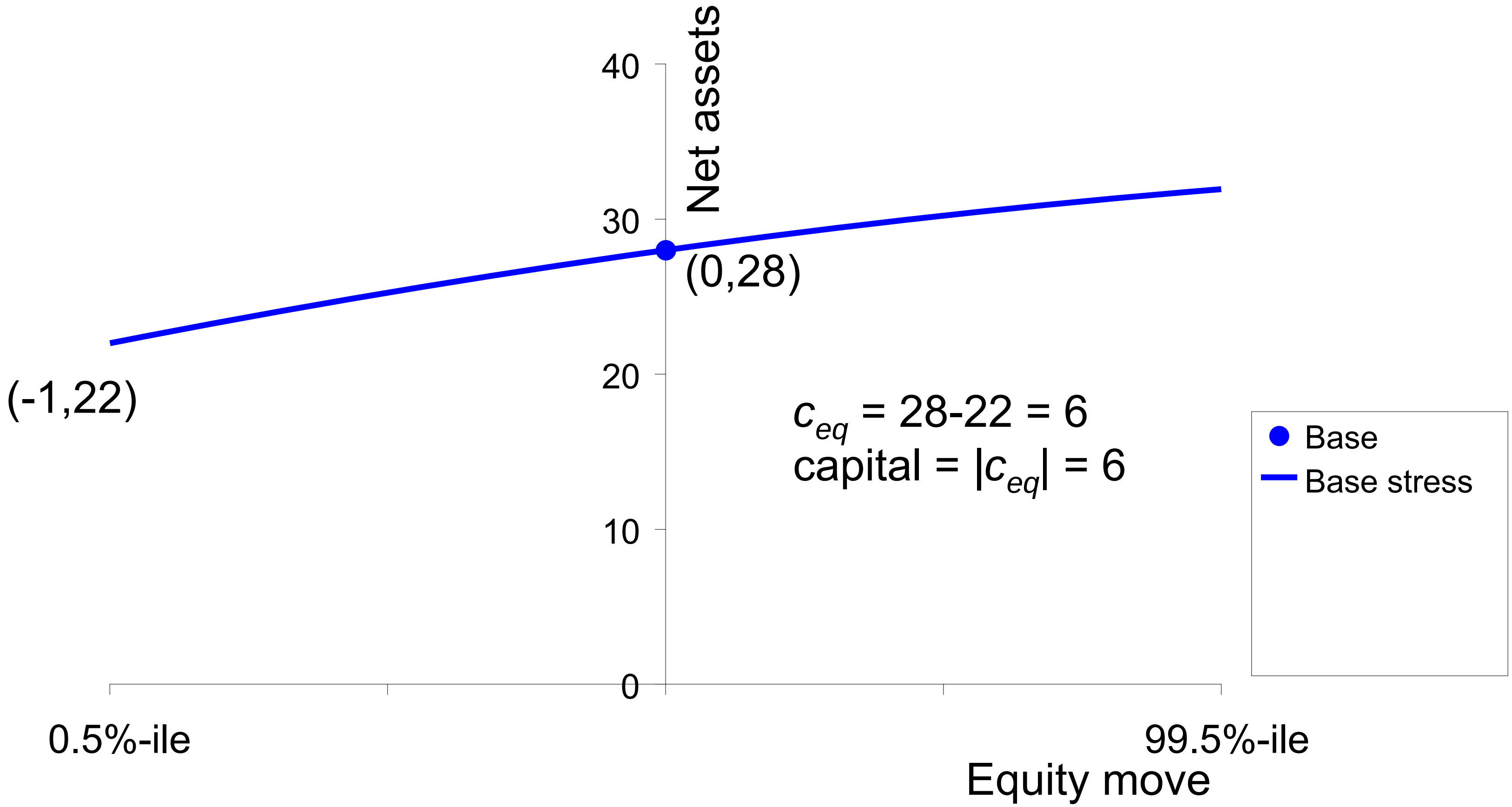
2007 GIRO Convention
Session D07
11:45-12:45 Tuesday 4 October

Overview

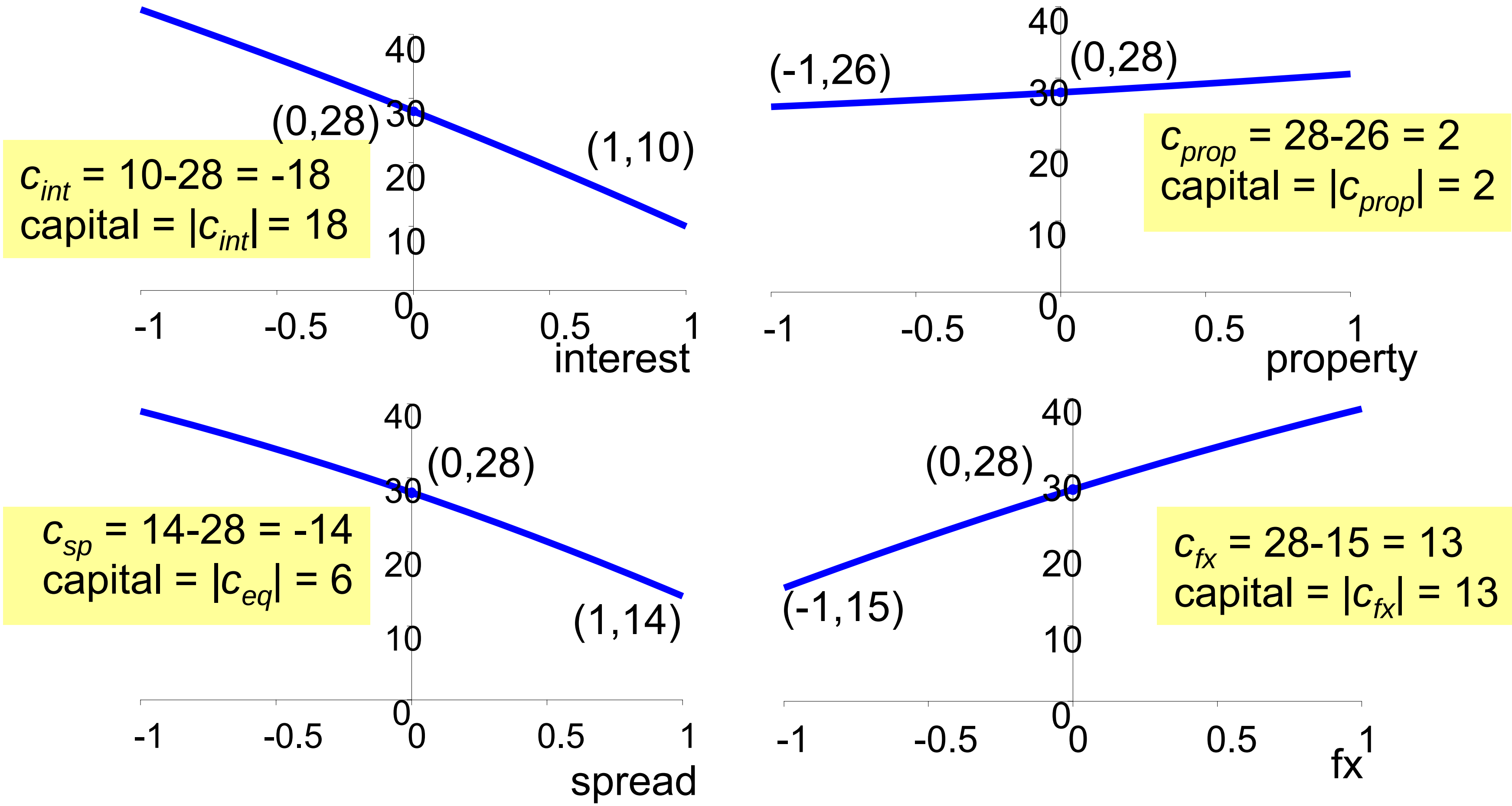
- Capital calculation under Solvency II (QIS 3)
  - Stress tests
  - Aggregation
- Analysis of Change Components
  - Elliptically contoured distributions
- Internal Hedges
  - Analytical capital compared to QIS3
  - The “absolute capital” bias
- Investigating Interactions
  - Combined stress events
  - Risk geographies

Capital Calculation under Solvency II

Calculation of Equity Capital



Capital: Other Risks



Stress Test Result Summary

Stress test	c_i	$\lvert c_i \rvert$
Interest	-18	18
Equity	6	6
Property	2	2
Spread	-14	14
Foreign Exchange	13	13

Assumption: net assets are monotone in each risk factor
Note convention: c_i is a finite difference approximation to net asset gradient, under a choice of units so for each driver, 0.5%-ile = -1 and 99.5%-ile = +1.
Only the absolute value $|c_i|$ is required for QIS 3.

QIS3 Aggregation Formula

$$C_{agg} = \sqrt{\sum_{i=1}^{n} C_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i} r_{ij}\left|C_i C_j\right|} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\left|C_i C_j\right|}$$

where

- C_{agg}: aggregate capital
- C_i: individual capital amounts
  - Signed: positive for increasing functions, negative for decreasing functions
- $\{r_{ij}\}$: Correlation matrix

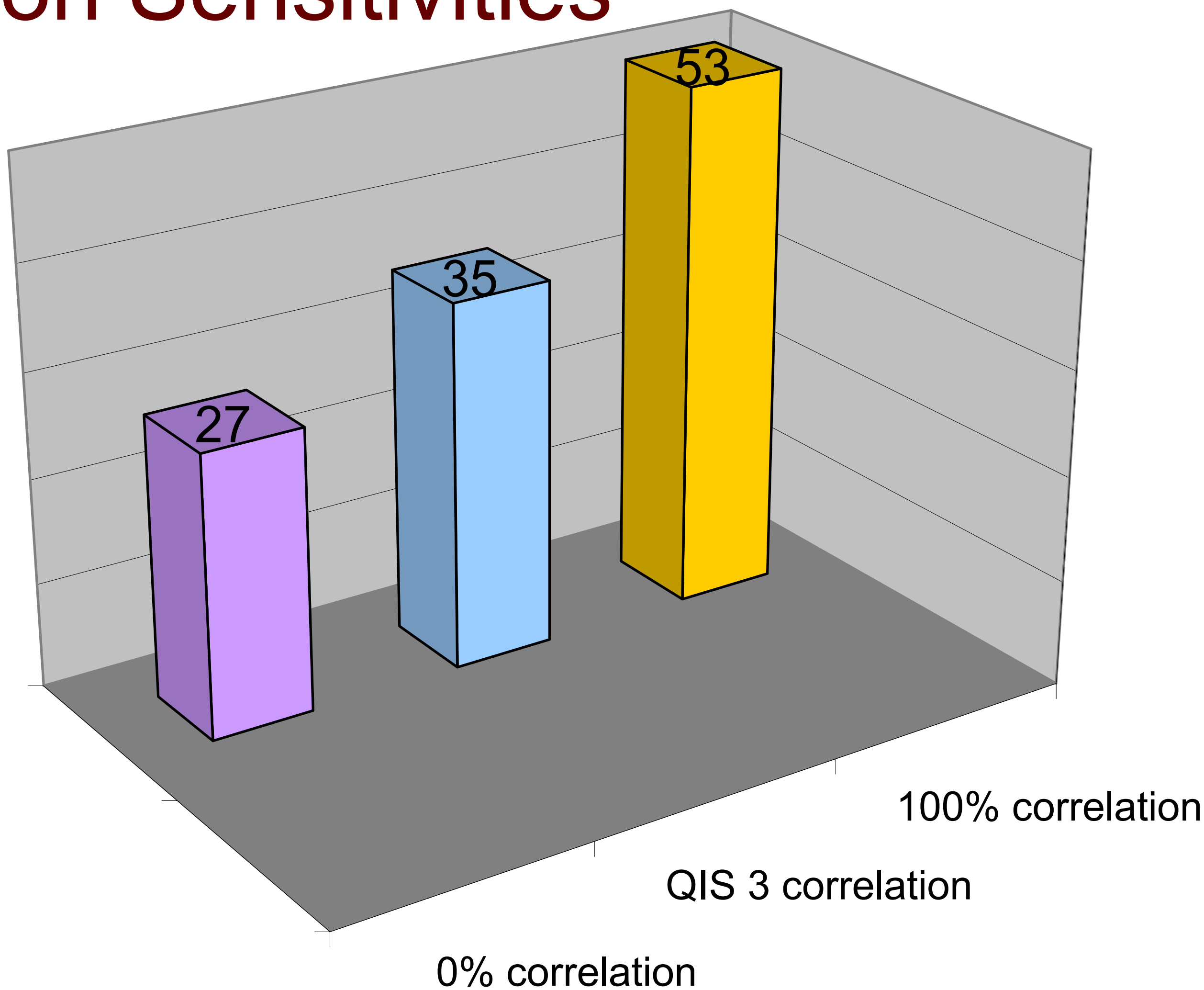
QIS3 Risk Correlation Matrix $\{r_{ij}\}$

	Mkt_{int}	Mkt_{eq}	Mkt_{prop}	Mkt_{sp}	Mkt_{fx}
Mkt_{int}	1	0	0.5	0.25	0.25
Mkt_{eq}	0	1	0.75	0.25	0.25
Mkt_{prop}	0.5	0.75	1	0.25	0.25
Mkt_{sp}	0.25	0.25	0.25	1	0.25
Mkt_{fx}	0.25	0.25	0.25	0.25	1

QIS 3 Capital Example

$$\sqrt{\begin{pmatrix}18 & 6 & 2 & 14 & 13\end{pmatrix}\begin{pmatrix}1 & 0 & 0.5 & 0.25 & 0.25\\ 0 & 1 & 0.75 & 0.25 & 0.25\\ 0.5 & 0.75 & 1 & 0.25 & 0.25\\ 0.25 & 0.25 & 0.25 & 1 & 0.25\\ 0.25 & 0.25 & 0.25 & 0.25 & 1\end{pmatrix}\begin{pmatrix}18\\ 6\\ 2\\ 14\\ 13\end{pmatrix}} = 35$$

Correlation Sensitivities



Motivation

- We have been given a standard **formula** for aggregating capital amounts
- … but no word about the **model** that this formula corresponds to
- If we don't understand the model, we can't say whether we like or dislike the formula
- The model underlying a standard formula is a useful benchmark for discussion of internal models.

Analysis of Change Components

Drivers and Profits

Before shock		After shock			Analysis of Change (AoC components)
Interest rate	X_{int}	Interest rate	X_{int}	$+\Delta X_{int}$	ΔY_{int}
Equity level	X_{eq}	Equity level	X_{eq}	$+\Delta X_{eq}$	$+\Delta Y_{eq}$
Property	X_{prop}	Property	X_{prop}	$+\Delta X_{prop}$	$+\Delta Y_{prop}$
Spread	X_{sp}	Spread	X_{sp}	$+\Delta X_{sp}$	$+\Delta Y_{sp}$
FX level	X_{fx}	FX level	X_{fx}	$+\Delta X_{fx}$	$+\Delta Y_{fx}$
Net assets Y		Net assets $Y+\Delta Y$			Total ΔY

For solvency purposes, we are interested in the distribution of ΔY
This is built up from the distribution of change to each risk factor.
Later in this workshop, we will think about the X's too.

What model?

- Recall the standard deviation of a sum

$$Stdev\left(\sum_i Y_i\right) = \sqrt{\sum_{i,j} r_{ij} Stdev(Y_i) Stdev(Y_j)}$$

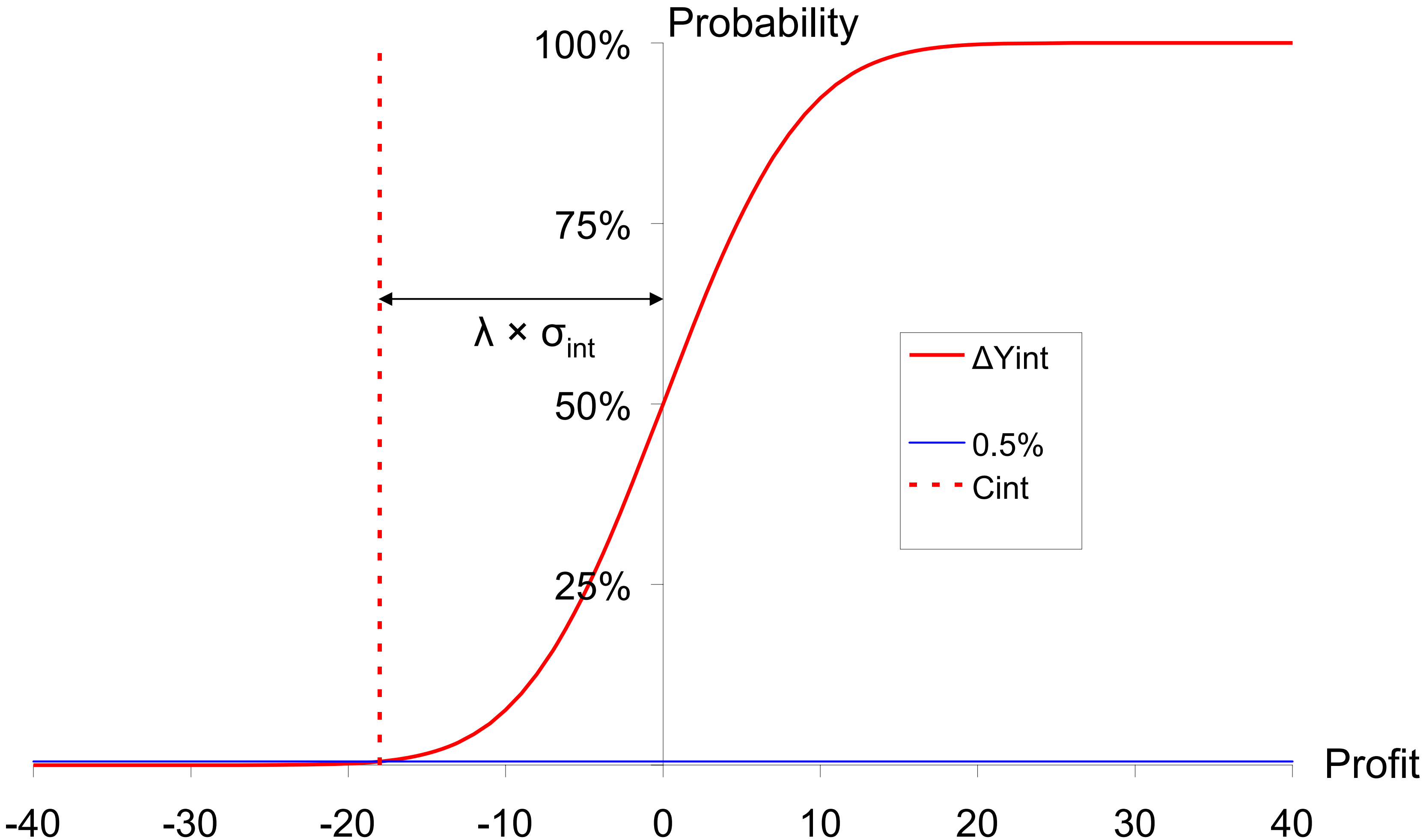
- Obviously, then, for any λ > 0:

$$\lambda Stdev\left(\sum_i Y_i\right) = \sqrt{\sum_{i,j} r_{ij} \{\lambda Stdev(Y_i)\}\{\lambda Stdev(Y_j)\}}$$

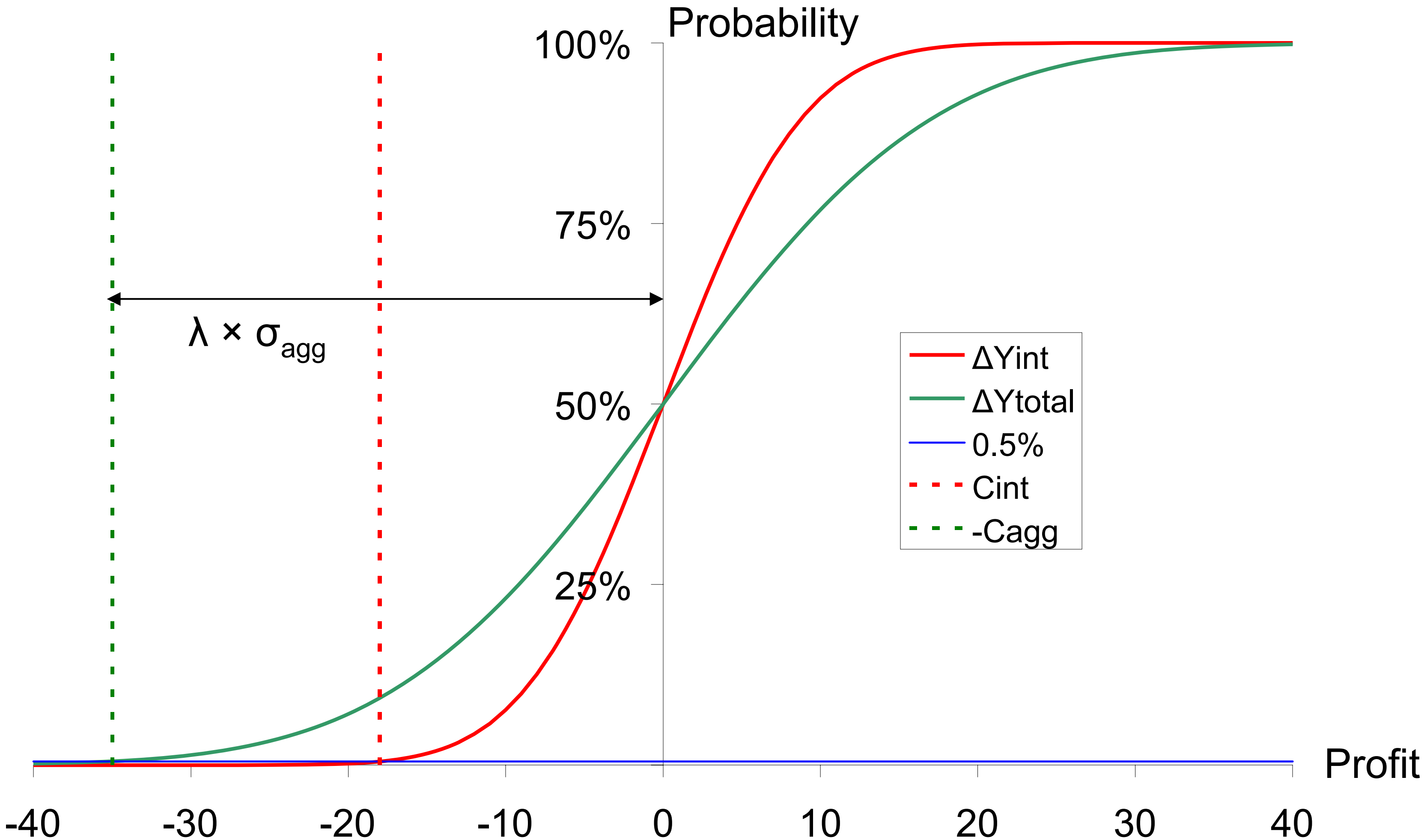
- Suppose $|c_i|$ is a multiple λ of standard deviation

$$C_{agg} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \left|C_i C_j\right|}$$

Cumulative Probability: Interest Rates



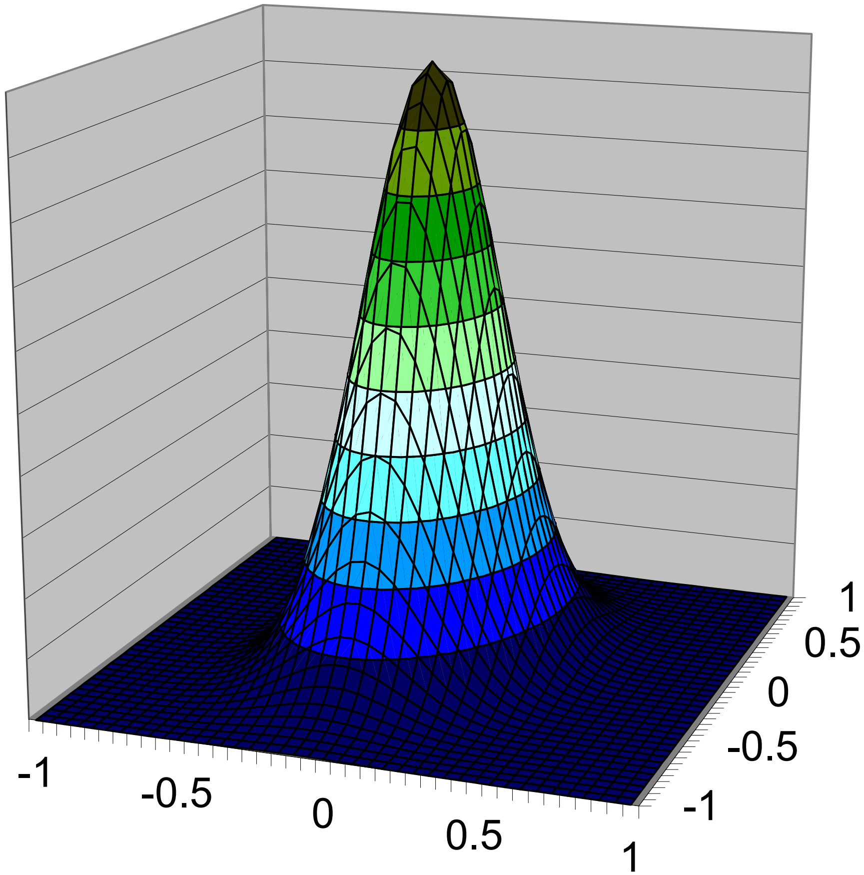
Cumulative Probability: Aggregate

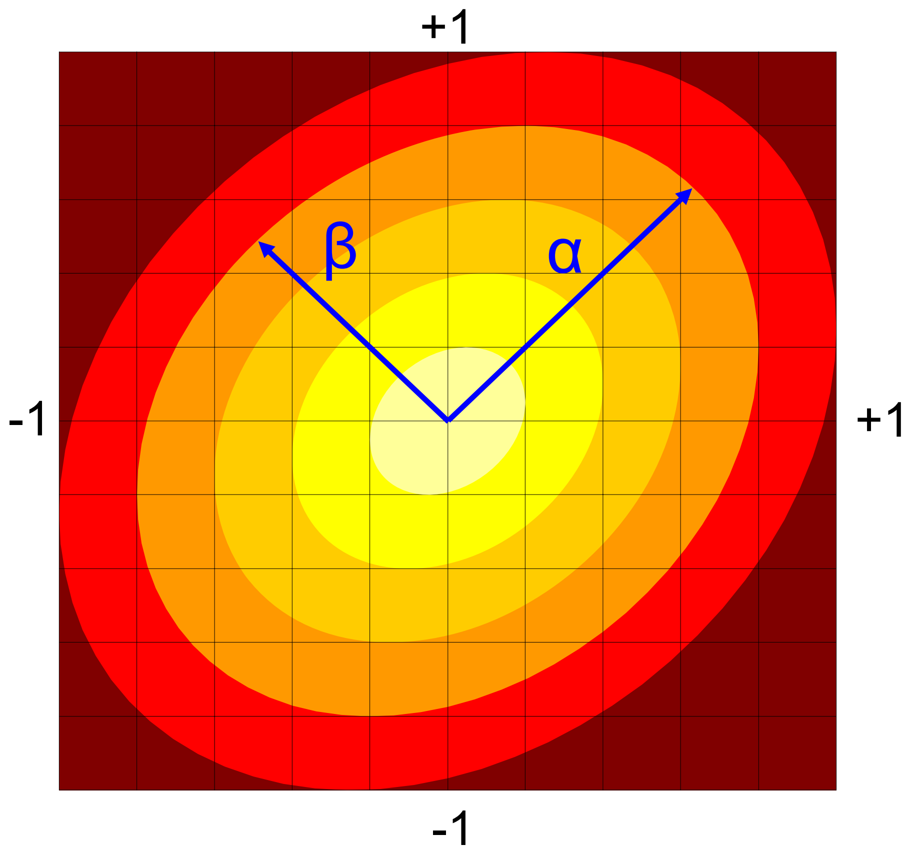


Model existence

- Vital property:
  - For each component, p-%-ile = mean - λ * stdev
  - For the total, p%-ile = mean - λ * stdev
  - Same value of λ in each case
- Does a model displaying the above properties exist?
  - Yes, multivariate normal works with $\lambda = 2.58$ ($p=0.5\%$)
- More generally, the AoC components can follow an elliptically contoured distribution
  - Multivariate normal
  - Multivariate t
  - Laplace
  - …

Elliptically Contoured Distributions





$$r = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}$$

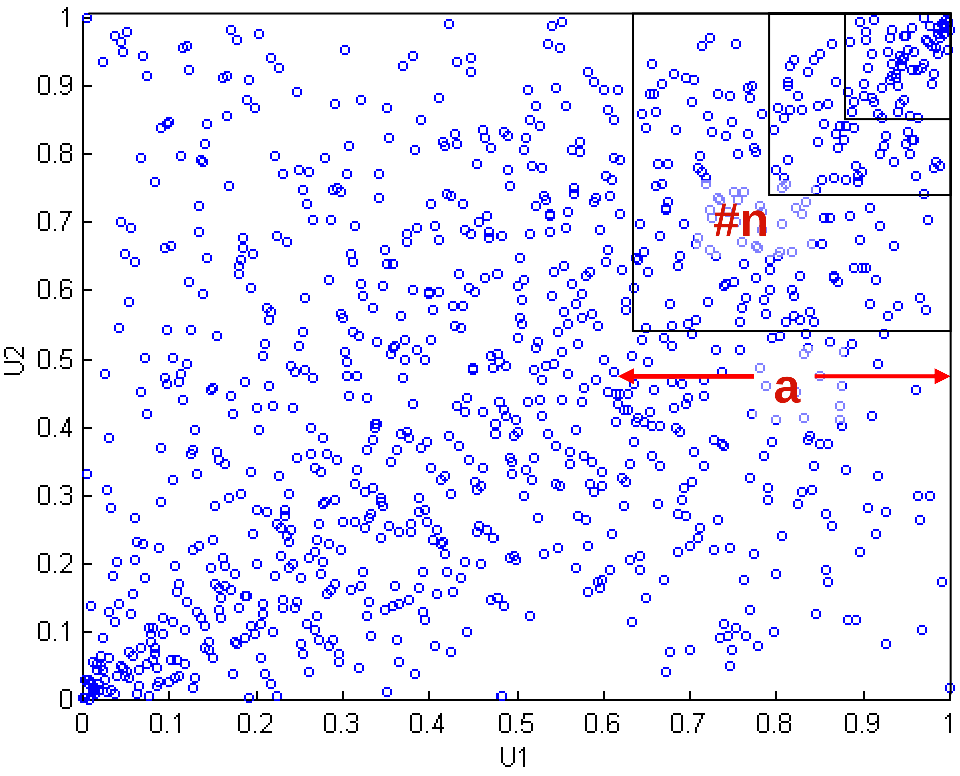
Tail dependence according to CEIOPS

- "Further analysis is required to assess whether linear correlation, together with a simplified form of tail correlation, may be a suitable technique to aggregate capital requirements for different risks." (CfA 10.138)
- “When selecting correlation coefficients, allowance should be made for tail correlation. To allow for this, the correlations used should be higher than simple analysis of relevant data would indicate.” (CEIOPS CP20)

Assumptions and Extensions

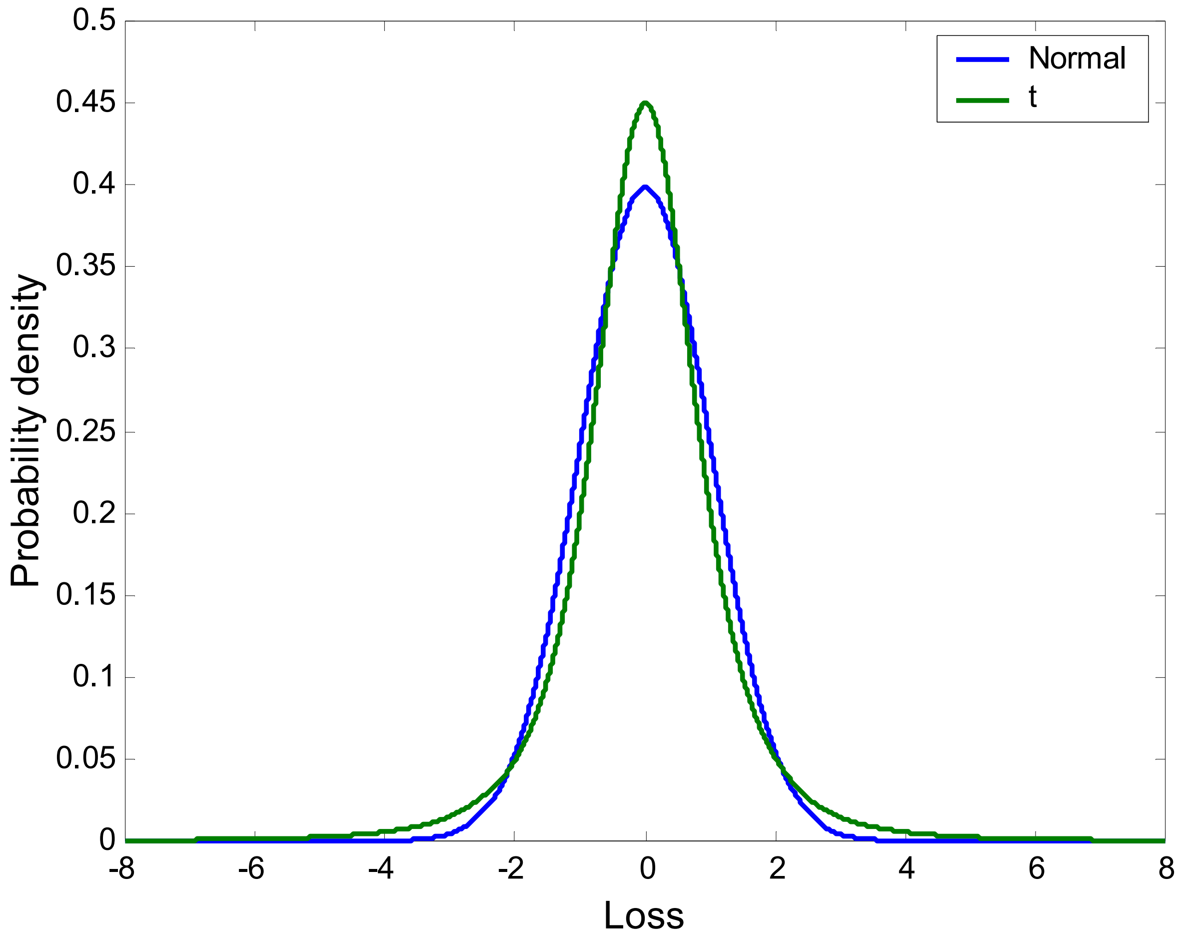
Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	

Asymptotic tail dependence

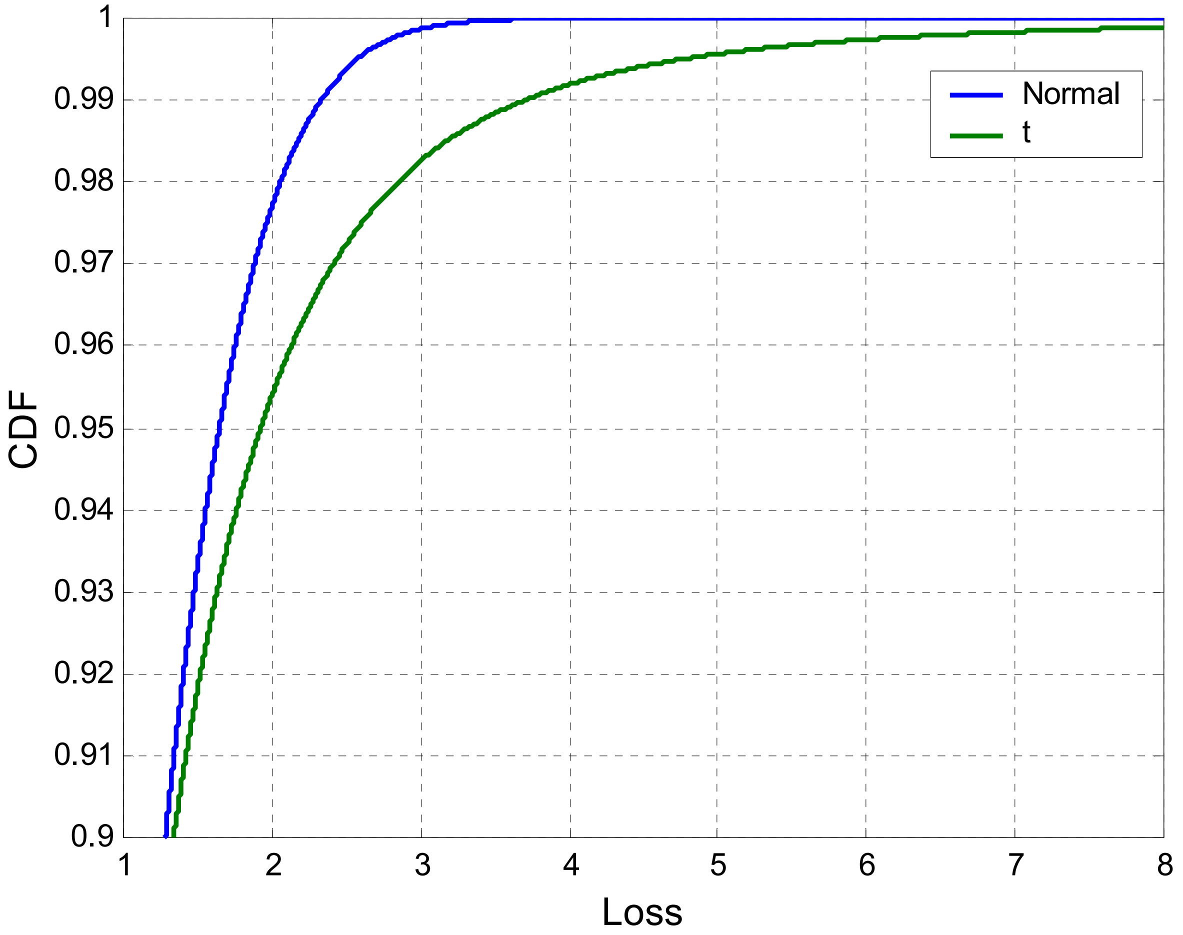


$$\lambda = \lim_{a \to 0} \frac{n}{a}$$

Elliptical distributions – heavy tails

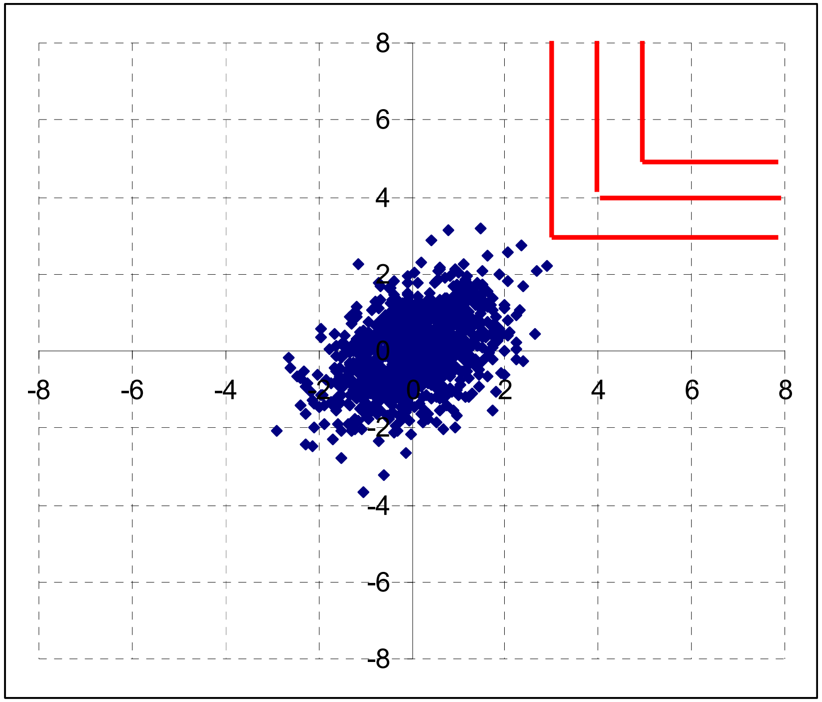


Elliptical distributions – heavy tails



Elliptical distributions – tail dependence

Normal



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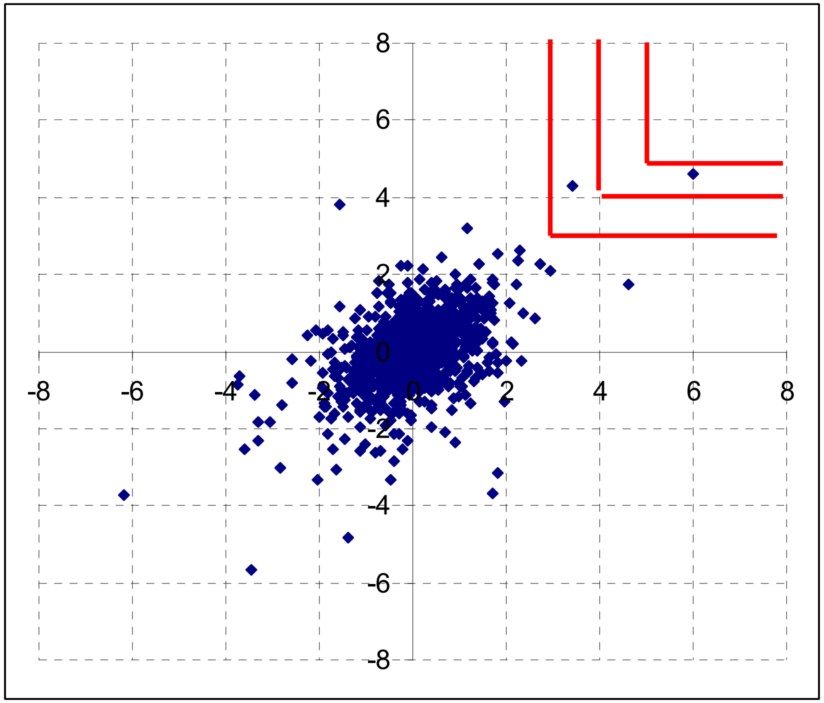


Elliptical distributions – summary

- Elliptical distributions are perfectly capable of demonstrating tail dependence
  - E.g. the t-distribution does
- Heavy tails $\Leftrightarrow$ tail dependence
- “Linear correlation” and “tail dependence” refer to completely different things
  - CEIOPS got it wrong

Assumptions and Extensions

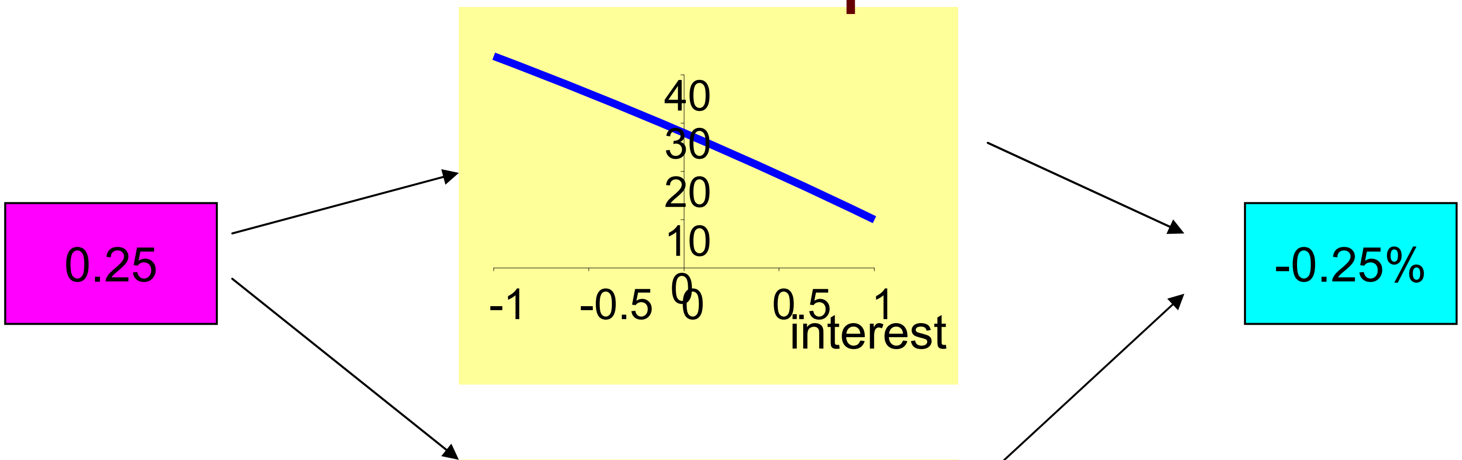
Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	Consistent with QIS3 methodology

Internal Hedges

Interest / FX correlation

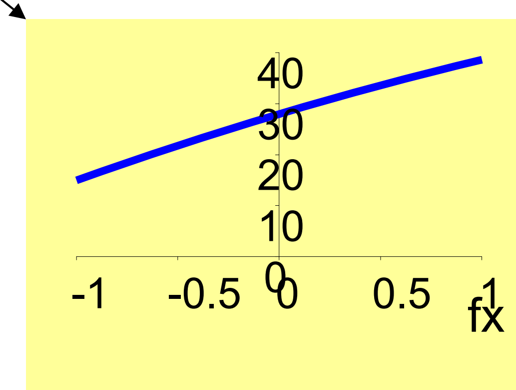
	Mkt_{int}	Mkt_{eq}	Mkt_{prop}	Mkt_{sp}	Mkt_{fx}
Mkt_{int}	1	0	0.5	0.25	0.25
Mkt_{eq}	0	1	0.75	0.25	0.25
Mkt_{prop}	0.5	0.75	1	0.25	0.25
Mkt_{sp}	0.25	0.25	0.25	1	0.25
Mkt_{fx}	0.25	0.25	0.25	0.25	1

25% Correlation: Interpretation



Correlation between interest and FX moves ($\Delta X_{int,}\ \Delta X_{fx}$). When X_{int} increases, X_{fx} probably increases.

Can be tested empirically. Same assumption valid for all firms, as this relates to external market moves.



Internal Hedge

Correlation between interest and FX AoC ($\Delta Y_{int,}\ \Delta Y_{fx}$). When Y_{int} increases, Y_{fx} probably **decreases**.

Effect varies from firm to firm according to the sign of their exposure to each risk factor.

Assumptions and Extensions

Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	Consistent with QIS3 methodology
No internal hedges	

Allowing for Internal Hedges

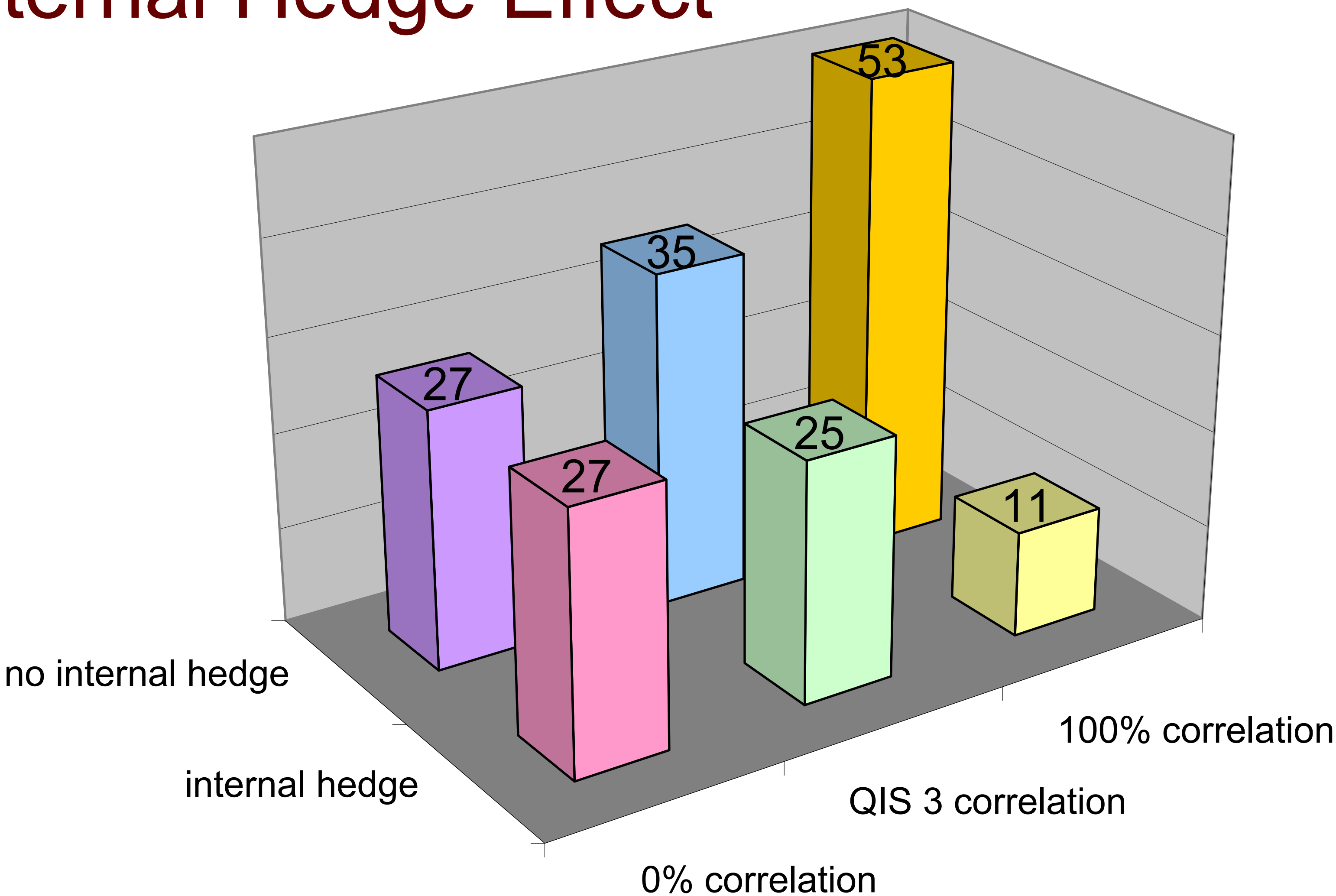
With internal hedging – use the signed capital amounts in the aggregation:

$$C_{agg} = \sqrt{\sum_{i=1}^{n} C_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i} r_{ij} C_i C_j} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} C_i C_j}$$

Compare this to no internal hedging (QIS3) with “absolute capital” bias.

$$C_{agg} = \sqrt{\sum_{i=1}^{n} C_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i} r_{ij} \left| C_i C_j \right|} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \left| C_i C_j \right|}$$

Internal Hedge Effect



Assumptions and Extensions

Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	Consistent with QIS3 methodology
No internal hedges	Signed capital for aggregation

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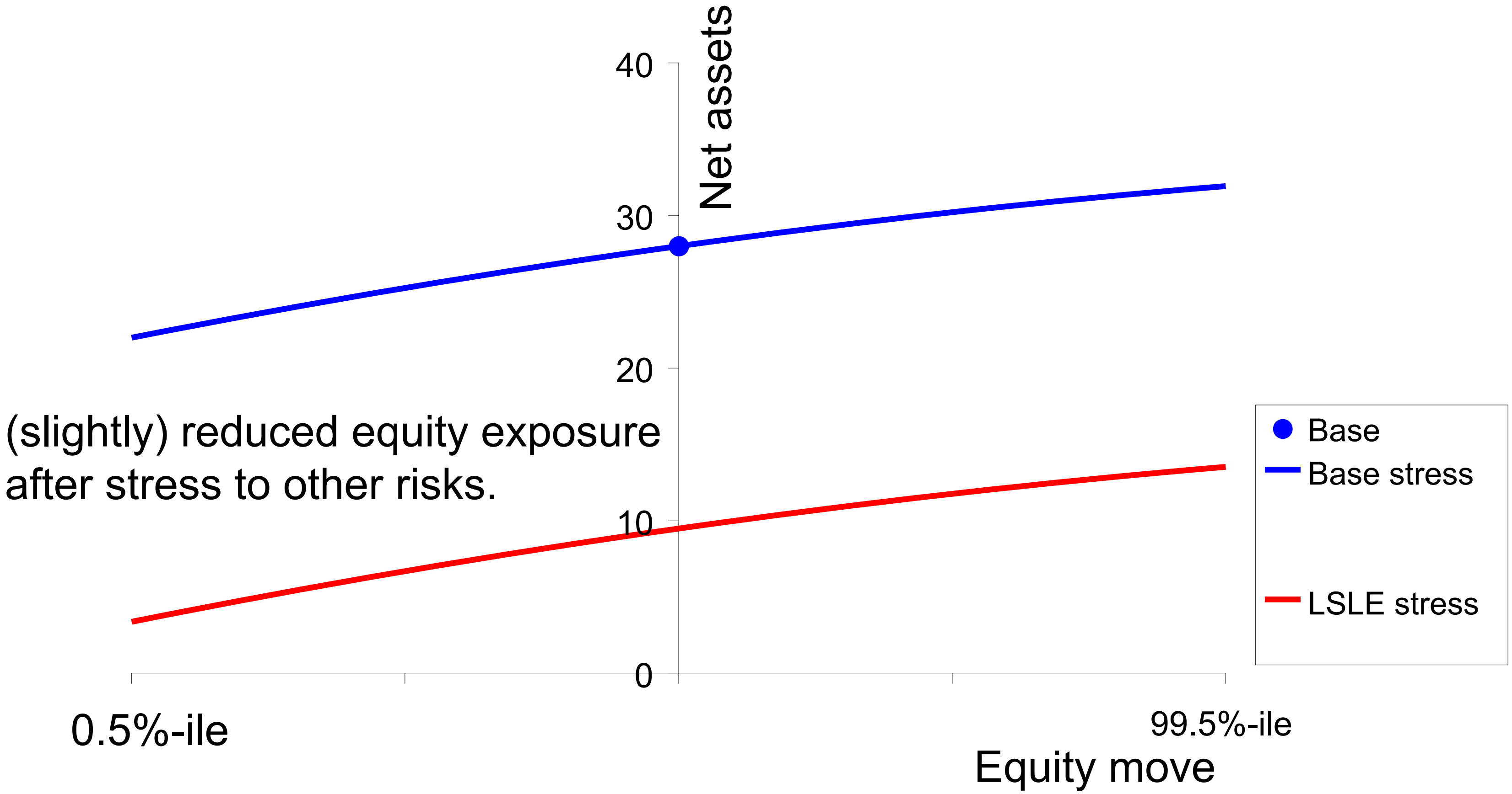
Investigating Interactions

Drivers and Profits

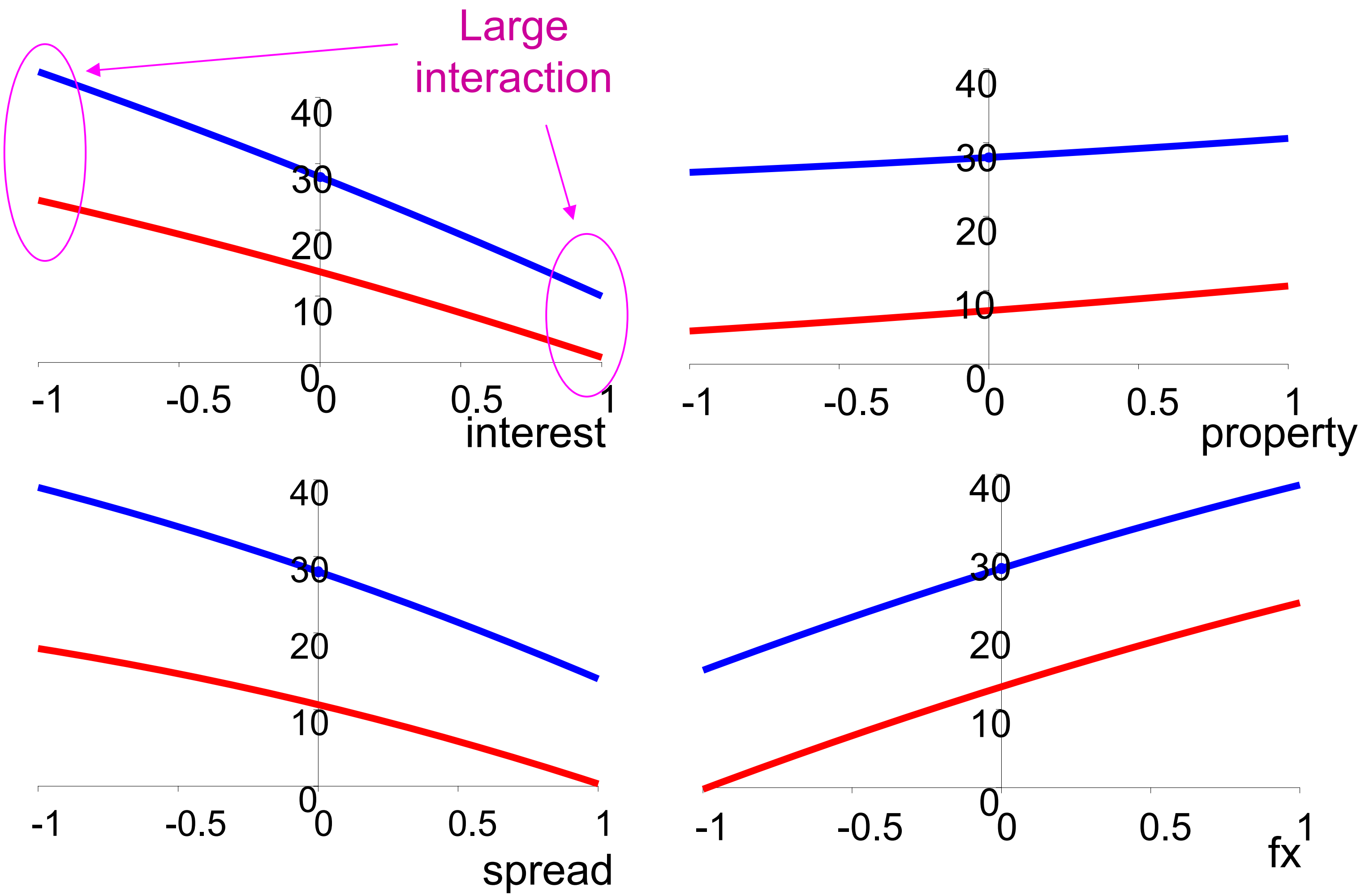
Before shock			After shock			Analysis of Change
Interest rate	X_{int}	→	Interest rate	X_{int}	$+\Delta X_{int}$	ΔY_{int}
Equity level	X_{eq}		Equity level	X_{eq}	$+\Delta X_{eq}$	$+\Delta Y_{eq}$
Property	X_{prop}		Property	X_{prop}	$+\Delta X_{prop}$	$+\Delta Y_{prop}$
Spread	X_{sp}		Spread	X_{sp}	$+\Delta X_{sp}$	$+\Delta Y_{sp}$
FX level	X_{fx}		FX level	X_{fx}	$+\Delta X_{fx}$	$+\Delta Y_{fx}$
↓			↓			
Net assets Y			Net assets Y+ΔY			Total ΔY

We estimate the distribution of ΔY_i by stressing one variable at a time.
But in AoC, ΔY_i depends on the moves of the previous X_i.
This is called an *interaction.* Interaction is the reason why the order matters in analysis of change.

Equity Interaction



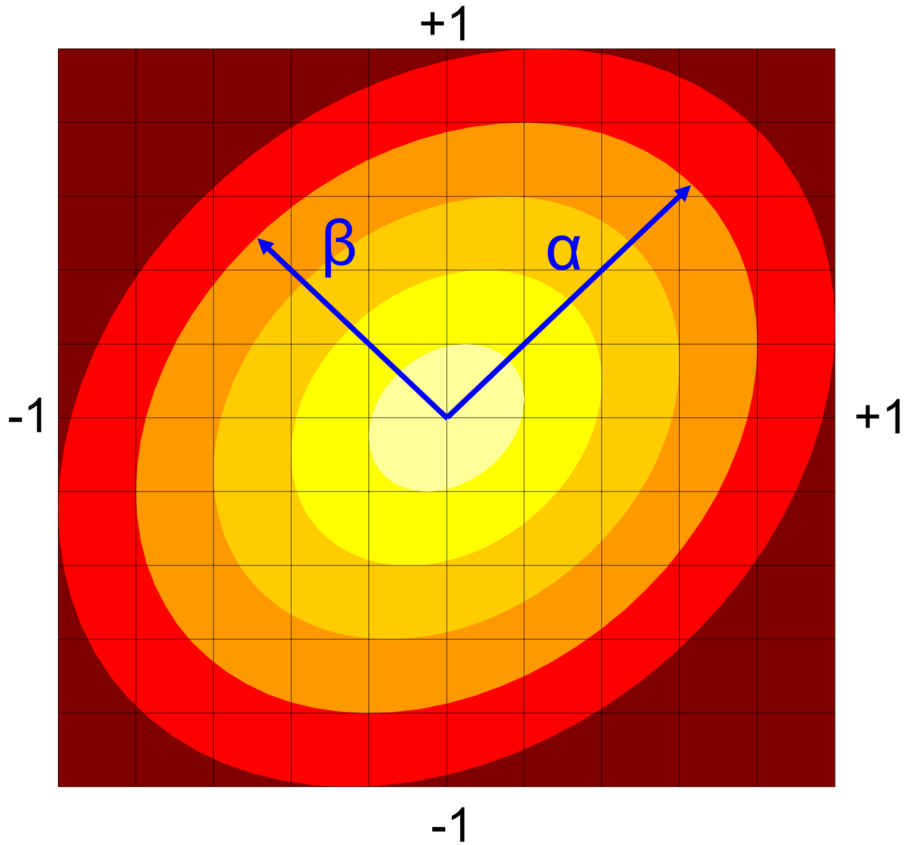
Interactions: Other Risks

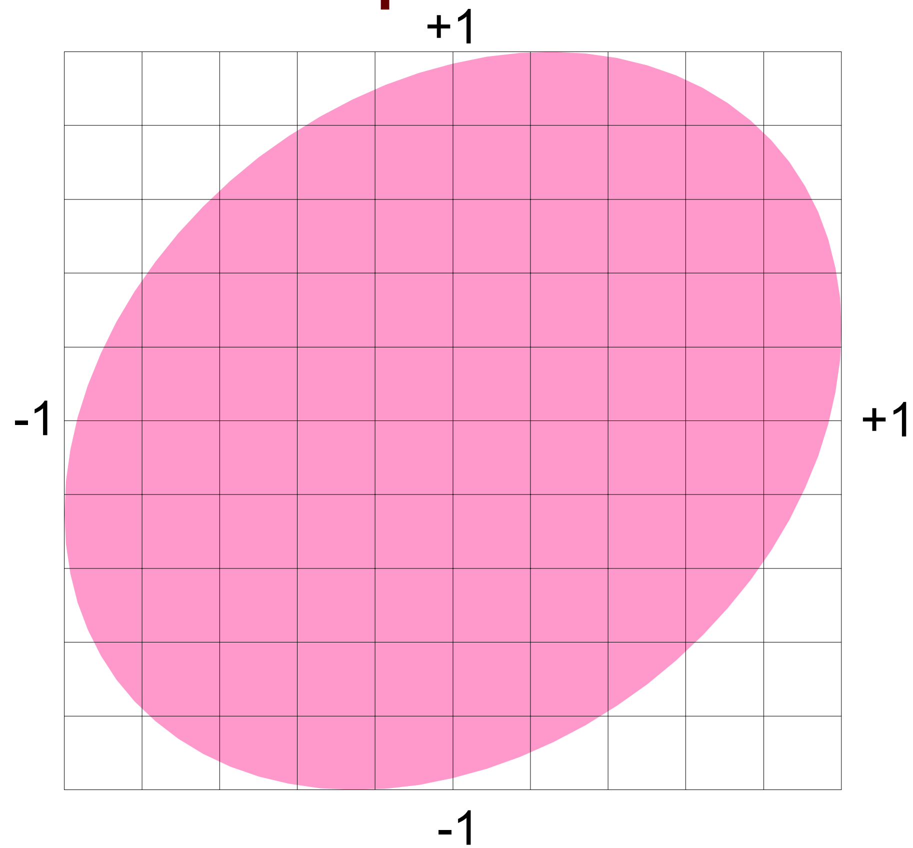


Assumptions and Extensions

Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	Consistent with QIS3 methodology
No internal hedges	Signed capital for aggregation
Linear response No interactions	

Remember that Contour Map?



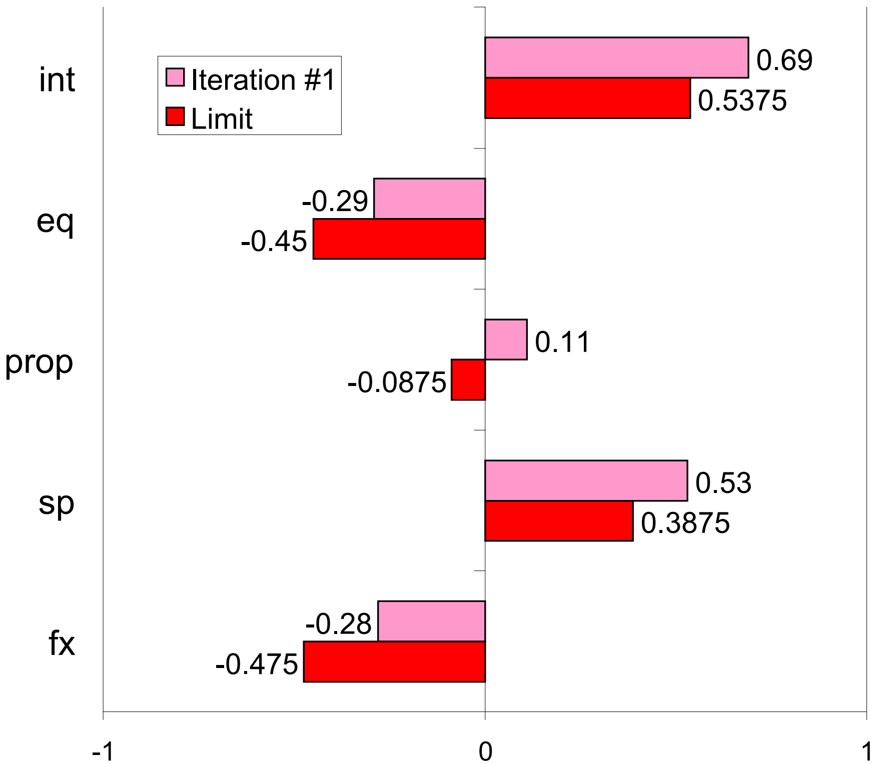


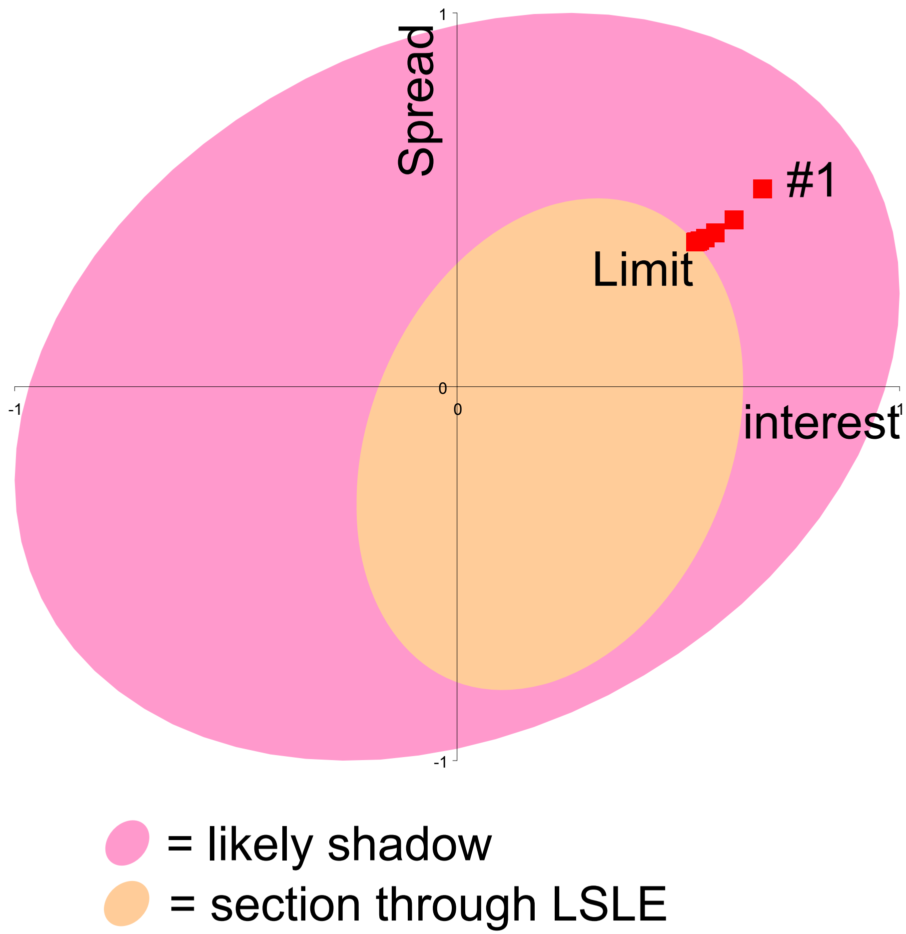
The *likely locus* is the blob touching ±1 in each direction.

The LSLE

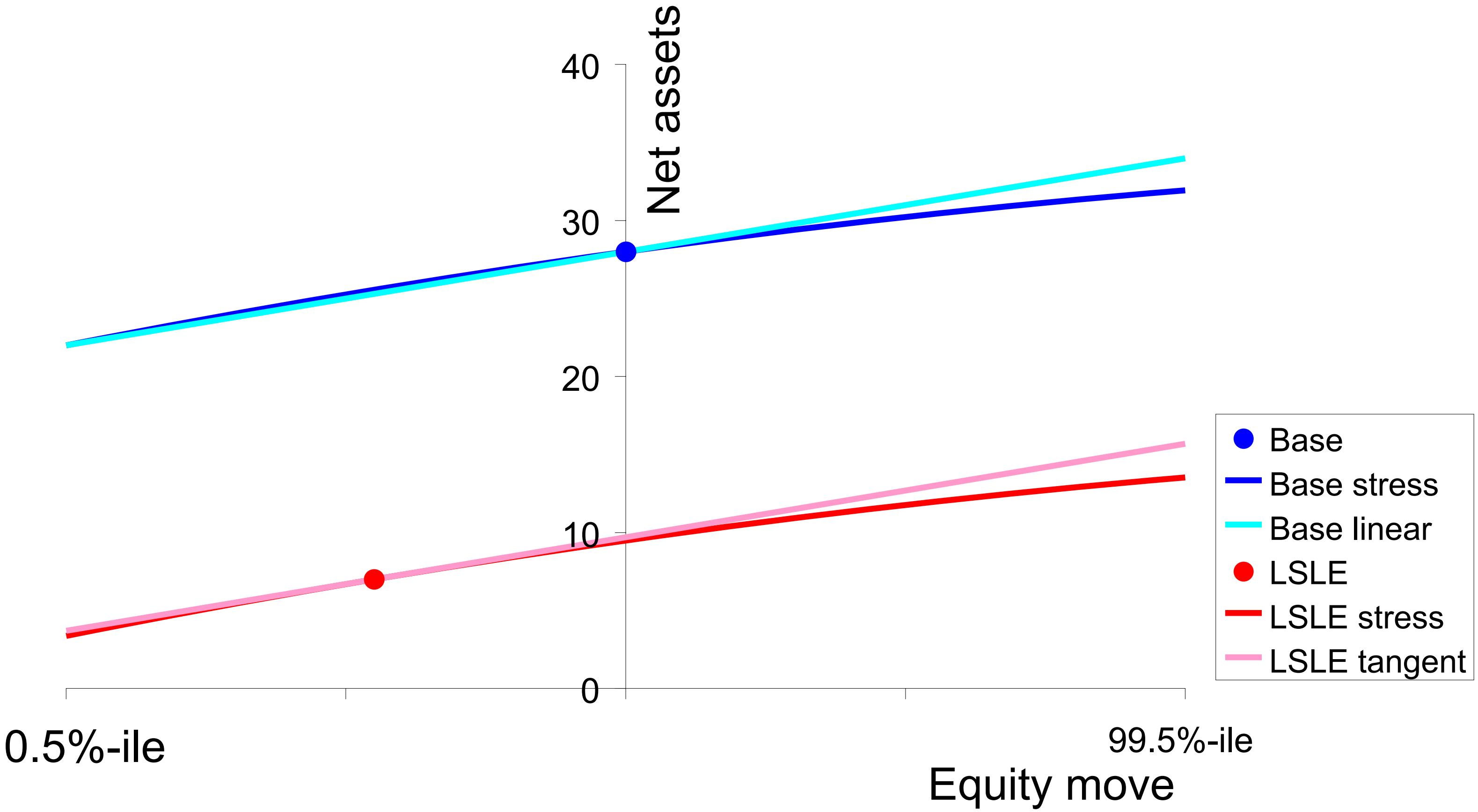
- LSLE is the "Least Solvent Likely Event"
- Minimises net assets over the likely locus
- Computed using iterative / hill-climbing approach
- We will see why:
  - 0.5%-ile $\{Y(X)\} \approx Y(X_{LSLE})$
  - Where X_{LSLE} minimises $Y(X)$ over likely X.
  - So, capital required = Y(0) – Y(LSLE)
- This result is robust, even with interactions

Iterating to Find LSLE

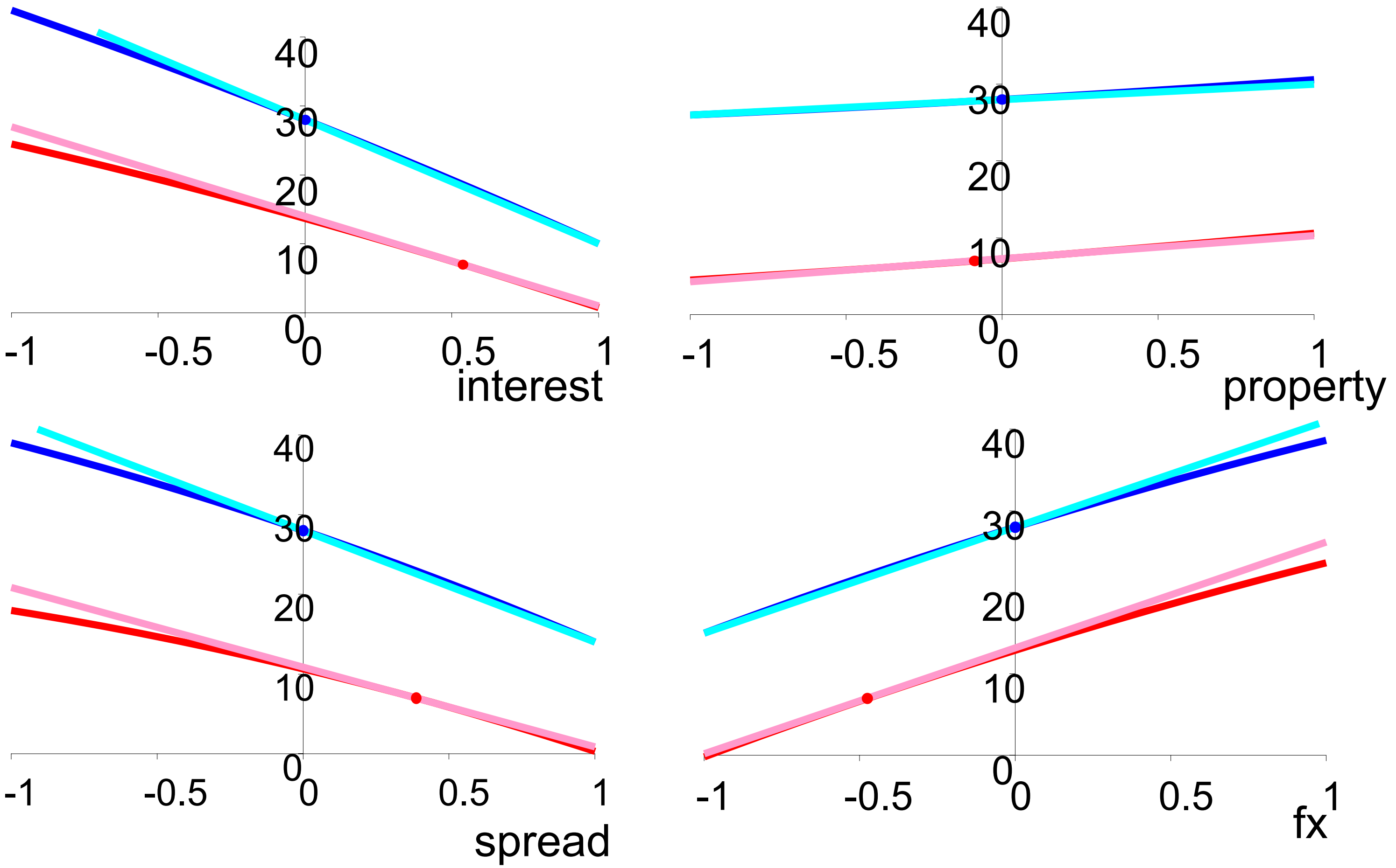




Linear Approximations: Equity



Linear Approximations: Other Risks



Two Linear Approximations

	Finite difference fitting x=0 And stress tests	Linear expansion around LSLE	
Y(0)	28	27	
c_{int}	-18	-13	
c_{eq}	6	6	
c_{prop}	2	3	
c_{sp}	-14	-10	
c_{fx}	13	13	
Y(LSLE)	3.85	7	Always equal for expansion about LSLE
0.5%-ile	3	7	
Required capital	25	21	

Linear expansion about the LSLE is most relevant for capital calculation, because it is more accurate in the region that is likely and painful.

Assumptions and Extensions

Solvency II (QIS 3) Assumption	How to Extend the Assumption
Heavy-tailed distributions Tail dependency	Consistent with QIS3 methodology
No internal hedges	Signed capital for aggregation
Linear response No interactions	Capital requirement = Base net assets – LSLE net assets LSLE = least solvent likely event

Assumptions and Extensions

Solvency II (QIS 3) Assumption	How to Extend the Assumption
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Linear response No interactions	Capital requirement = Base net assets – LSLE net assets LSLE = least solvent likely event
Non-elliptical contours Asymmetric distributions	Asymmetric likely locus

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