JIA 71 (1943) 0068-0095

NOTES ON INTERPOLATION (PART II).*—III (i) CONTINUED. THE ORIGIN OF THE THROW-BACK DEVICE. IV. AITKEN'S NEW METHOD OF INVERSE INTERPOLATION. V. THE CONNEXION OF THE THROW-BACK WITH STIRLING'S AND BESSEL'S FORMULAE

BY G. J. LIDSTONE, LL.D., ETC.

III (i) (*continued*)

49. The throw-back device, discussed in paras. 34–43, is usually associated with the name of Dr L. J. Comrie, who invented it independently and has greatly developed it: but in the interests of historical accuracy it is desirable to record that it was previously or concurrently suggested by other writers. The writer is indebted to Comrie himself for the information that the device seems to have been first given and used by the eminent astronomer Prof. E. W. Brown, F.R.S. (lately deceased), in the Introduction to his classic *Tables of the Motion of the Moon*, Vol. 1, p. 110 (1919). It was also proposed independently by an American actuary, Mr Kingsland Camp, F.A.S., *T.A.S.A.* Vol. XXIX, p. 216 (1928), almost concurrently with its first publication by Comrie. To quote the words of D. C. Fraser (*Newton and Interpolation*, p. 69) in relation to Everett's formula: "This is an example of what has continually happened in this subject, of formulas being given and forgotten [or overlooked] and rediscovered."

50. As Brown's work is rather inaccessible it will be well to record his remarks *in extenso*. He wrote (loc. cit.):

> Denote two consecutive half-daily values of either coordinate by F_0 and F_1, the first, third and fifth differences between F_0, F_1 by Δ^{i}, Δ^{iii}, Δ^{v} and the second and fourth differences lying on the same lines as F_0, F_1 by Δ^{ii}_0, Δ^{ii}_1, Δ^{iv}_0, Δ^{iv}_1. Bessel's formula for any value F_n lying between F_0, F_1 may be written
>
> $$F_n = F_0 + n\Delta^{i} + \tfrac{1}{4}n(n-1)\{\Delta^{ii}_0 + \Delta^{ii}_1 - \tfrac{1}{12}(n+1)(2-n)(\Delta^{iv}_0 + \Delta^{iv}_1)\}$$
> $$+ \tfrac{1}{6}n(n-1)(n-\tfrac{1}{2})\{\Delta^{iii} - \tfrac{1}{20}(n+1)(2-n)\Delta^{v}\}$$
>
> as far as fifth differences inclusive.

* Continued from Vol. LXVIII, pp. 267–96, referred to hereafter as Part I. The paragraphs are numbered in sequence.

The required values of n are 1/12, 2/12, ..., 11/12. For the first six of these, $(n+1)(2-n)/12$ has the values

$$\tfrac{299}{1728},\quad \tfrac{308}{1728},\quad \tfrac{315}{1728},\quad \tfrac{320}{1728},\quad \tfrac{323}{1728},\quad \tfrac{324}{1728},$$

and the same values for the latter six, taken in reverse order. Their range is small. If we use the value 318/1728 instead of any one of them, the errors of the whole coefficient of $\Delta_0^{iv}+\Delta_1^{iv}$ will be

$$\tfrac{209}{995328},\quad \tfrac{200}{995328},\quad \tfrac{81}{995328},\quad -\tfrac{64}{995328},\quad -\tfrac{175}{995328},\quad -\tfrac{216}{995328}.$$

The largest of these produces an error less than $(\Delta_0^{iv}+\Delta_1^{iv})/4600$, and this produces errors which are never greater than $0^s{\cdot}0015$ in right ascension or than $0''{\cdot}02$ in declination. [The fraction $318/1728 \doteqdot {\cdot}184$ ....]

The coefficient of Δ^v is always less than ·001 and the corresponding maximum errors caused by the neglect of Δ^v are always less than $0^s{\cdot}001$ and $0''{\cdot}01$ respectively. *Footnote.* The formula shows, nevertheless, that Δ^v can be included with Δ^{iii} by means of the common factor 0·11. [Comrie's value is ·108.]

51. Camp said, loc. cit., p. 221:

"The suggestion may now be advanced, that when the higher orders of differences are small (and this is usually the case), it is entirely practicable to omit them and substitute for the differences of lower order, adjusted values which eliminate the need for the higher orders. This would save space in the printing and be more convenient for the user."

He also pointed out the advantage of the suggestion in simplifying inverse interpolation. Suggesting specifically the substitution of $(\delta^2 u+\kappa\delta^4 u)$ for the sum of the terms involving δ^2 and δ^4, he found the value of κ from "the condition that the sum of the squares of the errors within the range for which they are used (for this case -1 to $+1$) be a minimum". If for brevity we write Everett's formula* as

$$u_x=\begin{cases}\epsilon_0 u_1+\epsilon_2\delta^2 u_1+\epsilon_4\delta^4 u_1\ \ldots,\\ +e_0 u_0+e_2\delta^2 u_0+e_4\delta^4 u_0\ \ldots,\end{cases}$$

[where

$$\epsilon_{2n}=(x+n)_{(2n+1)}\ \text{ and }\ e_{2n}=-(x+n-1)_{(2n+1)}=(1-x+n)_{(2n+1)}],$$

Camp's solution may be presented as follows (note that Camp's k is our $-\kappa$):

* It is known that ϵ_{2n} and e_{2n} are the same function of x and $(1-x)$ respectively: they are of degree $2n+1$, while their sum—the coefficient of the mean δ^{2n} in Bessel's formula—is of degree $2n$.

Then $\{e_2\delta^2u_0+e_4\delta^4u_0-e_2(\delta^2u_0+\kappa\delta^4u_0)\}$

is the expression whose square is to be minimized for values of x lying between -1 and $+1$. Simplifying it somewhat, [the integral of]

$$\{e_4-\kappa e_2\}^2(\delta^4u_0)^2$$

is to be minimized with respect to κ. Therefore, as the derivative with respect to κ of the coefficient of $(\delta^4u_0)^2$ for any one point x is [twice]

$$-e_2e_4+\kappa e_2^2,$$

the sum [integral] of the values of this for the possible values of x between -1 and $+1$ must come to 0.

In this way Camp finds $k=-\kappa=11/60=\cdot18\dot{3}$.

52. It is not clear to the writer why the calculation is based on one only of the two lines of Everett's formula, each line being, as we have seen, of degree higher by one than the degree of the effective coefficient of δ^{2n}: nor why the integration is taken from -1 to $+1$, the usual Everett range being 0 to 1.* The resulting value of κ differs—very slightly, it is true—from that found by taking the actual coefficients of δ^2 and δ^4, viz. $x(x-1)/2$ and $(x+1)x(x-1)(x-2)/24$. If for the latter we substitute $\kappa.x(x-1)/2$ the squared error is

$$\tfrac{1}{4}[x^2(x-1)^2(x^2-x-2-\kappa/12)^2],$$

and minimizing the integral of this between 0 and 1, we find

$$\kappa=-31/168=-\cdot1845.$$

53. Milne-Thomson, *Calculus of Finite Differences* (1933), p. 71, finds $\kappa_3=-13/120=-\cdot108\dot{3}$ and $\kappa_4=-191/924\doteqdot-\cdot207$ by a different method, viz. from the condition that the integrated deviation over the range 0 to $\frac{1}{2}$ or 0 to 1 shall vanish. These values agree nearly with those found by the method of least squares as used in para. 52; but this coincidence seems to arise from the particular form of the coefficients since the principles involved in the two methods are quite different. It would seem that in principle the Least Squares method is safer and to be preferred. For

* Actually the second point makes no difference, for the integrand is an even function, and to minimize $\int_0^1$ is the same as to minimize $\int_{-1}^{1}=2\int_0^1$.

a zero *mean* error is consistent with large *actual* errors of different sign; whereas the Least Squares method tends to keep down the *largest* numerical errors of either sign. This is what we want; for the true principle (adopted by Comrie) is to fix κ so that the *worst* error may be as small as possible. After κ_2 this was done by trial and error; but the Least Square method gives a useful and very close approximation by direct calculation. Actually indeed sufficiently good results can be obtained (as suggested in para. 37, *ante*) by finding κ from the coefficients at the point $y = \pm\frac{1}{4}$, $x = \frac{1}{4}$ or $\frac{3}{4}$. We then have a zero error at the four points $x = 0, \frac{1}{4}, \frac{3}{4}, 1$. The following are numerical values yielded by this process, compared with Comrie's "best values" and the earlier least-square values. It will be seen that the three sets of values are barely distinguishable:

	$-\kappa_2$	$-\kappa_3$	$-\kappa_4$	$-\kappa_5$	$-\kappa_6$	$-\kappa_7$	$-\kappa_8$
Approx.	·182	·109	·206	·147	·218	·169	·2243
Comrie	·184	·108	·208	·147	·218	·169	·2246
Least Sq.	·185	·108	·207	—	—	—	—

It may be remembered that a somewhat rough value of κ does not introduce error into an interpolation, but merely restricts the range of δ^{2n+2} over which the residual error is negligible.

54. The first published Table (after Brown's) in which modified differences, on the "throw-back" principle, were tabulated were given in the *British Association Tables*, Vol. I (1931).

54*a*. Comrie's throw-back appears, so far, to have been applied only to Bessel's formula, but it may be remarked that the same principle is applicable to Stirling's formula. If in that formula the coefficient of the *t*th difference (or mean difference) be represented by s_t, we have

$$\lambda_{2n} \equiv s_{2n+2}/s_{2n} = -\frac{n^2 - x^2}{(2n+1)(2n+2)},$$

$$\lambda_{2n+1} \equiv s_{2n+3}/s_{2n+1} = -\frac{(n+1)^2 - x^2}{(2n+2)(2n+3)}.$$

These λ's are similar to the κ's appearing in the Bessel throw-back. Like the κ's, the λ's vary only slightly with x, when x lies in the central unit-range, and we may adopt suitable mean values of λ, and so use the throw-back principle precisely as with Bessel's formula. These mean values can be found in the same way as the Bessel κ's.

IV

55. Aitken's extension of his Quadratic Cross-Means method to *inverse* interpolation (foreshadowed in Part I, para. 22) has recently been published in the *Proc. Roy. Soc. Edinb.* Vol. LVIII (1938), pp. 161–75,* and copies of the paper have been placed in the Libraries of the Institute and the Faculty. The process is remarkably effective, and readers may be glad to have an account of it. This we give to some extent in our own way, and with some variations and additional matter.

56. The problem may be stated thus. Having given equidistant values of a function u, to find the argument x corresponding to a non-tabular value u_x. Just as in direct interpolation, it is desirable (at least, theoretically) that $|x|$† may be as small as possible, and by a suitable choice of origin x may be made to lie between $-\frac{1}{4}$ and $+\frac{1}{4}$. Thus, suppose u_x lies between u_t and u_{t+1} (where the tabular interval is taken as unity), and divide the interval t to $t+1$ into quarters. Then (i) if x falls in the first quarter we take the origin at t so that u_t becomes u_0 and x is positive: (ii) if x is in the central two quarters we take the origin at $t+\frac{1}{2}$ so that $u_{t+\frac{1}{2}}$ (which is not tabulated) becomes u_0, and x may be positive or negative: (iii) if x is in the last quarter we take the origin at $t+1$ so that u_{t+1} becomes u_0 and x is negative. The given values may then be symmetrically disposed in pairs about the central value u_0; thus in cases (i) and (iii)

$$u_0;\ u_{\pm 1},\ u_{\pm 2},\ u_{\pm 3}\ \ldots\ u_{\pm h}\quad (h=1,\ 2,\ 3\ \ldots),$$

and in case (ii)

$$(u_0);\ u_{\pm\frac{1}{2}},\ u_{\pm\frac{3}{2}},\ u_{\pm\frac{5}{2}}\ \ldots\ u_{\pm h}\quad (h=\tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{5}{2}\ \ldots).$$

In the second form the central value u_0 is not a tabular value and must therefore be found by interpolation if it is to be used.

* The paper can be obtained separately at the price of 1*s*. 3*d*. It also contains a method (not here discussed) of forming differential coefficients of successive orders at non-tabular points. [Cf. Comrie, *Interpolation and Allied Tables*, pp. 803–5].

Note the following *erratum*:

p. 164. u_{-3} should read 0·78767 ....

† $|x|$ is the modulus of x, i.e. its numerical value without sign. Cf. *J.I.A.* Vol. LI, p. 133.

But in practice u_0 is not generally used in this case; this question is discussed later (para. 62).

57. From any pair of values $u_{\pm h}$ we may obtain an approximate value of x by inverse linear interpolation, or proportional parts. We find

$$x \doteq \frac{u_x - u_{-h}}{(u_h - u_{-h})/2h} - h \equiv v_h.$$

Or if $\omega_h = u_x - u_h$ an alternative form is

$$v_h = \frac{h\,(\omega_{-h} + \omega_h)}{\omega_{-h} - \omega_h}.$$

This function v_h has important and interesting properties.

(i) $$v_x = x.$$

This appears at once on substitution, and it is the kernel of Aitken's method.

(ii) v_0 takes the indeterminate form 0/0, but defining it as $\mathop{\mathrm{Lt}}_{h\to 0} v_h$ we find

$$v_0 = (u_x - u_0)/u'_0,$$

where the accent denotes differentiation.

(iii) $$v_h = v_{-h},$$

i.e. the function does not change when $-h$ is substituted for h: it is therefore an even function of h or a function of h^2 (see the expansion in Note A, following para. 69). To such a function the Cross-Means method can be applied without preliminary linear interpolation, and two orders of differences are brought in at each step.

(iv) Since v_h is an even function its derivatives and central differences of odd order are all zero for $h=0$. In particular its first derivative at the point 0 is zero: hence near that point the function changes slowly, i.e. by increments approximately proportional to the *squares* of the increments of h.

58. If we take an approximate value of x, say $x+\epsilon_1$, and calculate $v_{x+\epsilon_1}$ by direct interpolation, we shall get (instead of x exactly) a result which may be written

$$v_{x+\epsilon_1} = x + \epsilon_2,$$

and it will be shown that, subject to conditions which will usually be fulfilled, the new error ϵ_2 is much smaller than ϵ_1. Repeating the process by interpolating for $v_{x+\epsilon_2}$ we shall get $x+\epsilon_3$, where ϵ_3 is much smaller than ϵ_2. Thus we approach $v_x = x$ very rapidly,* and in fact $\epsilon_1, \epsilon_2 \ldots$ are approximately in G.P. For if accents denote differentiation we have from (iv) above

$$v_{x+\epsilon_1} \doteqdot v_0 + \tfrac{1}{2}(x+\epsilon_1)^2 v'', \text{ or neglecting } \epsilon_1^2$$
$$\doteqdot v_0 + \tfrac{1}{2}(x^2 + 2x\epsilon_1)\, v_0'',$$

while $$x = v_x \doteqdot v_0 + \tfrac{1}{2}x^2 v_0'',$$

so that ϵ_2 is approximately $\epsilon_1 x v_0''$. Now x is numerically less than $\frac{1}{2}$, and v_0'' is generally small,† so in general ϵ_2 is much smaller than ϵ_1. Similarly, repeating the process, we shall get

$$\epsilon_3 \doteqdot \epsilon_1 (x v_0'')^2.$$

Thus as stated the errors decrease in G.P. (except so far as disturbed by higher derivatives when ϵ has become very small), and the quickness of convergence depends primarily on the smallness of x and v_0''. Hence $\epsilon_2, \epsilon_3 \ldots$ are specially small if x is small, which is the reason for so fixing the origin that $|x| < \frac{1}{4}$ (para. 56); but even if x is not kept within that range, $\epsilon_2, \epsilon_3 \ldots$ are generally small and rapidly decreasing because of a small value of v_0''.

59. It is not in practice necessary to push the full iterative process very far. There are [see next para.] limits imposed by the data on the accuracy with which x can be found; and the method is so powerful that when a certain stage has been reached the remaining figures of x, within those limits, can be found more quickly by a simple linear process. See Note B, *infra*, and the worked examples which are given later.

60. The degree of accuracy with which x is obtained is that of an interpolated v. It is shown in Note B, *infra*, that if the tabular values of u and the value of u_x are rounded off to the nearest unit in the last place, the consequential "tabular error" in an interpolated v may reach a maximum value of about $\frac{3}{2}/\mathrm{Y}$, where Y is the smallest value of $(u_h - u_{-h})/2h$ used in forming the v's. This is reduced to $\frac{3}{4}/\mathrm{Y}$ if u_x itself is not rounded off but exact. There is

* The process is therefore an iterative one, as to which Whittaker and Robinson (*Calculus of Observations*, p. 81) say: "A pleasing characteristic of iterative processes...[is] that a mistake...does not invalidate the whole calculation"; though it will usually slow down the approach to the correct result.

† It appears from Note A, *infra*, that $v_0'' \doteqdot -u_0''/u_0'$.

also to be considered the "residual error" due to the approximate nature of the interpolation. It seems hardly practicable to deduce an explicit remainder-term: in practice (to quote Aitken, loc. cit. p. 162), "the convergence of the interpolation is indicated to the eye of the computer by the convergence towards equality of [two or more] consecutive cross-means in the same column at any stage; the process is stopped at the stage when such entries agree to an assigned number of digits", not exceeding the number free from the tabular error just discussed.

61. The calculation of the v's is illustrated, with full notes, in the examples given below, after para. 69. The subsequent working process is as follows. Let the first two v's be $v_a = \xi$ and v_b (where a and b may be 0, 1; or 0, $\frac{1}{2}$; or $\frac{1}{2}$, $\frac{3}{2}$). Then ξ is a first approximation to x and a better one will be v_ξ, found by the Quadratic Cross-Means process as

$$v_\xi = \begin{vmatrix} v_a & a^2 - \xi^2 \\ v_b & b^2 - \xi^2 \end{vmatrix} \div (b^2 - a^2) = v_a - \frac{a^2 - \xi^2}{b^2 - a^2}(v_b - v_a);$$

the second form saves one multiplication but involves writing down figures, which is unnecessary with the direct process (Part I, para. 11, p. 274). If the result, taken to a few digits only, is ξ_1 we repeat the process with ξ_1 in place of ξ, getting as result a better value ξ_2. This part of the process takes a very short time: Aitken himself finds that "such trials take only a few seconds to perform". We then work out v_{ξ_2} fully to the required number of places, and find in practice that this differs from ξ_2 only in the later places. The complete solution is then

$$x = \xi_2 + (v_{\xi_2} - \xi_2)/(1 - 2\xi_2 V),$$

where $V = \frac{1}{2} v'_{\xi_2}$ may be found as indicated in Note B or by the rather longer alternative process given in Aitken's paper.

62. In Aitken's examples he uses the values $v_{\frac{1}{2}}$, $v_{\frac{3}{2}}$, $v_{\frac{5}{2}}$ ..., while on p. 175 he suggests using v_1, v_2, v_3 ...;* in both cases omitting v_0. Thus the process of finding v_ξ is one of *extra*polation. It would appear that greater convergence would be secured by incorporating v_0, thus interpolating instead of extrapolating for v_ξ and

* I.e. when u_x is close to u_0. This basis considerably increases the maximum tabular error (see Note B).

thereby greatly reducing the coefficients of neglected differences;* this might be of importance if the differences of u and therefore of v were but slowly convergent. Indeed, if for practical reasons v_0 is excluded it would seem that, even if x is near to a tabular point, it is better to use the system $v_{\frac{1}{2},\frac{3}{2}\ldots}$ than the system $v_{1,2\ldots}$. The principal object of using the latter system would be (para. 56) to make $|x|<\frac{1}{4}$ because this results in more rapidly diminishing errors in successive iterative approximations (para. 58). But in practice the direct iterative process is not carried very far (para. 59); and the rapidity of convergence in the interpolation of v_ξ is more important. Thus if v_0 is excluded—as it generally will be in practice because it requires rather special calculation—the best rule seems to be to use the $v_{\frac{1}{2},\frac{3}{2}\ldots}$ system, as in Aitken's examples, even though $|x|$ may approach $\frac{1}{2}$. If, however, it is desired to bring in v_0 it must be calculated by the formula [para. 57 (ii)]

$$v_0=(u_x-u_0)/u_0',$$

and it will be necessary to calculate u_0', and u_0 when it falls in the middle of an interval, by well-known formulae based on the u's and/or their differences, or by the Quadratic Means process, which Aitken ingeniously adapts to the purpose.

63. As a first illustration (Example VI) we take the following.†

* For example, if we use the system $v_{1,2,3\ldots}$ with $x=\cdot 1$ the coefficient of $\triangle^6 v$ is

$$(\cdot 01-1)(\cdot 01-2^2)(\cdot 01-3^2)\fallingdotseq -36,$$

while if we bring in v_0, i.e. use the system $v_{0,1,2\ldots}$, the coefficient is only

$$\cdot 01\,(\cdot 01-1)(\cdot 01-2^2)\fallingdotseq \tfrac{1}{25}.$$

Similarly, with the system $v_{\frac{1}{2},\frac{3}{2},\frac{5}{2}\ldots}$ the coefficient is

$$(\cdot 01-\cdot 25)(\cdot 01-2\cdot 25)(\cdot 01-6\cdot 25)\fallingdotseq -3\cdot 4,$$

while if we bring in v_0 and use the system $v_{0,\frac{1}{2},\frac{3}{2}\ldots}$ the coefficient is

$$\cdot 01\,(\cdot 01-\cdot 25)(\cdot 01-2\cdot 25)\fallingdotseq \tfrac{1}{188}.$$

† This is the example which is worked by Woolhouse's powerful method in the writer's Note, *J.I.A.* Vol. XLV (1911), p. 491. In this method the values of $u', u'' \ldots$ are found in terms of differences and the equation expressed as

$$(u_t-u_0)/u_0'=t+r_2t^2+r_3t^3+\ldots, \quad \text{(This is our } v_0\text{)}$$

and the first approximation to t is the appropriate (smaller) root, say ω, of

$$(u_t-u_0)/u_0'=t+r_2t^2.$$

Then t is expanded in a rapidly convergent series in powers of ω, thus:

$$t=\omega-h_3\omega^3-h_4\omega^4\ldots.$$

The method is systematic and very effective, but it necessitates the formation of differences if not tabulated.

Given the values of $a_{\overline{30}|}$ at 2 %, $2\frac{1}{2}$ %, ... $4\frac{1}{2}$ %, find the rate (in fact $3\frac{1}{6}$ %) at which the value is 19·1848276. This value falls between the 3 % and $3\frac{1}{2}$ % values, so the origin is taken as $3\frac{1}{4}$ %, and the interval $\frac{1}{2}$ % as unit: thus the true value of x is $(3\frac{1}{6}-3\frac{1}{4})/{\cdot}5=-{\cdot}1\dot{6}$. The data and the calculation of $v_{\frac{1}{2}}$, $v_{\frac{3}{2}}$ ... are shown in the annexed Table (p. 81). In this case the value of $v_0=(u_x-u_0)/u_0'$ has been worked out, *J.I.A.* Vol. XLV, p. 496, and it is brought into the Table. In the first place x is determined from the system $v_{\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}}$ and is found, correct to 7 places (the maximum number that the data will yield with certainty), to be $-{\cdot}1666667$. A second calculation, based on the system $v_{0,\frac{1}{2},\frac{3}{2},\frac{5}{2}}$, is also given and leads to the same result, but the quicker convergence is noticeable. This arises largely from the fact that in this example the tabular intervals are rather wide.

64. As a second illustration (Example VII) we take Aitken's first example, but worked on the system $v_{0,1,2\ldots}$. The Table gives the data and the calculation of the v's except v_0 which was specially calculated by the methods indicated in para. 62; the even central differences of u will be found if required in Comrie's *Interpolation and Allied Tables*, Ex. 5, p. 934. In this case u_x lies between u_0 and u_1 in Aitken's table (loc. cit. p. 164) and is nearer to his u_1; we therefore shift the origin and call his u_1 our u_0. Thus x is negative and proves to be $-{\cdot}263\ldots$; this becomes ·736... when referred to Aitken's origin, but it should be noted that analytically x is $-{\cdot}263\ldots$ in his work also. A comparison of Ex. VII with Aitken's working shows that the introduction of u_0 has secured a definite increase of convergence in the subsequent work.

65. Example VII, involving u's taken to 10 digits and a very accurate determination of x, shows well the great power and rapidity of Aitken's method when the new technique has been mastered; it must be remembered that in such a case *any* method must involve some considerable amount of calculation. But as Aitken remarks (p. 175): "For many practical purposes, when x is required to a few digits only, the method will give x correctly with no more than the first few trial cross means." Comparing the new method with the Linear Cross-Means plan (see Part I, paras. 20–1, Exs. II and III), it is first to be noticed that the calculation of the v's (which are linear approximations to x) is of the same form as that of the approximate values of x found in the first

stage of the linear method; but the new method involves only half as many such values. The number of stages required to reach a required degree of approximation is also about halved, since the quadratic method eliminates two orders of differences at each stage; and the divisors entering into the calculation are simple integers instead of long decimals. On the whole the saving of labour may be considerable if a high degree of accuracy is desired, involving the use of a considerable number of terms. Where, however, this is not the case some may prefer the rather more straightforward routine of the linear method. There is one point in which the linear method may be considered to have a slight advantage. The process of interpolation is stopped (see para. 60) when the entries in any column agree to an assigned number of digits. In the absence of a calculated remainder-term this criterion is slightly safer with the linear method than with the quadratic: for in the former, when the terms are properly arranged, the true value lies *between* successive values in a column; while in the latter the process is one of extrapolation and the true value lies behind the first entry in a column.

66. Aitken's method does not require the formation or use of differences, and this is one of its advantages when differences are not tabulated. But when they are tabulated some workers may prefer the more usual routine of the well-tried process of subtabulation followed by linear approximation. As this is only incidentally alluded to in *Mathematics for Actuarial Students* we may take this opportunity to illustrate it. It is thus described in Milne-Thomson's *Calculus of Finite Differences*, para. 4·6, p. 99. "A few figures of the argument are found, and the values of the function for this and one or two adjacent arguments [differing only in the last place] are calculated. Using these functional values we find some more figures of the argument [by linear interpolation], and then [if necessary] repeat the process..." with finer intervals until we have found as many places as we wish or the data will yield. If the primary tabular interval be taken as unit and the approximate value of x, say ξ, is found to within $(\cdot 1)^n$, the second differences of u will be reduced by subtabulation in the ratio

$1:(\cdot 1)^{2n}$, and so will not affect an interpolated value of u by more than $(\cdot 1)^{2n}/8$ of the primary second difference, and this will be negligible if it does not affect the last place of the u's.

67. The best interpolation formula from which to find the approximate value ξ is this form of Bessel's (Comrie, *Barlow's Tables*, 3rd edition, p. x, reviewed *T.F.A.* Vol. XIII, p. 476):

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{4}(\Delta u_1 - \Delta u_{-1}),$$

a simple and useful formula which is accurate to about $1/125$ of the *third* difference. It gives

$$x = \frac{u_x - u_0}{\Delta u_0 - \frac{1-x}{4}(\Delta u_1 - \Delta u_{-1})}.$$

Starting with the rough value $(u_x - u_0)/\Delta u_0$ and substituting in the denominator of the foregoing expression we get a greatly improved value, and if necessary the process may be repeated.* In the case of our Ex. VI, which we shall take as an illustration, we get

$$(u_x - u_0)/\Delta u_0 = -\cdot 4156/-1\cdot 208 = \cdot 344,$$

which is not very good because second differences are large, and then

$$\xi = -\cdot 4156 \Big/ \left[-1\cdot 2084 + \frac{\cdot 656}{4} \times -\cdot 02298\right] = \cdot 3335.$$

Another step would give $\cdot 3333\ldots$ but is unnecessary: it is quite sufficient to use $\cdot 333$ and $\cdot 334$, for by thus dividing the primary tabular interval by 10^3 we reduce second differences in the ratio $1:(\cdot 1)^6$, i.e. to a negligible figure if 7 digits in x will suffice.

68. The calculation of $u_{\cdot 333}$ and $u_{\cdot 334}$ can be effected by any of the usual central-difference formulae. When two parallel interpolations have to be performed (and a third if it is desired to verify the whole work by calculating u_x) a convenient alternative method is that suggested by W. F. Sheppard (Article 'Interpolation'; *Encyc. Brit.*, 11th ed., Vol. XIV, p. 707, § 3). This method is specially

* In the case of Exs. 2 and 3, Chap. v of *Maths. for Actuarial Students*, Part II, the first step (after the first rough trial value) gives the excellent approximations ·7646 and ·0430 respectively.

convenient when ξ goes to more than the 3 decimal places for which tables of coefficients are available by direct entry. We put u_x into the form

$$u_x = u_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \dots$$

and then beginning with the last term to be included, in our case A_4x^4, we construct in sequence

$$\begin{aligned} A'_3 &= A_3 + xA_4, \\ A'_2 &= A_2 + xA'_3, \\ A'_1 &= A_1 + xA'_2, \\ u_x &= u_0 + xA'_1. \end{aligned}$$

Here A_1, A_2 ... are the same for all values of x and are the values of $u'_0, u''_0/2!, u'''_0/3! \dots$, formed by the usual formulae for differential coefficients in terms of differences. A useful check, found by putting $x=1$, is that the algebraic sum of A_1, A_2 ... taken as far as will affect the last recorded place of u_x, is Δu_0. Also if (as often happens) $\Delta^t u$ and $\Delta^{t+1} u$ are of alternate signs, the sum of the arithmetic values without sign, $\Sigma \mid A_t \mid$, is $\pm\Delta u_{-1}$.

69. The working is shown in Example VIII appended. The correct value $x = \cdot 3333333$ is brought out and verified by computation of u_x, agreeing with the datum. This value of x is measured from an origin at the *beginning* of the tabular interval, and it thus indicates the same position as the value found in para. 63, Ex. VI, viz. $-\cdot 1666667$ measured from the point $\cdot 5$, the *middle* of the interval.

Examples VI and VII: calculation of v_h

h	u_h	$\begin{cases} u_{-h}-u_x \\ u_x-u_h \end{cases}$	$\begin{cases} u_{-h}-u_h \\ (u_{-h}-u_h)/2\,\lvert h\rvert \end{cases}$	$-v_h$
$-\frac{1}{2}$	19·6004414	0·4156138		
$+\frac{1}{2}$	18·3920454	0·7927822	1·2083960	·15606159
$-\frac{3}{2}$	20·9302926	1·7454650	3·6382593	
$+\frac{3}{2}$	17·2920333	1·8927943	1·2127531	·06074167
$-\frac{5}{2}$	22·3964556	3·2116280	6·1075671	
$+\frac{5}{2}$	16·2888885	2·8959391	1·2215342	−·12922040
$-\frac{7}{2}$	24·0158380	4·8310104	8·6433870	
$+\frac{7}{2}$	15·3724510	3·8123766	1·2347696	−·41247931

$u_x = 19{\cdot}1848276.$

0	·3867873682	·0239965396	·0906346235*	·264761287
−1	·4796611346	·0688772268	·1813649819	
+1	·2982961527	·1124877551	·09068249095	·240457269
−2	·5772156649	·1664317571	·3633050520	
+2	·2139106129	·1968732949	·09082626300	·167581142
−3	·6797722790	·2689883712	·5463986938	
+3	·1333735852	·2774103226	·09106644897	·046240693

$u_x = {\cdot}4107839078.$

Notes

The number marked * is the value of u_0', found as described in the text, paras. 62–3.

$$-v_h = [\text{2nd line of pair, col. 3}] \div [\text{2nd line of pair, col. 4}] - h.$$

The first line of each pair in col. 4 is the sum of the pair in col. 3.

The third col. may be formed thus on the arithmometer. Set up on the upper register the arithmetic complement of u_x, with enough 9's on the left for carrying purposes. Set up u_{-h} on the slide and give an addition turn, recording the result: give a subtractive turn, restoring the complement of u_x to the upper register, and proceed similarly with the next value of u_{-h} down to u_0. Then set up u_x on the upper register and u_h on the slide giving a subtractive turn followed by an addition turn, i.e. reversing the order of the operations.

Example VI: interpolation of v_ξ

h	$-v_h$	1st stage	2nd stage	3rd stage	$\frac{1}{2}$ Q.P.
$\frac{1}{2}$	·15606159				·11112222
$\frac{3}{2}$	·06074167	·166|65375			1+ ,,
$\frac{5}{2}$	−·12922040	·166|62865	…667|69		3+ ,,
$\frac{7}{2}$	−·41247931	·166|59118	…667|65	..73	5+ ,,

$$-v_{\cdot 1666} = \cdot 16666773.$$

[Note A, case (iii).] $2\xi V = \cdot 1666 \times [\cdot 15606 - \cdot 16667]/\cdot 11112 = -\cdot 0159$,

or case (iv), $2\xi V = \cdot 1666 \times (\cdot 06074 - \cdot 15606) = -\cdot 0159$.

$$\delta = (\cdot 16666773 - \cdot 1666)/1\cdot 0159$$
$$= \quad 6667,$$
$$x = -\cdot 1666666|7 = x + \delta.$$

Notes

With the $v_{\frac{1}{2}}$, $v_{\frac{3}{2}}$ ... scheme the semi-quadratic parts [col. headed $\frac{1}{2}$ Q.P.] are $\frac{1}{2}\{(\frac{1}{2})^2 - x^2\} = (\frac{1}{2}+x)(\frac{1}{2}-x)/2$ and the same increased by 1, 3, 5 ....

Taking $v_{\frac{1}{2}} = \cdot 156$ as a first approximation, the next is $v_{\cdot 156}$. For this the $\frac{1}{2}$ Q.P. is $\cdot 656 \times \cdot 344/2 = \cdot 113$, and we get for $v_{\cdot 156}$

$$\begin{vmatrix} \cdot 156 & \cdot 113 \\ \cdot 061 & 1\cdot 113 \end{vmatrix} = \cdot 156 + \cdot 113 \times \cdot 095 = \cdot 167.$$

We next try $v_{\cdot 167}$ for which the $\frac{1}{2}$ Q.P. is $\cdot 667 \times \cdot 333/2 = \cdot 111$

$$\begin{vmatrix} \cdot 1561 & \cdot 111 \\ \cdot 0607 & 1\cdot 111 \end{vmatrix} \quad \text{or } \cdot 1561 - \cdot 1111\ (\cdot 0607 - \cdot 1561) = \cdot 1667,$$

and we take ·1666, which gives slightly easier $\frac{1}{2}$ Q.P., and work out $v_{\cdot 1666}$ by the ordinary Quadratic Cross-Means routine, as above.

Note B shows that, since u is taken to 7 places and $Y \doteq 1\cdot 20$, the maximum error in v_h is about 1 in the seventh place. We nevertheless take in that and an additional place as a guard, rejecting this place at the finish and taking $x = -\cdot 1666667$ with a possible error of a unit in the last place. In this case it is evident that the error E of the final linear approximation (see end of Note B) is completely negligible.

Example VII: interpolation of v_ξ

h	$-v_h$	1st stage	2nd stage	Q.P.	
0	·264761287				$-$·0692110864
1	·240457269	·263079180		$1-$	,,
2	·167581142	..79801	·263078987	$4-$	,,
3	·046240693	..80837	·263078987	$9-$	,,

$$v_{\cdot 26308} = \cdot 263078987.$$

[Note A, case (i).] $2\xi V = (\cdot 263079 - \cdot 264761\ldots) \times 2 \div \cdot 26308$

$$= -\cdot 001682/\cdot 1315 = -\cdot 0128,$$

or case (ii), $2\xi V = 2 \times \cdot 26308 \times (\cdot 24076 - \cdot 26476) = -\cdot 0128.$

$$\delta = (\cdot 263078987 - \cdot 26308)/1\cdot 0128$$

$$= -\ldots\ldots 1000,$$

$$x = \cdot 26307900|0 = \xi + \delta.$$

Notes

With the $v_0, v_1 \ldots$ scheme we take the full (not semi-) quadratic parts as $-x^2$, $(1-x^2)$, $(4-x^2)\ldots$. Taking as a first approximation $v_0 \fallingdotseq \cdot 264$, the square of which is ·0697.., the next approximation is $v_{\cdot 264}$, viz.

$$\begin{vmatrix} \cdot 2648 & -\cdot 0697 \\ \cdot 2405 & 1- \quad ,, \end{vmatrix} \quad \text{or } \cdot 2648 + (\cdot 0697 \times -\cdot 0243) = \cdot 2631,$$

and using this value, $\cdot 2631^2$ is ·06922 and the next approximation is $v_{\cdot 2631}$, viz.

$$\begin{vmatrix} \cdot 26476 & -\cdot 06922 \\ \cdot 24046 & 1- \quad ,, \end{vmatrix} \quad \text{or } \cdot 26476 + (\cdot 06922 \times -\cdot 02430) = \cdot 26308,$$

and we then work out, as above, $v_{\cdot 26308}$ by the ordinary quadratic-means routine.

Note B shows that, since u is taken to 10 places and $Y \fallingdotseq \cdot 09$, the maximum error in v_h is about 1 in the 9th place. We nevertheless keep that place as a guard, rejecting it at the finish. Investigating the error E of the final linear approximation by the rule given in Note A, we find $m=9$, $n=6$, $p=1$ and $r=7$, and the error does not affect the 12th place. [Here $\delta < (\cdot 1)^6$; $A_2 \fallingdotseq V \fallingdotseq 2\cdot 4 \times (\cdot 1)^3$; $A_4 \fallingdotseq 6\cdot 2 \times (\cdot 1)^7/(\cdot 0692 \times 3) \fallingdotseq 3 \times (\cdot 1)^6$, by the Rules in Note A.]

Example VIII: data as in Ex. VI

	$\xi=\cdot333$	$\xi=\cdot334$	$\xi=\cdot3333333$
A_4	744	744	744
ξA_4	248	248	248
A_3	−23170	−23170	−23170
Σ	−22922	−22922	−22922
$\xi\Sigma$	− 7630	− 7656	− 7641
A_2	606532	606532	606532
Σ	598902	598876	598891
$\xi\Sigma$	199434	200025	199630
A_1	−12668046	−12668046	−12668046
Σ	−12468612	−12468021	−12468416
$\xi\Sigma$	− 4152048	− 4164319	− 4156138
u_0	196004414	196004414	196004414
$\Sigma=u_\xi$	191852366	191840095	$191848276=u_x$
u_x	191848276	191852366	
	4090 ÷	12271	

$=\xi=\cdot3333$ of the interval ·001,

$x=\cdot333|3333$.

A_1	−12668046
A_2	606532
A_3	− 23170
A_4	744
A_5	− 20 (not used)
Σ with signs	$-12083960=\Delta u_0$
Σ ex. signs	$13298512=\Delta u_{-1}$

Note. A_1, A_2 ... are found by the usual central-difference formulae for differential coefficients. The differences of u and the actual working out will be found in *J.I.A.* Vol. XLV, p. 496.

Note A.* On the Expansion of v_h

We may put v_h in the form

$$v_h = \frac{(u_x - u_0) - \frac{1}{2}(u_h - 2u_0 + u_{-h})}{(u_h - u_{-h})/2h}.$$

Denote differentiation by accents. Expand u_h and u_{-h} by Maclaurin's Theorem, and divide numerator and denominator by u_0'. Then, putting $v_0 \equiv (u_x - u_0)/u_0'$, $\lambda_2 = u_0''/2!\,u_0'$, $\lambda_3 = u_0'''/3!\,u_0'$ and so on, we get

$$v_h = \frac{v_0 - h^2\lambda_2 - h^4\lambda_4 - h^6\lambda_6 - \dots}{1 + h^2\lambda_3 + h^4\lambda_5 + h^6\lambda_7 + \dots},$$

from which we get by ordinary division

$$v_h = v_0 - h^2(\lambda_2 + v_0\lambda_3) - h^4[\lambda_4 - \lambda_2\lambda_3 - v_0(\lambda_3^2 - \lambda_5)]$$
$$- h^6[\lambda_6 - \lambda_4\lambda_3 + \lambda_3^2\lambda_2 - \lambda_5\lambda_2 + v_0(\lambda_3^3 - 2\lambda_3\lambda_5 + \lambda_7)] - \dots,$$

which may be written briefly as

$$v_h = v_0 + A_2 h^2 + A_4 h^4 + A_6 h^6 + \dots,$$

where A_2, A_4, A_6 ... involve x through

$$v_0 = x + x^2\lambda_2 + x^3\lambda_3 + x^4\lambda_4 + \dots.$$

The coefficients of h^8 etc. are very complicated, but can be expressed simply as determinants (see *J.I.A.* Vol. LI, p. 43; *T.F.A.* Vol. XIII, p. 271; Todhunter's *Theory of Equations*, p. 291). In this form A_{2n} is the following determinant of the $(n+1)$th order, viz.

$$\begin{vmatrix} 1 & 0 & 0 & \dots & v_0 \\ \lambda_3 & 1 & 0 & \dots & -\lambda_2 \\ \lambda_5 & \lambda_3 & 1 & \dots & -\lambda_4 \\ \lambda_7 & \lambda_5 & \lambda_3 & \dots & -\lambda_6 \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

The convergence of the series depends largely on λ_2, λ_3 ... being a rapidly diminishing sequence. This is generally the case with a well-ordered table of u_x, suitable for direct and inverse interpolation.

Having found v_ξ, where ξ is close to x, a near value v may be found with close approximation by means of a linear adjustment of v_ξ, based on two values v_a and v_b, one of which should preferably be v_ξ itself. Omitting terms which are evidently insignificant, we have

$$\begin{aligned} v_{\xi+\delta} &= v_0 + A_2(\xi^2 + 2\xi\delta + \delta^2) + A_4(\xi^4 + 4\xi^3\delta + \dots), \\ v_\xi &= v_0 + A_2\xi^2 \qquad\qquad\quad + A_4\xi^4, \\ \text{Diff.} &= \qquad A_2(2\xi\delta + \delta^2) \qquad + A_4(4\xi^3\delta + \dots). \end{aligned}$$

* The matter in Notes A and B is not contained in Aitken's paper.

Also
$$v_a = v_0 + A_2 . a^2 + A_4 a^4,$$
$$v_b = v_0 + A_2 . b^2 + A_4 b^4,$$
$$V \equiv \frac{v_b - v_a}{b^2 - a^2} = \quad A_2 \quad + A_4 (a^2 + b^2).$$

Hence
$$v_{\xi+\delta} = v_\xi + 2\xi\delta V + E,$$
where
$$E = \delta^2 A_2 + 2\xi\delta (2\xi^2 - b^2 - a^2) A_4.$$

E will usually be negligible: this point will be dealt with later.

Since, by the properties of the function, $v_x = x$ we have, when $x = \xi + \delta$,
$$x = v_{\xi+\delta} = \xi + \delta = v_\xi + 2\xi\delta V + E,$$
whence
$$\delta = \frac{v_\xi - \xi}{1 - 2\xi V} + \frac{E}{1 - 2\xi V}.$$

Now $2\xi V < V \risingdotseq \frac{1}{2} v'_\xi$, a small quantity, so the denominator $1 - 2\xi V$ differs little from unity, and the numerator $v_\xi - \xi$ is very small. Thus the error in δ due to neglecting the second term will be very nearly the error in v_ξ increased by E. When ξ is very near to x, so that δ is very small, the term E will be negligible; and δ, also $\xi + \delta = x$, will be correct to as many places as the interpolated v_ξ. In any particular case the magnitude of E may be estimated by means of approximate values of A_2 and A_4,* viz.
$$A_2 \risingdotseq V = (2\xi V)/2\xi,$$
$$A_4 \risingdotseq -d/Q_1 (Q_3 - Q_2),$$
$$\risingdotseq -4d/\tfrac{1}{2}Q_1 (\tfrac{1}{2}Q_3 - \tfrac{1}{2}Q_2),$$
where d is the difference between the first two values in the first column of interpolated values of v_ξ; Q_1, Q_2 and Q_3 the first three Quadratic Parts, and $\frac{1}{2}Q$, etc. the corresponding Semi-Quadratic Parts. A very rough value of A_4 is sufficient. The following general rule may be formulated: it is slightly too rigorous.

If v_ξ is accurate to the mth place, $\delta < (\cdot 1)^n$, $A_2 < (\cdot 1)^p$, $A_4 < (\cdot 1)^r$, the values of δ and x are correct to m places if $2n + p$ and $n + r$ are both greater than m. In practice this condition will generally be fulfilled.

The following are the values of $2\xi V$ and E in practical cases.

If the values used are v_0, v_1, v_2 ..., as in the Paper, Ex. VII:

(i) $a = 0$, $b = \xi$. $\quad 2\xi V = 2 (v_\xi - v_0)/\xi,$
$$E \;= \delta^2 A_2 - 2\delta\xi^3 A_4.$$

(ii) $a = 0$, $b = 1$. $\quad 2\xi V = 2\xi (v_1 - v_0),$
$$E \;= \delta^2 A_2 - 2\delta\xi (1 - 2\xi^2) A_4.$$

* $A_4 \risingdotseq \triangle^2 v$ (the 2nd divided difference of the v's): this could be found directly, but is easily seen to have the value given, involving smaller figures.

If the values used are $v_{\frac12}$, $v_{\frac32}$, $v_{\frac52}$ ..., as in Aitken's work:

(iii) $a=\frac12$, $b=\xi$. $\quad 2\xi V=\xi\,(v_{\frac12}-v_\xi)/\tfrac12\,(\tfrac14-\xi^2)$,

$$E \quad =\delta^2 A_2-4\delta\xi\,.\,\tfrac12\,(\tfrac14-\xi^2)\,A_4,$$

where $\frac12(\frac14-\xi^2)$ is the first Semi-Quadratic Part in the interpolation table.

(iv) $a=\frac12$, $b=\frac32$. $\quad 2\xi V=\xi\,(v_{\frac32}-v_{\frac12})$,

$$E \quad =\delta^2 A_2-2\delta\xi\,(\tfrac52-2\xi^2)\,A_4.$$

Instances of (i), (ii), (iii), (iv) are given in the Examples. As instances of (iii) and (iv) we take also the first example in Aitken's paper.

(iii)		(iv)	
$v_{\frac12}$	·741622	$v_{\frac32}$	·790105
v_ξ	·736921	$v_{\frac12}$	·741622
	·004701	$2V$	·048483
	÷		×
$\frac14-\xi^2$	·09693	ξ	·23692
	=		=
$2V$	·04851	$2\xi V$	·01149
	×	$1-2\xi V$	·98851
ξ	·23692		
$2\xi V$	·01149		
$1-2\xi V$	·98851		

The values agree with each other, and with Aitken's divisor.

Note B. On the Tabular Errors of the v's

Let the values of u be recorded to the nearest unit in the nth place: then in terms of that unit the error in any u is between $-\frac12$ and $+\frac12$. Now

$$v_h=\frac{u_x-u_{-h}}{(u_h-u_h)/2h}-h,$$

and as h is exact the error of v_h will be that of the fraction. If a, b and c are the actual errors (in units of the last place) in u_{-h}, u_h and u_x, we may write down the following scheme:

	True value	Recorded value	
u_{-h}	A	$A+a$	Put $C-A=X$
u_h	B	$B+b$	$(B-A)/2h=Y$
u_x	C	$C+c$	= divided-difference per unit interval*
v_h	$\dfrac{X}{Y}$	$\dfrac{X+c-a}{Y+(b-a)/2h}$	then $v_h=X/Y$

* It is assumed that an additional place is taken in forming Y, so that the rounding-off of v is not combined and confused with a further rounding-off of Y. The point does not arise with the alternative form

$$v_h=\{h\,(u_x-u_{-h})/(u_h-u_{-h})\}-h.$$

Thus the error in v_h is

$$\frac{X+c-a}{Y+(b-a)/2h}-\frac{X}{Y}$$

$$=\frac{(c-a)\,Y-(b-a)\,X/2h}{Y\,[Y+(b-a)/2h]\doteqdot Y^2}$$

$$\doteqdot [cY-a\,(Y-X/2h)-bX/2h]/Y^2$$

$$=[c-a\,(1-X/2hY)-bX/2hY]/Y$$

$$=[c-a\,(1-v_h/2h)-bv_h/2h]/Y.$$

Since v_h is positive and less than 1, this expression will have its greatest numerical value when

$$c=\pm\tfrac{1}{2},\ a=\mp\tfrac{1}{2},\ b=\mp\tfrac{1}{2},$$

when it becomes $\qquad \pm(\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}v_h/2h+\tfrac{1}{2}v_h/2h)/Y$

$$=\pm 1/Y.$$

This is reduced to $\pm\tfrac{1}{2}/Y$, if $c=0$, i.e. if u_x is an exact value. If Y is varying much we must take the smallest value we are dealing with.

As an illustration we take the u's of Aitken's first example (loc. cit. p. 164) cut down to 3 places and assumed to have approximately the extreme errors stated above.

	To 3 places	*Assumed exact values*	
u_0	·480	·47951	
u_1	·387	·38651	
u_x	·411	·41149	
v_h	·069/·093	·06802/·093	$1/Y=1/{\cdot}093=10{\cdot}6$
	$=\cdot742$	$=\cdot731$	Error 11 units of the last place.

If u_x has no error the error in v is reduced to about 5 units in the last place.* In Aitken's second example (p. 174), $1/Y\doteqdot 1/{\cdot}032\doteqdot 30$, reduced to 15 if u_x is exact. This agrees with his statement, p. 174, that the 7th decimal may be almost [possibly more than] a unit in error.

Since an interpolated value of v_x is a linear blend of individual v's, the error in v_x will be a similar blend of the individual errors, and in practice this will often be *less* than the greatest individual error involved. But for the purpose of measuring accuracy the worst possible combination must be considered. If the interpolated value is based on v_0, v_1, v_2 ... and these have individual errors e_0, e_1, e_2, the error e_x in v_x may be expressed as follows by Lagrange's Interpolation Formula:

$$e_x=C_0e_0+C_1e_1+C_2e_2+\dots.$$

* We cannot follow Aitken's statement (loc. cit. p. 172) that the tabular errors of the v's do not exceed $\tfrac{1}{4}$ in the 9th place, that is $2\tfrac{1}{2}$ in the 10th and last place.

Now algebraically $\Sigma C = 1$, but the C's will not all have the same sign, and the maximum numerical error of e_x will arise when every e has its maximum numerical value with the same sign as the corresponding C. Considering the numerator and denominator of C, from Lagrange's Formula, it is readily seen that (the given v's being $v_0, v_1, v_2 \ldots$, and x between 0 and 1) the signs are as follows:

$$\left.\begin{array}{lccccc} \text{Numerator} & \pm & \mp & \mp & \mp & \cdots \\ \text{Denominator} & \pm & \mp & \pm & \mp & \cdots \\ \text{C and } e & + & + & - & + & \cdots \end{array}\right\} \begin{array}{l}\text{Upper or lower signs as the number} \\ \text{of terms is even or odd.}\end{array}$$

Assuming that $e_0, e_1 \ldots$ have the value 1 with the signs indicated above, the value of e_x may be found by using the Quadratic Means formula, as for the main interpolation. The value of $|x|$ being less than $|\frac{1}{2}|$ we shall give the results for $|x| = \cdot 1, \cdot 2, \cdot 3, \cdot 4$ and $\cdot 5$ (see below). The numbers represent the maximum errors in the interpolated v_x (the errors in the given values being taken as unity) when 3 or 4 v's are used in the interpolation. It is seen that the tabular errors of v are not much increased in interpolation. This is because v_0 has been used and v_x is close to v_0: if v_0 is not used, but the calculation is based on $v_{\frac{1}{2}}, v_{\frac{3}{2}}, v_{\frac{5}{2}} \ldots$ as in Aitken's paper, the process is one of *extrapolation*, and the maximum error in the interpolated value may be as high as 1·4 times the maximum error in the given v's if three are used, and 1·5 times if four are used. If $v_1, v_2, v_3 \ldots$ are used, the maxima are considerably greater.

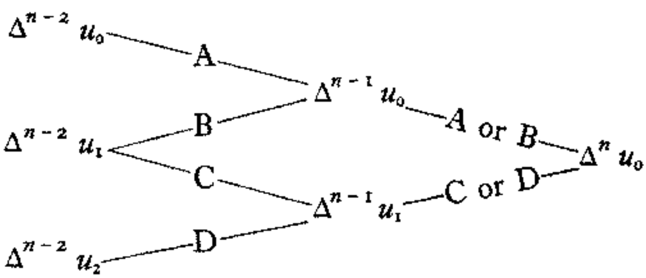
To sum up: if the u's, including u_x, are correct to the nearest unit in the nth place, the maximum error in the original v's is $1/Y$, and in the interpolated v's $\not> 1{\cdot}06/Y$, if the values $v_0, v_1, v_2 \ldots$ are used as the basis; or $1/Y$ and $1{\cdot}5/Y$ if $v_{\frac{1}{2}}, v_{\frac{3}{2}} \ldots$ are used. These values are to be halved if the value of u_x (the argument of which is to be found) is exact. As Aitken remarks, loc. cit. p. 172, "this maximal error, or even half of it, is in the highest degree improbable in practice".

Table (basis, $v_{0, 1, 2, 3}$)

ξ	3 terms	4 terms
·1	1·002	1·003
·2	1·006	1·011
·3	1·014	1·024
·4	1·022	1·040
·5	1·031	1·055

V

70. Suppose that we base an interpolated value of u_x on the given values $u_0, u_1 \dots u_n$, arranged in that order, using any formula involving only tabular differences, not mean differences: thus we may use the advancing-difference formula, the receding-difference formula, the Gauss forward or backward formula, or any other "zig-zag" formula. All of these will give identically the same result, and all will end with the difference $\Delta^n u_0$. From Sheppard's Rules (see *Maths. for Act.*, Part II, Chap. III, para. 7 and references there given) it is easily seen that, corresponding to the four possible routes A, B, C, D shown in the following diagram of the apex of the difference triangle:



there will be four forms for the last pair of terms, viz.

(A) $$x_{(n-1)}\Delta^{n-1}u_0 + x_{(n)}\Delta^n u_0 = x_{(n-1)}\left[\Delta^{n-1}u_0 + \frac{x-n+1}{n}\Delta^n u_0\right],$$

(B) $$(x-1)_{(n-1)}\Delta^{n-1}u_0 + x_{(n)}\Delta^n u_0 = (x-1)_{(n-1)}\left[\Delta^{n-1}u_0 + \frac{x}{n}\Delta^n u_0\right],$$

(C) $$(x-1)_{(n-1)}\Delta^{n-1}u_1 + (x-1)_{(n)}\Delta^n u_0 = (x-1)_{(n-1)}\left[\Delta^{n-1}u_1 + \frac{x-n}{n}\Delta^n u_0\right],$$

(D) $$(x-2)_{(n-1)}\Delta^{n-1}u_1 + (x-1)_{(n)}\Delta^n u_0 = (x-2)_{(n-1)}\left[\Delta^{n-1}u_1 + \frac{x-1}{n}\Delta^n u_0\right].$$

71. Let the coefficient of $\Delta^n u_0$ in the [ ] be represented by c, distinguished if necessary as c_A, c_B, etc. Then we may allow approxi-

mately for $\Delta^n u_0$, without using its actual coefficient, by giving to c any convenient approximate value k (for example a fraction in low terms). For a particular x, *any* value of k between 0 and $2c$ will be better than the value $k=0$ which corresponds to the entire omission of the Δ^n term. Conversely, any given k will be better than 0 for some range of x, viz.

Case A. $x >$ or $< n(1+\frac{1}{2}k)-1$ as k is +ve or −ve.

Case B. $x >$ or $< \frac{1}{2}nk$,,

Case C. $x >$ or $< n(1+\frac{1}{2}k)$,,

Case D. $x >$ or $< 1+\frac{1}{2}kn$,,

72. In using the method the coefficient of $\Delta^{n-1}u_\alpha$ (where α is 0 or 1) is to be multiplied into

$$[\Delta^{n-1}u_\alpha + k\Delta^n u_0] \fallingdotseq \Delta^{n-1}u_{\alpha+k},$$

assuming that $\Delta^n u$ may be regarded as practically constant. Thus the effect may be regarded as equivalent to a shift of the line of $\Delta^{n-1}u_\alpha$, upwards if k is negative, downwards if k is positive.

73. If x has a range of unity, say from $t-\frac{1}{2}$ to $t+\frac{1}{2}$, the selected value of k should be as near as convenient to the true value of c at the midpoint t; e.g. c_A to $-(n-t-1)/n = -1+(t+1)/n$, c_B to t/n, c_C to $(t-n)/n = -1+t/n$. If these exact values be taken the maximum numerical error will be $\pm 1/2n$, which diminishes as n increases.

74. *The advancing difference formula.* This falls in Case A, and if as usual x falls between 0 and 1, i.e. $t=\frac{1}{2}$, the best value of k will be $-1+3/2n$, which gives the following values:

$n=2 \quad k=-\cdot 25 \qquad n=4 \quad k=-\cdot 62$

$n=3 \quad k=-\cdot 50 \qquad n=5 \quad k=-\cdot 70$

so that in general, if we wish to end with an $(n-1)$th difference, the result will be considerably improved by taking $k=-\frac{1}{2}$, i.e. replacing $\Delta^{n-1}u_0$ by

$$\Delta^{n-1}u_0 - \tfrac{1}{2}\Delta^n u_0 \fallingdotseq \tfrac{1}{2}[\Delta^{n-1}u_{-1} + \Delta^{n-1}u_0].$$

Or if a high order of differences is involved we might take $k=-\frac{2}{3}$, i.e. use

$$\Delta^{n-1}u_0 - \tfrac{2}{3}\Delta^n u_0 \fallingdotseq \tfrac{2}{3}\Delta^{n-1}u_{-1} + \tfrac{1}{3}\Delta^{n-1}u_0.$$

75.* *The Gauss formulae.* (For brevity we shall use G^F and G^B to denote the Gauss forward-difference and backward-difference formulae respectively.) Consider Cases B and C of para. 70 and assume x to lie within the range of $\frac{1}{2}$ on each side of the line $\frac{1}{2}n$. Then $t=\frac{1}{2}n$, and our value of k will be $\frac{1}{2}n/n=+\frac{1}{2}$ in Case B and $(\frac{1}{2}n-n)/n=-\frac{1}{2}$ in Case C. Thus in Case B we shift $\Delta^{n-1}u_0$ *downwards* by half a space, i.e. replace it by

$$\tfrac{1}{2}(\Delta^{n-1}u_0+\Delta^{n-1}u_1),$$

this mean difference falling on the central line. In Case C we shift $\Delta^{n-1}u_1$ *upwards* by half a space, so replacing it by the same mean difference. Thus in each case the last two terms will be replaced by

$$(x-1)_{(n-1)}\,\tfrac{1}{2}(\Delta^{n-1}u_0+\Delta^{n-1}u_1).$$

But since the two formulae both include the same $(n-2)$th difference their sums up to that term are identically equal. Adding the common mean-difference term just given, the total of this and all preceding terms will be the same for Case B and Case C and therefore also for the mean of the two†. But this mean represents precisely:

(1) Stirling's formula when n is even and $n-1$ odd, x falling between $\frac{1}{2}n-\frac{1}{2}$ and $\frac{1}{2}n+\frac{1}{2}$: these are mid-interval points.

(2) Bessel's formula when n is odd and $n-1$ even, x falling between $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n+1)$: these are tabular points.

76. Thus Stirling's and Bessel's formulae are particular cases of what may be called (para. 78) the one-step throw-back, x being placed in the central unit-range with the effect of giving k the most convenient values $\pm\frac{1}{2}$, leading to simple means of differences. It is also clear that the benefits of Stirling's and Bessel's formulae may be obtained without the labour of forming all the odd or even mean differences: the same results are obtained by using G^F or G^B with ordinary differences except the *final* mean difference. This was pointed out and illustrated by D. C. Fraser, in his instructive

* The work in the remaining paragraphs may be read in conjunction with D. C. Fraser's illuminating paper, *J.I.A.* Vol. L (1916), p. 15, and with the writer's paper, *T.F.A.* Vol. IX (1922), p. 246.

† The difference diagram shows that if n is even both G^F and G^B start with the term $u_{\frac{1}{2}n}$, but if n is odd G^F begins with $u_{\frac{1}{2}(n-1)}$ and G^B with $u_{\frac{1}{2}(n+1)}$.

paper above referred to. To adapt his precept (loc. cit. p. 27), applicable when x is in the central unit range:

> The point to be remembered is that the best results are obtained by ending on a mean *odd* difference in line with the central u, or by ending on a mean *even* difference in line with the centre of the middle interval: in each case this mean difference is in line with the preceding ordinary difference. It is necessary to add that the rules may not be applicable if the differences of the order at which we stop show wide fluctuations. The order of the final mean difference should be so chosen that the first neglected *term* shall be small; and this largely turns on the first omitted *difference* being small, since its coefficient is made small by the adoption of the throw-back.

Thus the formation of all the odd or even mean differences is of advantage only when Comrie's throw-back is used. It may be mentioned that the coefficients in G^F are given *inter alia* to 8 decimal places, for differences up to the 6th and for 3 decimal places in x, in Chappell's *Interpolation Coefficients*, privately published.

77.* The investigation of paras. 75–6 fixes the position of the final mean difference and the preceding ordinary difference and hence the values of u to be used; but as seen in para. 70 there is considerable latitude as to the particular u with which the formula begins. In practice it is convenient to use the G^F or G^B formula, modified by the substitution of a mean difference for an ordinary difference in the last term only, according to the following rule, which is Fraser's (loc. cit. pp. 25–7) otherwise expressed. Cf. *T.F.A.* Vol. IX, p. 252, para. 12.

> Fix the origin so that u_0 is the *nearest* value to u_x, x thus ranging between $-\frac{1}{2}$ and $+\frac{1}{2}$. If x is positive, use the modified G^F formula, and if x is negative use the modified G^B formula; in either case beginning with u_0, proceeding as usual by *ordinary* differences but ending with a *mean* difference in line with the preceding ordinary difference.

When the final mean difference is of *even* order, the following alternative (involving only one formula) may perhaps be regarded

* In framing this paragraph the writer has had much benefit from consultation with Mr Fraser.

as simpler [see Woolhouse, *J.I.A.* Vol. XI, p. 69 (1863), and G. King, *Life Contingencies*, pp. 448–50, where an example is given]:

> Fix the origin so that u_x falls between u_0 and u_1, x thus ranging between 0 and 1. Use the modified G^F formula, beginning with u_0, proceeding as usual by ordinary differences but ending with a *mean* difference in line with the preceding ordinary difference. Since
>
> $$(x+t)_{(2t)}=(-x+t-1)_{(2t)} \quad \text{and} \quad (x+t)_{(2t+1)}=-(-x+t)_{(2t+1)},$$

the coefficients in the G^B formula for u_x are numerically the same as those in the G^F formula for u_x, a relation which also follows at once from writing the u's in reverse order (see Whittaker and Robinson's *Calculus of Observations*, pp. 37–8, para. 22, or their *Interpolation*, same ref.): thus Chappell's tables of coefficients can be used in all cases. It is this relation that makes it convenient (though not necessary), when the last difference is of odd order and x is negative, to use the G^B formula rather than the equivalent G^F formula, which is equally available.

---

78. The use of an approximate k is equivalent to a throw-back of one step, from one difference to the immediately preceding difference, instead of a Comrie throw-back of two steps, from one odd or even difference to the preceding odd or even difference. We have seen that Bessel's and Stirling's formulae are essentially based on this one-step process, and that it may be capable of useful extension to other cases. There is, however, a distinction to be noted as regards the value of k. If C is the coefficient of $\Delta^{n-1}u$, the error resulting from an approximate k is $(c-k)\,C\Delta^n u$. We have found k so that, for a given range of x, the value of $(c-k)$ and therefore of the error may approximately vanish in the middle of the range; whereas Comrie's κ is such that the *worst* value of the deviation of $(c-\kappa)\,C$ may be numerically as small as possible. This is the more strict criterion but for our present purpose the difference is not of great practical importance; for our k has small error near the middle of the range of x where C is most sensible, while at the extremes of that range, where k has its greatest error, C is small and the error in k has little effect.

Editorial Note

Mr Kingsland Camp has submitted the following additional remarks on his derivation of the throw-back coefficient outlined in para. 51 above.

He points out that each difference in Everett's formula is used throughout two intervals, and that if a function is not too extensive and not recorded to too many decimal places it can be compactly tabulated in short columns of figures whose higher differences, say of the fourth order, might easily change rapidly. Then the best value of the throw-back coefficient for modifying second differences would minimise especially the errors involving the larger of the fourth differences.

Suppose, for example, $\delta^4 u_1 = r\delta^4 u_0$. Then, by the reasoning outlined in *T.A.S.A.* Vol. XXIX, pp. 222–3, the error is a multiple of $\delta^4 u_0$ involving r and a fifth-degree expression in x. The maximum errors occur where the derivative vanishes. Mr Kingsland Camp finds that, when $r=1$, his value of k (11/60) gives a maximum error of $\cdot 00053\delta^4 u_0$, and that the value found in para. 51 above (31/168) gives a maximum error of $-\cdot 0005\delta^4 u_0$; so that the latter is slightly superior, as was to be expected since its method of calculation assumes equal fourth differences. On the other hand, when $r=2$, the maximum errors are $-\cdot 0012\delta^4 u_0$ and $-\cdot 0013\delta^4 u_0$ respectively, so that in this case the value 11/60 for k is slightly the better. The difference between the two values of k is, however, so small that the choice between them is hardly likely to be a matter of great importance.