

Investigating some Measures of Market Efficiency

- A Technical Note

Andrew Smith
Of the Stochastic Investment Modelling Working Party
andrewsmith8@deloitte.co.uk

Comments from: Jeroen van Bezooen, Keith Guthrie, Robert Howie, Peter Ludvik and Shyam Mehta are gratefully acknowledged.

1. Introduction

Many consider efficiency as a Boolean property – a market is either efficient or it is not. A more detailed investigation reveals that efficiency is a matter of degree. Investors can either make small expected profits from taking risks, or they can make large expected profits.

In this note we express no view on whether markets are efficient or not. Instead, we try to measure objectively whether a given stochastic model describes an efficient market. We develop quantitative measures of market efficiency which can be used to classify models according to the degree of market inefficiency they imply. Model builders may want to use our efficiency measures in order to construct models which deliberately reflect either efficient or inefficient markets.

2. Utility – Based Efficiency Measures

Sharpe Ratios

The simplest measure of efficiency is the optimised Sharpe ratio. This can be calculated over a finite time horizon, in a market with finitely many assets.

Let us suppose we have $n+1$ assets, and that the risk free return factor (ie 1 plus the risk-free rate) is r . Let us suppose that the return factors for the n risky assets have mean vector m and non-singular variance-covariance matrix V .

We can ask the following question:

"What is the maximum expected return on a portfolio, if the standard deviation of the return does not exceed σ ?"

We will see that the answer takes the form:

$$\text{maximum expected return} = r + S\sigma$$

where S is the *Optimised Sharpe Ratio*. We can regard S as a measure of market inefficiency, in the sense that if S is large, then investors can obtain a large expected profit from taking a small risk. For a one-year model, for example, we might expect a diversified equity portfolio to have a standard deviation around 16%, with a risk premium over bonds of perhaps 4%. If this is an optimal portfolio, then we would

have an optimised Sharpe ratio of $S = 0.25$. Expert opinions would vary on the size of this; we could argue for larger risk premiums or smaller volatilities. More generous estimates of then optimised Sharpe ratio might reach $S = 0.5$. Experts would still regard this as an efficient market.

If, however, we had a model that generated $S = 5$, we would all agree that was an inefficient market. It is hard to think of reasons why an efficient market should grant risk premiums as high as 5 standard deviations on a one-year horizon. So somewhere, perhaps around $S = 1$ lies the median expert consensus value of S which divides efficient and inefficient markets.

In our example, I claim S is given by the matrix formula:

$$S = \sqrt{(m - r\mathbf{1})^T V^{-1} (m - r\mathbf{1})}$$

where $\mathbf{1}$ is the unit vector of 1's. This enables us to compute whether or not a stated set of assumptions is consistent with market efficiency.

To prove this, let us denote by X the random vector of returns on the risky assets. With an initial investment of 1, suppose we allocate a vector p to each of the risky assets. The remaining amount to invest, $1 - p \cdot \mathbf{1}$ should be allocated to cash. The return factor on this portfolio is then given by $(1 - p \cdot \mathbf{1})r + p \cdot X$. The mean is $r + p \cdot (m - r\mathbf{1})$ and the variance is $\sigma^2 = p^T V p$. Now, by the Schwartz inequality, we know that

$$p \cdot (m - r\mathbf{1}) \leq \sqrt{p^T V p} \sqrt{(m - r\mathbf{1})^T V^{-1} (m - r\mathbf{1})}$$

and this is a best possible result. It follows that

$$r + p \cdot (m - r\mathbf{1}) \leq r + \sqrt{p^T V p} \sqrt{(m - r\mathbf{1})^T V^{-1} (m - r\mathbf{1})} = r + S S$$

which was to be proved.

Certainty Equivalent

The mean-variance framework presents a number of shortcomings, not least of which is the symmetric nature of standard deviation as a measure of risk. This symmetry means that a remote chance of a beneficial outcome could increase overall risk as measured by standard deviation. Such a result is counterintuitive and undesirable.

An alternative measure of efficiency, which circumvents such problems, starts by considering an increasing, convex utility function U . We can consider the problem of choosing p to maximise the expected utility. In other words, we seek p to solve

$$\max_p \mathbf{E}U[(1 - p \cdot \mathbf{1})r + p \cdot X]$$

The *optimised certainty equivalent* is the constant return factor \mathbf{r} that has the same expected utility as the optimised portfolio. Algebraically, \mathbf{r} is the solution to the equation:

$$U(\mathbf{r}) = \sup_p \mathbf{E}U[(1 - p \cdot \mathbf{1})r + p \cdot X]$$

The main difficulty with certainty equivalents is their computation. Unless U and the distributions of X are exceptionally tractable, we can only rarely develop suitable closed forms for the certainty equivalent \mathbf{r} .

3. Option Pricing Measures of Efficiency

Use of Density Functions

There is an alternative approach to measuring efficiency which does not rely on utility functions. Instead, it relies on measuring the difference between true statistical distributions and those implicit in market prices.

As before, let X denote the vector of stochastic returns. We suppose this has (true) probability density function $f(x)$.

Now let us consider the pricing of an arbitrary financial derivative, whose payoff is some function $h(X)$. In the absence of arbitrage (and subject to some further technical conditions) the price will always be representable in the form:

$$\text{price} = \frac{1}{r} \int_{\mathbb{R}^n} h(x) g(x) d^n x$$

where $g(x)$ is the *risk neutral* density. The same risk neutral density applies for all derivatives h .

Constraints on Risk-Neutral Laws

We cannot choose risk neutral laws arbitrarily. One constraint is that they must exactly price the original assets at 1. This implies in particular that

$$\mathbf{1} = \frac{1}{r} \int_{\mathbb{R}^n} x g(x) d^n x$$

Measures of Efficiency Based on Risk Neutrality

We can consider the efficiency of the market in terms of the differences between f and g . If $g=f$ then all assets are priced at the risk free rate, and there is no reward for taking risks. If on the other hand, f is different from g then the expected return factor on a derivative h is given by

$$\text{expected return factor} = \frac{\int_{\mathbb{R}^n} h(x) f(x) d^n x}{\int_{\mathbb{R}^n} h(x) g(x) d^n x} r$$

We could generate a high mean return by picking h to be large where $f \gg g$, and small elsewhere. The extent of the risk premiums available will depend on how different are f and g .

One possible measure of the difference between f and g is to use integrals of the form:

$$\int_{\mathbb{R}^n} f(x)^{1-\theta} g(x)^{\theta} d^n x = \mathbf{E} \left[\left(\frac{g(x)}{f(x)} \right)^{\theta} \right]$$

where the expectation is taken under the real world probability law f . If $\theta=0$ or $\theta=1$ then this integrates to unity. If $0<\theta<1$ then the integral is equal to 1 at most, by Hölder's inequality. Equality only occurs when $f=g$. Likewise for $\theta<0$ or $\theta>1$, we obtain an integral greater than 1, and perhaps infinite.

This suggests we can define a positive function $\mathbf{g}(\theta)$ by

$$\exp\left[-\frac{\mathbf{q}(1-\mathbf{q})}{2}\mathbf{g}(\mathbf{q})^2\right] = \int_{\mathfrak{R}^n} f(x)^{1-\mathbf{q}} g(x)^{\mathbf{q}} d^n x$$

For each value of θ , $\mathbf{g}(\theta)$ provides a measure of market inefficiency based on the difference between f and g . We investigate values of these efficiency measures, seeking to establish whether any particular value of θ makes sense for particular applications.

Entropy Concepts

When $\theta = 0$ or $\theta = 1$ our definition of $\mathbf{g}(\theta)$ reduces to $1=1$. However, we can still define $\mathbf{g}(\theta)$ by taking appropriate limits.

Taking θ close to zero, we have to first order:

$$\exp\left[-\frac{\mathbf{q}(1-\mathbf{q})}{2}\mathbf{g}(\mathbf{q})^2\right] \approx 1 - \frac{\mathbf{q}}{2}\mathbf{g}(0)^2$$

$$\int_{\mathfrak{R}^n} f(x)^{1-\mathbf{q}} g(x)^{\mathbf{q}} d^n x \approx \int_{\mathfrak{R}^n} f(x) \left[1 + \mathbf{q} \ln\left(\frac{g(x)}{f(x)}\right)\right] d^n x$$

Equating coefficients of θ , we find

$$\mathbf{g}(0) = \sqrt{2 \int_{\mathfrak{R}^n} f(x) \ln\left(\frac{f(x)}{g(x)}\right) d^n x}$$

Now if g were a uniform distribution, the integral would be what physicists call the *entropy* of f , that is, a measure of how irregular the true distribution is. More generally, we can think of this as a measure of the entropy of f relative to g .

Taking θ close to 1, we have a similar set of arguments, the final result of which is:

$$\mathbf{g}(1) = \sqrt{2 \int_{\mathfrak{R}^n} g(x) \ln\left(\frac{g(x)}{f(x)}\right) d^n x}$$

which is an entropy measure for g relative to f .

Gaussian Example

Let us now take an example where the underlying distribution of X is Gaussian, with mean m and variance-covariance matrix V . Under the risk neutral law, X must have mean $r\mathbf{1}$, and for we assume that the same variance-covariance matrix V applies to both the real world and risk neutral laws. Using the formula provided in the appendix, it now follows that, for all θ , we have

$$\mathbf{g}(\mathbf{q}) = \sqrt{(r\mathbf{1} - m)^T V^{-1} (r\mathbf{1} - m)}$$

We notice that in this case, the efficiency measure turns out to be exactly equal to the optimised Sharpe ratio \mathbf{S} .

4. Relationships between Utility and Option Pricing Efficiency Measures

The really useful property of the risk-neutral efficiency measures is that, firstly we can compute them, and secondly they act as upper bounds for certainty equivalents. In this section, we establish these relationships.

We use Y to denote the final payoff of a portfolio, that is,

$$Y = (1 - p)1 + pX.$$

Then Y is also a function of Z . It has the important property that its mean under the risk neutral law is the risk free rate, so that

$$\int_{\mathfrak{R}^n} y(x) g(x) d^n x = r$$

We now consider some common utility functions and create bounds on the certainty equivalent in terms of the inefficiency measure $\mathbf{g}(\theta)$.

Logarithmic Utility

We consider first an investor who wants to maximise $\mathbf{E} \log(Y - c)$ over portfolios Y . This requires that Y always exceeds c , so this cannot be possible if c exceeds the risk free rate r . Thus, we may assume $c < r$.

Now, by Jensen's inequality, and the concavity of the logarithm, we have

$$\int_{\mathfrak{R}^n} \log \left[\frac{g(x)}{f(x)} (y(x) - c) \right] f(x) d^n x \leq \log \left[\int_{\mathfrak{R}^n} \frac{g(x)}{f(x)} (y(x) - c) f(x) d^n x \right]$$

We can factorise the log on the left hand side, and cancel the f 's on the right hand side, to give

$$-\frac{1}{2} \mathbf{g}(0)^2 + \mathbf{E} \log[Y - c] \leq \log(r - c)$$

We can express this in terms of certainty equivalents. In particular, picking Y to optimise the left hand side, we find:

$$r \leq c + (r - c) \exp \left[\frac{1}{2} \mathbf{g}(0)^2 \right]$$

Exponential Utility

Optimising an exponential utility means minimising $\mathbf{E}[\exp(-\alpha Y)]$. Here again, we use Jensen's inequality, to deduce:

$$\int_{\mathfrak{R}^n} \exp \left[\log \left(\frac{f(x)}{g(x)} \right) - \alpha y(x) \right] g(x) d^n x \geq \exp \int_{\mathfrak{R}^n} \left[\log \left(\frac{f(x)}{g(x)} \right) - \alpha y(x) \right] g(x) d^n x$$

On the left hand side, we have a combination of an exp and a log, which we can multiply out. We can develop the right hand side analytically. This gives:

$$\mathbf{E} \exp[-\alpha Y] \geq \exp \left[-\frac{1}{2} \mathbf{g}(1)^2 - \alpha r \right]$$

Once again, choosing the optimal Y on the left hand side we have

$$r \leq r + \frac{\mathbf{g}(1)^2}{2\alpha}$$

We can see that larger α implies more risk aversion, and so a smaller certainty equivalent.

Sharpe Ratio

We now determine an inequality relating Sharpe ratios to inefficiency measures. We denote the expected return by μ and its standard deviation by σ .

As correlations can be at most 1, we know that

$$\int_{\mathfrak{R}^n} \left(1 - \frac{g(x)}{f(x)}\right) (y(x) - \mathbf{m}) f(x) d^n x \leq \sqrt{\int_{\mathfrak{R}^n} \left(1 - \frac{g(x)}{f(x)}\right)^2 f(x) d^n x} \sqrt{\int_{\mathfrak{R}^n} (y(x) - \mathbf{m})^2 f(x) d^n x}$$

The integrals simplify; taking each in turn we have

$$\begin{aligned} \int_{\mathfrak{R}^n} \left(1 - \frac{g(x)}{f(x)}\right) (y(x) - \mathbf{m}) f(x) d^n x &= \int_{\mathfrak{R}^n} (y(x) - \mathbf{m}) f(x) d^n x - \int_{\mathfrak{R}^n} (y(x) - \mathbf{m}) g(x) d^n x \\ &= \mathbf{m} - \mathbf{m} - r + \mathbf{m} \\ \int_{\mathfrak{R}^n} \left(1 - \frac{g(x)}{f(x)}\right)^2 f(x) d^n x &= \int_{\mathfrak{R}^n} \left(f(x) - 2g(x) + \frac{g(x)^2}{f(x)}\right) d^n x \\ &= 1 - 2 + \exp[\mathbf{g}(2)^2] - 1 \\ \int_{\mathfrak{R}^n} (y(x) - \mathbf{m})^2 f(x) d^n x &= \mathbf{S}^2 \end{aligned}$$

Now, substituting into the correlation inequality, we have

$$\mathbf{m} - r \leq \mathbf{S} \sqrt{\exp[\mathbf{g}(2)^2] - 1}$$

Taking the highest μ for a given σ , we deduce that the Sharpe ratio \mathbf{S} satisfies

$$\mathbf{S} \leq \sqrt{\exp[\mathbf{g}(2)^2] - 1}$$

We can verify this immediately in the case of multivariate normal distributions, where we saw that $\mathbf{g}=\mathbf{S}$. At least in this case, our inequality is fairly tight provided \mathbf{S} is small.

Power Law – Power between 0 and 1

Let us now consider a power, β , between 0 and 1. Let us suppose that an investor wishes to maximise $\mathbf{E}(Y-c)^\beta$. As the utility function is only defined on $Y \geq c$, this problem is only well posed when $c < r$. In this case, we know by Hölder's inequality that

$$\int_{\mathfrak{R}^n} [y(x) - c]^\beta f(x) d^n x \leq \left(\int_{\mathfrak{R}^n} f(x)^{\left(\frac{1}{1-b}\right)} g(x)^{\left(\frac{b}{1-b}\right)} d^n x \right)^{1-b} \left(\int_{\mathfrak{R}^n} [y(x) - c] g(x) d^n x \right)^b$$

We can now substitute for each term, and maximising the left hand side we have:

$$(r - c)^b \leq \exp \left[\frac{\mathbf{b}}{2(1-\mathbf{b})^2} \mathbf{g} \left(-\frac{\mathbf{b}}{1-\mathbf{b}} \right)^2 \right]^{1-b} (r - c)^b$$

Thus, finally we have the bound on the certainty equivalent:

$$r \leq c + \exp \left[\frac{1}{2(1-\mathbf{b})} \mathbf{g} \left(-\frac{\mathbf{b}}{1-\mathbf{b}} \right)^2 \right] (r - c)$$

We can see that risk tolerance increases as β tends up to 1. In this case, the certainty equivalent also increases without bound. As β tends to zero, we obtain the logarithmic result.

Power Law, with Negative Power

The final case we consider is one where an investor sees to minimise an expectation of the form $\mathbf{E}(Y-c)^{-\gamma}$ for some $\gamma > 0$. Once again, Hölder's inequality implies that

$$\int_{\mathfrak{R}''} f(x)^{\left(\frac{1}{1+g}\right)} g(x)^{\left(\frac{g}{1+g}\right)} d^n x \leq \left(\int_{\mathfrak{R}''} [y(x)-c]^{-g} f(x) d^n x \right)^{\left(\frac{1}{1+g}\right)} \left(\int_{\mathfrak{R}''} [y(x)-c] g(x) d^n x \right)^{\left(\frac{g}{1+g}\right)}$$

Once more, we can evaluate each term of the integral. On minimising the right hand side over choices of $y(x)$, we have

$$\exp\left(-\frac{g}{2(1+g)^2} g\left(\frac{g}{1+g}\right)^2\right) \leq (r-c)^{-\left(\frac{g}{1+g}\right)} (r-c)^{\left(\frac{g}{1+g}\right)}$$

Thus, rearranging, we finally have

$$r \leq c + (r-c) \exp\left(\frac{1}{2(1+g)} g\left(\frac{g}{1+g}\right)^2\right)$$

Interestingly, this has the same form as our previous result, although the proof was different.

Collation of Results

We can now pull together the results we have had so far. We can see that many possible arguments θ in the function $g(\theta)$ give us bounds for utility functions. In particular, we note the following ranges:

range for q	utility function
$\theta < 0$	power law; power between 0 and 1
$\theta = 0$	logarithm
$0 < \theta < 1$	power law; negative power
$\theta = 1$	exponential
$\theta = 2$	Sharpe ratio

The pattern we see is not surprising given what we know about the limiting definitions of transcendental functions. We know, for example, that logarithms can be approximated by powers close to zero, and that exponentials can be defined using limits of large negative powers.

Using Derivatives in Investment Portfolios

Derivatives are priced by the same risk neutral law as any other security. Indeed, it is only by examination of derivative prices that we can determine the risk neutral law. Our bounds on utility therefore apply equally to portfolios containing options. Indeed, when derivatives are included, our bounds are often attained.

5. Building Maximally Efficient Models

It might be interesting to measure inefficiency for its own sake. The real benefit of these measures arises if we use them to build better models. There are a number of situations in which one does not want to introduce inefficiencies into a model, or where certain specific inefficiencies are to be incorporated without introducing others by accident. As we have already seen, in this context the efficiency measures such as $g(\theta)$ based on risk neutral ideas are more tractable than efficiency measures based on certainty equivalents such as r . We now consider two applications of the concept of maximum efficiency.

Choosing Assumptions for Multinational Models

Choosing assumptions for multinational asset models is seldom easy. We may be able to model volatilities and correlations from historic data, but what is to be done about expected returns?

Sometimes the modeller has views about returns on certain asset classes, but may have less of a view on other less well known classes. How is one to choose neutral assumptions for those classes for which scant information is available?

One approach is to fix the volatilities and correlations empirically, assuming this are equal under the risk neutral and real world laws. The mean returns under the risk neutral laws must be equal to the risk free rate. The real returns under the real world law are unknown, or partially unknown.

The natural approach here is to fix the expected returns where we have a strong view, and to set the other returns by minimising the overall inefficiency of the model.

Completing Markets

Another application of our techniques is to complete models, that is, to develop pricing models given only a true probability law. For example, given a statistical model of share prices, we might want to consider the pricing of derivatives. This problem has been considered, for example, by Kemp (2001). One natural way to carry out such a completion is to select the risk neutral law in such a way as to minimise the total inefficiency. This means that we continue with any inefficiencies originally in the model, but that we seek to avoid adding any further when creating the option pricing model.

In the same vein, we may start with a statistical model of two points on the yield curve – for example, a short rate and a long bond yield. Often, it is necessary to interpolate such models in order to treat prices of other bonds. One way of doing this is to interpolate the given yield curve using some suitable functional form. Such approaches often introduce inefficiencies or even arbitrages into the model. The alternative is to construct a maximally efficient risk neutral law consistent with the observed price series. Given a risk neutral law, we then have a way of constructing the missing bond prices.

Appendix: Multivariate Normal Calculations

Functional Normal Example

We now consider an example where the return factor vector X is a function of some normally distributed vector Z . This would cover, for example the lognormal model underlying the formula of Black and Scholes.

We assume that Z has the following means and variances under the probability laws:

	real world	risk neutral
mean vector	m_f	m_g
variance-covariance matrix	V_f	V_g

The risk neutral law must still be contrived so that the mean of X is $r\mathbf{1}$.

We could write the efficiency measures in terms of integrals over X -space, but is simpler to transform the variables and work over Z -space instead.

Our inefficiency definition now becomes:

$$\begin{aligned} & \exp\left[-\frac{\mathbf{q}(1-\mathbf{q})}{2}\mathbf{g}(\mathbf{q})^2\right] \\ &= \frac{1}{(2\mathbf{p})^{n/2}|V_f|^{(1-\mathbf{q})/2}|V_g|^{\mathbf{q}/2}} \int_{\mathbb{R}^n} \exp\left[\begin{aligned} & -\frac{1-\mathbf{q}}{2}(z-m_f)^T V_f^{-1}(z-m_f) \\ & -\frac{\mathbf{q}}{2}(z-m_g)^T V_g^{-1}(z-m_g) \end{aligned}\right] d^n z \end{aligned}$$

After much manipulation, the inefficiency finally emerges as:

$$\mathbf{g}(\mathbf{q}) = \sqrt{\frac{(m_f - m_g)^T [(1-\mathbf{q})V_g + \mathbf{q}V_f]^{-1} (m_f - m_g)}{+\frac{1}{\mathbf{q}(1-\mathbf{q})} [\log((1-\mathbf{q})V_g + \mathbf{q}V_f) - \mathbf{q}\log|V_f| - (1-\mathbf{q})\log|V_g|]}}$$