GIRO 2003 Convention

Practical Issues in Dependences

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Contents

- Reasons for modelling dependences
- Ways dependences arise
- Methods of estimating dependences
- Limiting Theorems
- Simulating dependent random variables
- Choosing an appropriate dependence structure

Reasons for Modelling Dependences

- Pricing
- Solvency Assessment
- Capital Structure
- Portfolio Structure

How Do Dependences Arise?

- Factors affecting more than one variable
  - E.g. insurance cycle, economic factors
- Other Random Factors

Examples

- World Trade Centre causing losses to Property, Life, Workers Compensation, Aviation insurers
- ENRON causing losses to the stock market and to Surety Bonds, E&O and D&O insurers
- Dot.Com market collapse causing the stock market to fall and losses to insurers of financial institutions and D&O writers

Examples (continued)

- WTC / Enron / stock market losses causing impairment to reinsurers solvency, so increasing credit risk on RI recoveries
- Asbestos affecting many past liability years at once
- WTC and other R/I losses led to reduced capacity causing severe lack of R/I capacity in space market

Modelling Dependences

- Model factors causing dependences and its relation to other variables
  - Natural Hazards Models
  - Insurance Cycle
  - Economic Cycle and its effect on Property and Credit insurance
- After removing specific dependences use statistical methods
- Use Disaster Scenarios to examine the financial effect of extreme events

Purpose of Modelling

- Consider the effect of dependence
- Consider the whole dependence structure
- Examples
  - Solvency: need to look at tail dependence
  - Multi-Class Stop Loss pricing:
    - Attachment close to expected L/R: dependence may not be crucial
    - Attachment higher than expected L/R: dependence more important

Measures of Correlation

- Linear correlation

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}}$$

  - is not invariant under non-linear strictly increasing transformations

- Spearman's Rank correlation

$$\rho_s(X,Y)=\rho(F(X),F(Y))$$

  - is the linear correlation of the ranks
  - it is invariant under non-linear strictly increasing transformations

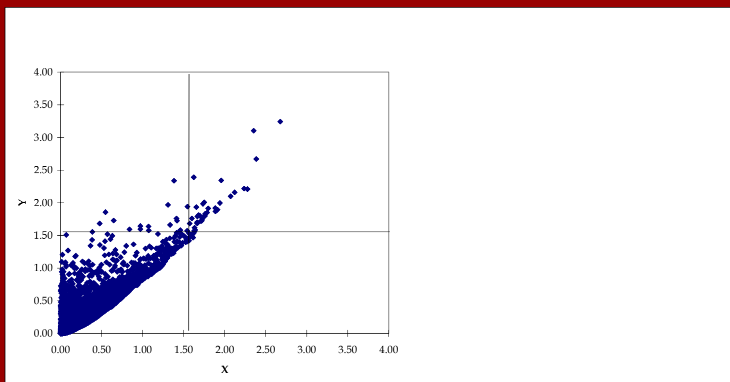
- Kendall's tau

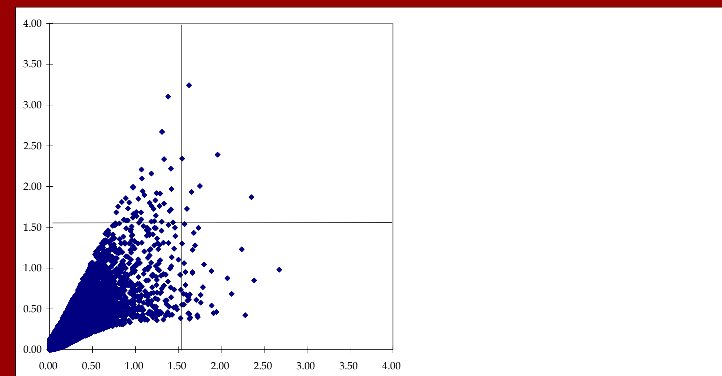
$$\tau(X,Y)=P((X_i-X_j)(Y_i-Y_j)>0)-P((X_i-X_j)(Y_i-Y_j)<0)$$

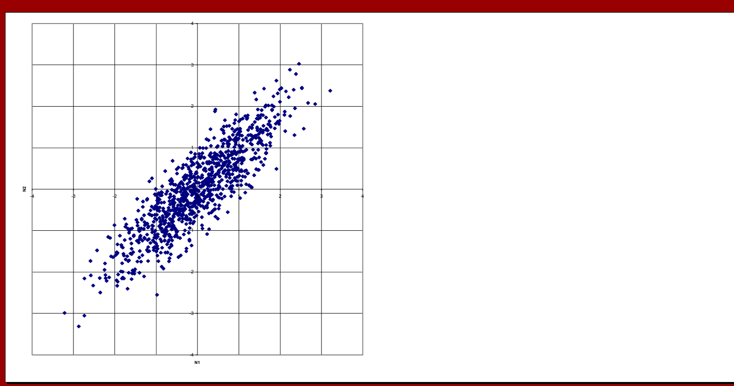
  - Where X_j,Y_j is an independent copy of X_i, Y_i
  - looks at the number of sign reversals
  - it is invariant under non-linear strictly increasing transformations

Same Correlation Different Dependence Structures

- Average does not tell the whole story
- Correlation does not tell the whole story







- In all graphs correlation is equal to 85%, but the dependence structure very different

Copula Definition

For m-variate distribution F with j th univariant margin F_j the copula associated with F is a distribution function

$$\boxed{\begin{array}{l} C:[0,1]^m \rightarrow [0,1] \text{ that satisfies} \\ F(\underline{X}) = C(F_1(X_1),..,F_m(X_m)) \end{array}}$$

(Note: if F is a continuous m-variate distribution the copula associated with F is unique)

This can also be represented via a density function

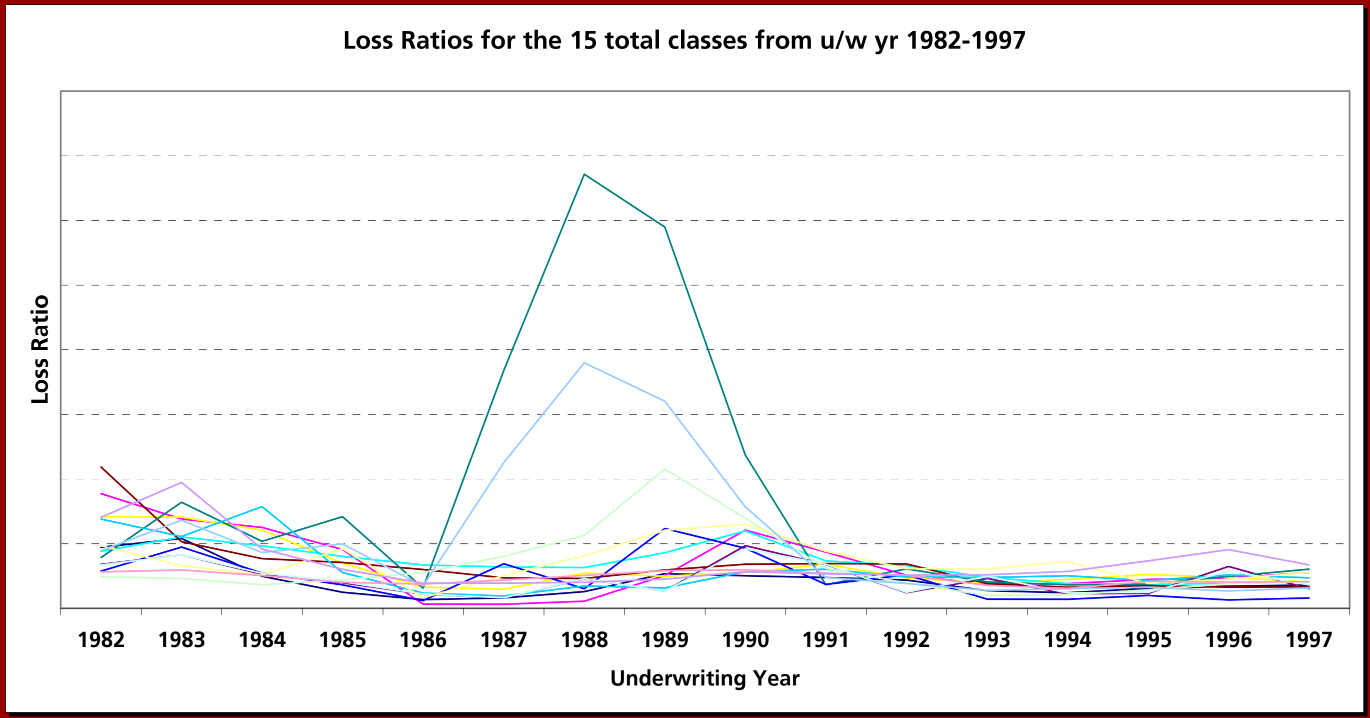
$$\boxed{c(u,v) = \frac{\partial C(u,v)}{\partial u\, \partial v}, \qquad 0 < u,v < 1}$$

Kendall's tau and rank correlation can be expressed as functions of C

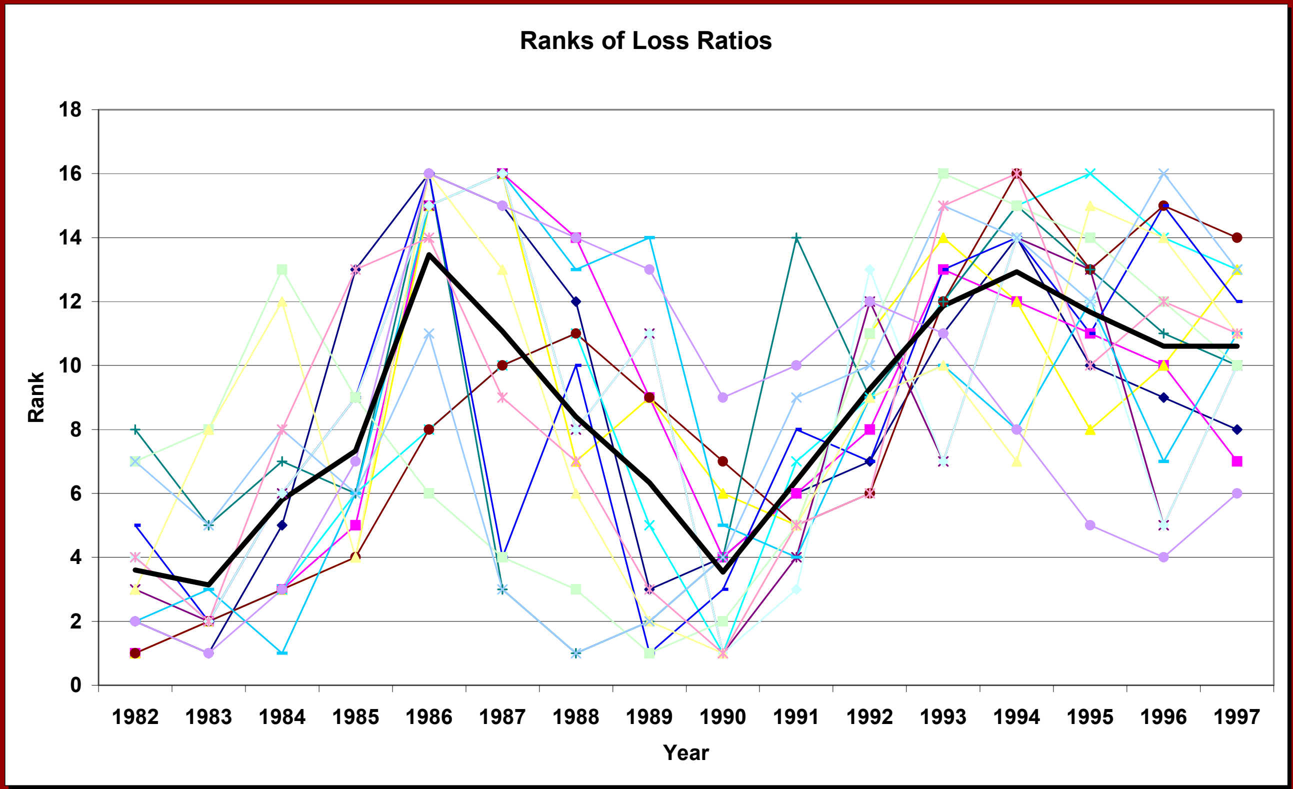
Estimation of Dependences

- Historical Data
  - Objective
  - Dependences change over time
  - Not enough data
- Educated Guesses
  - Incorporate qualitative information
  - Subjective
  - Not always easy to capture dependence structure
- Theorems about limiting cases

Historical Data Example: London Market Loss Ratios



Ranked Loss Ratios

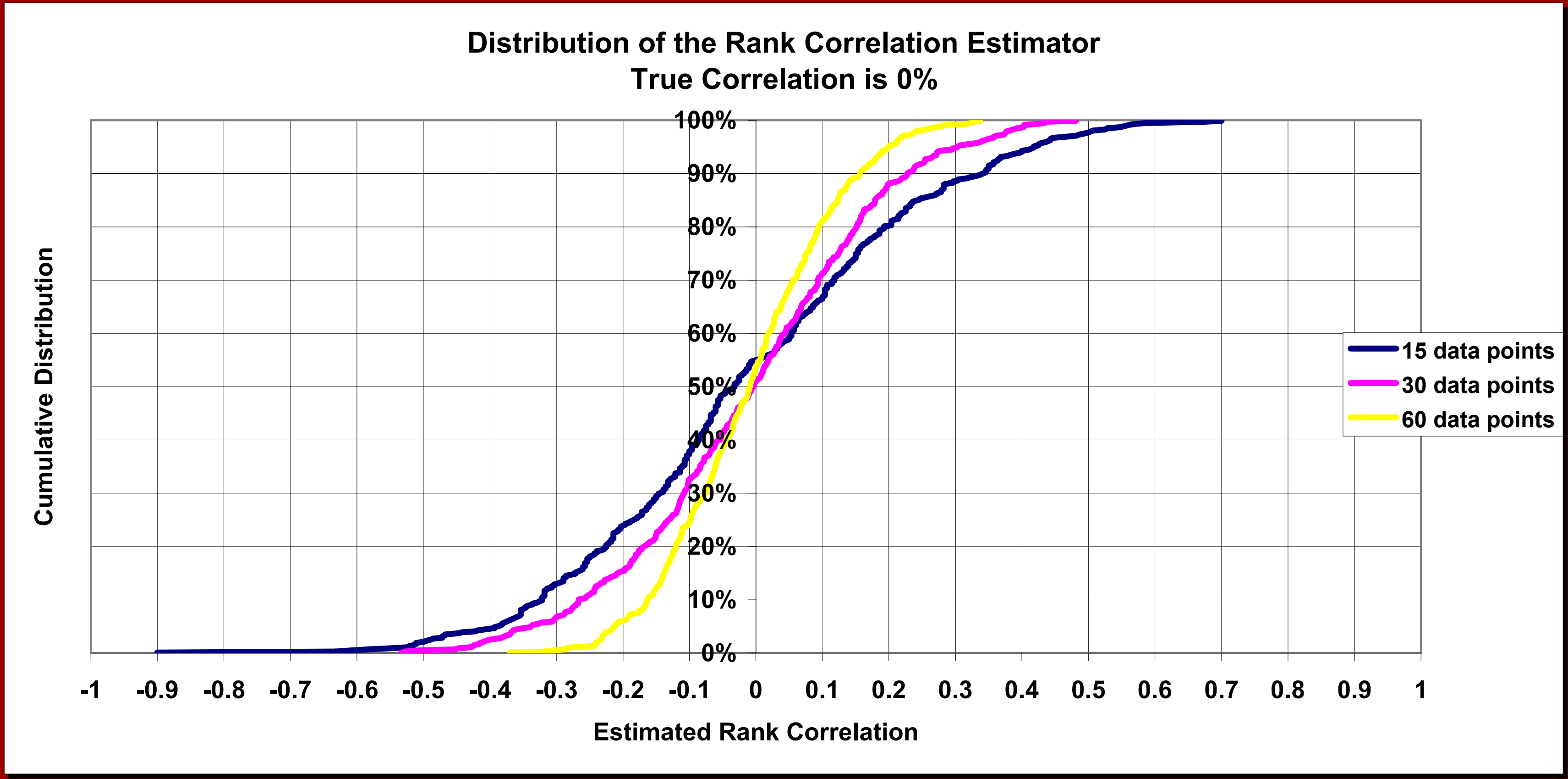


- **Average Rank Correlation is about 50%**
- **Average Rank Correlation is around 0% after “removing” premium cycle**
- **Proportional/Non Proportional**
- **Marine/All Other Classes**
- **Error in estimation is significant**

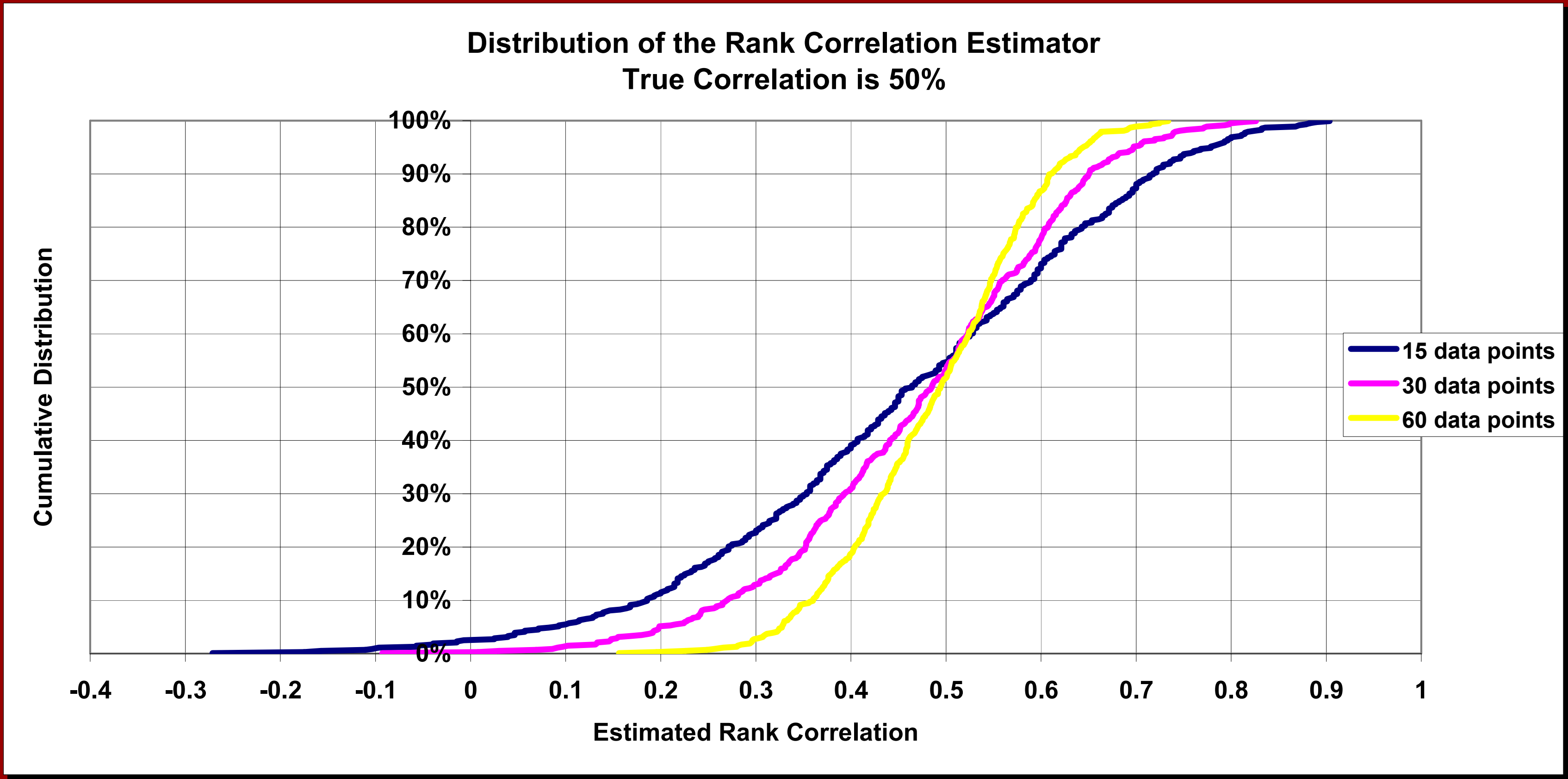
Distribution of Rank Correlation Estimators

- Distribution of the estimator has been Simulated
- Based on 500 simulations
- Dependence Structure as for Bivariate Normal

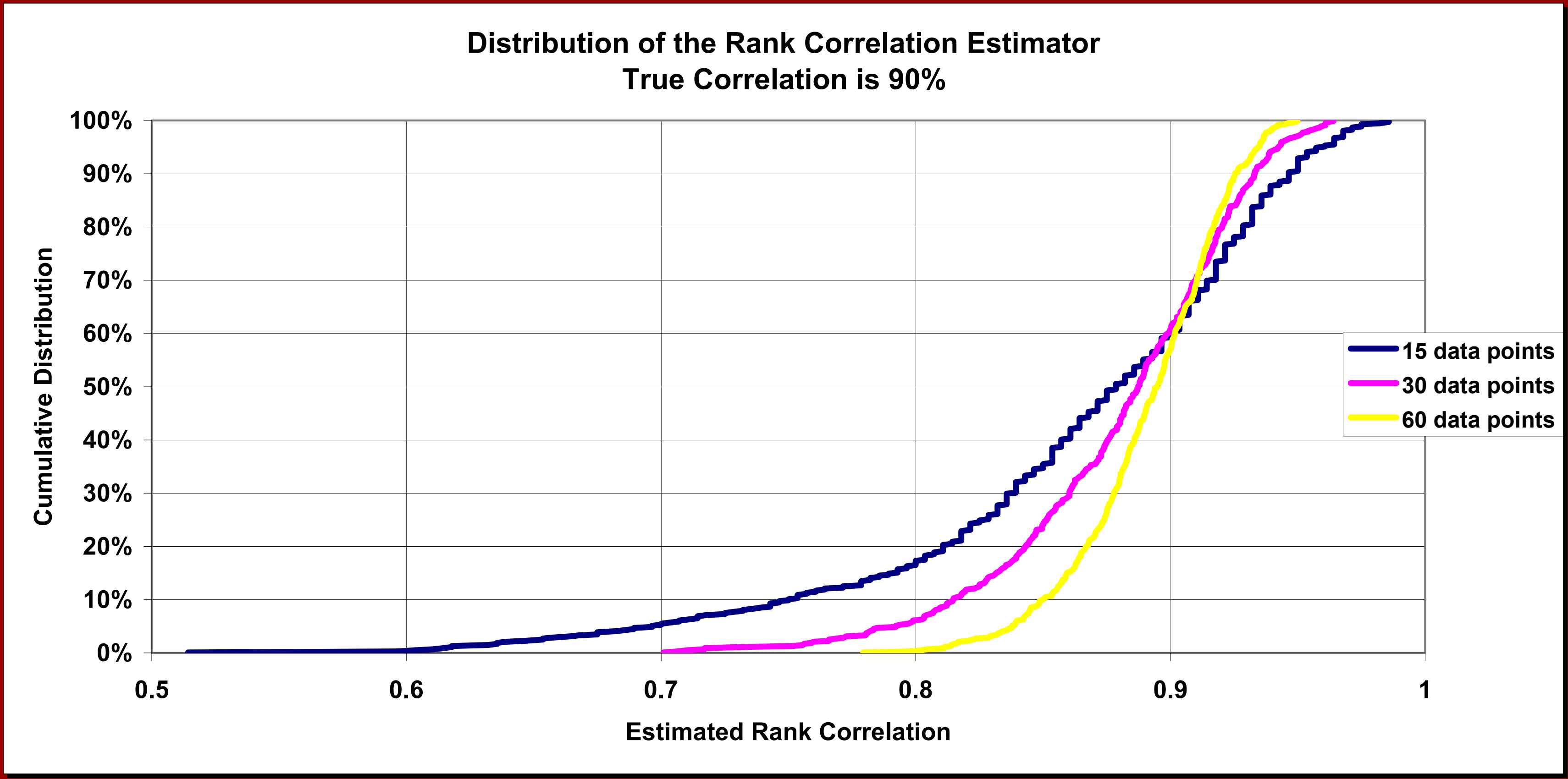
Distribution of Rank Correlation Estimators



Distribution of Rank Correlation Estimators

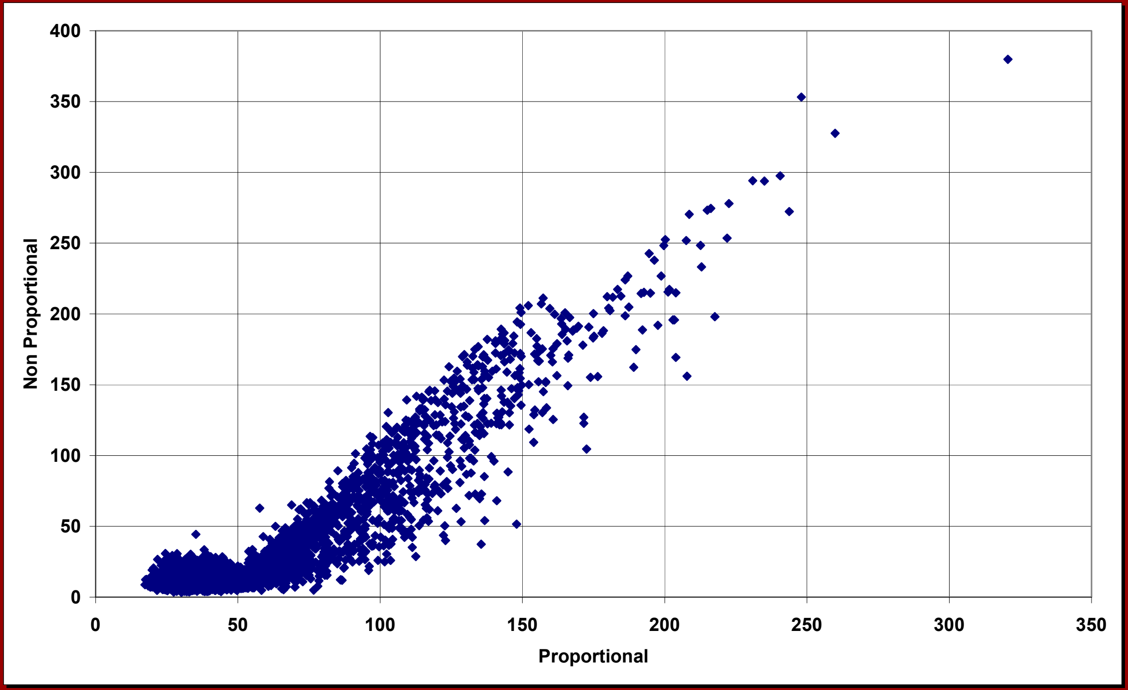


Distribution of Rank Correlation Estimators



Simulated Proportional – Non Proportional Amounts

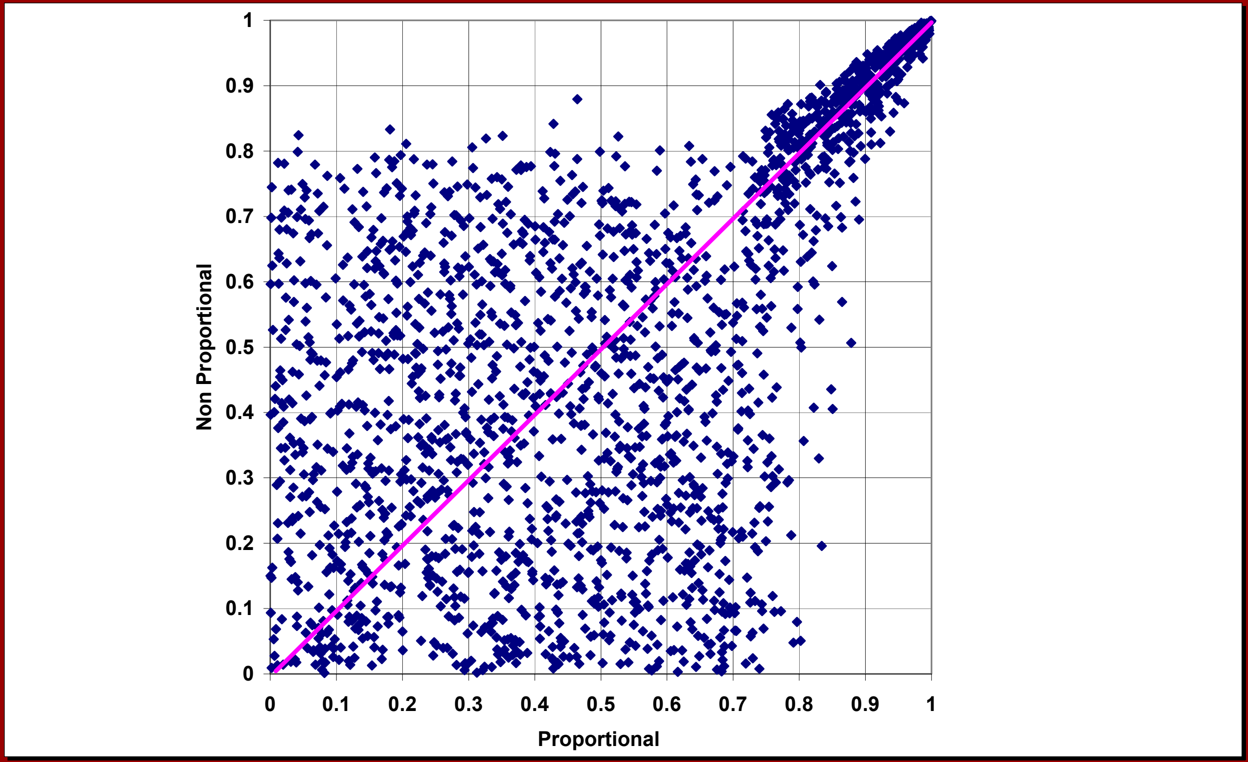
Attritional	
mean	60
stdev	15
Catastrophe	
Poisson lamda	0.3
Pareto	
minimum	50
maximum	250
a	1.1
b	0
Reinsurance	
proportional a	50%
XL	
limit	250
deductible	50
# Simulations	2000



- Pearson Correlation = 92%
- Dependence Structure not very clear

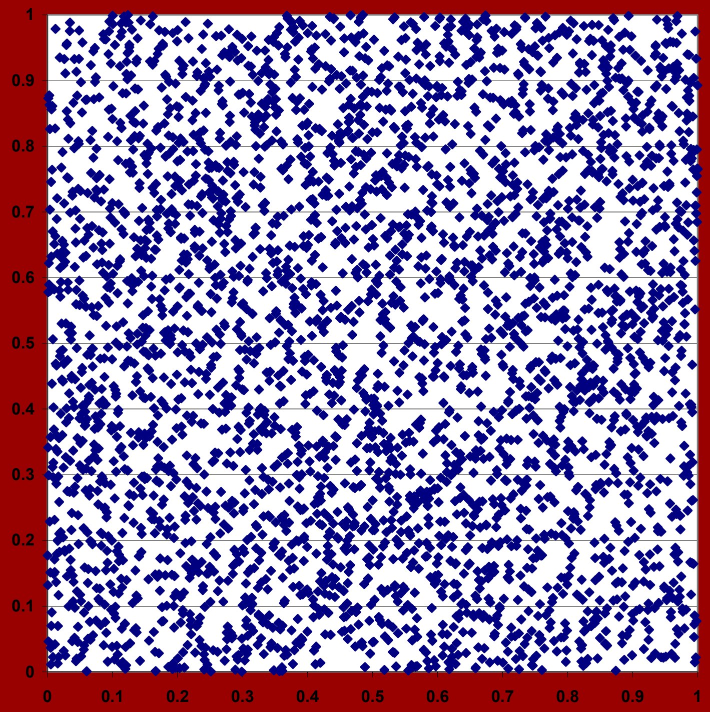
Simulated Proportional – Non Proportional Ranks

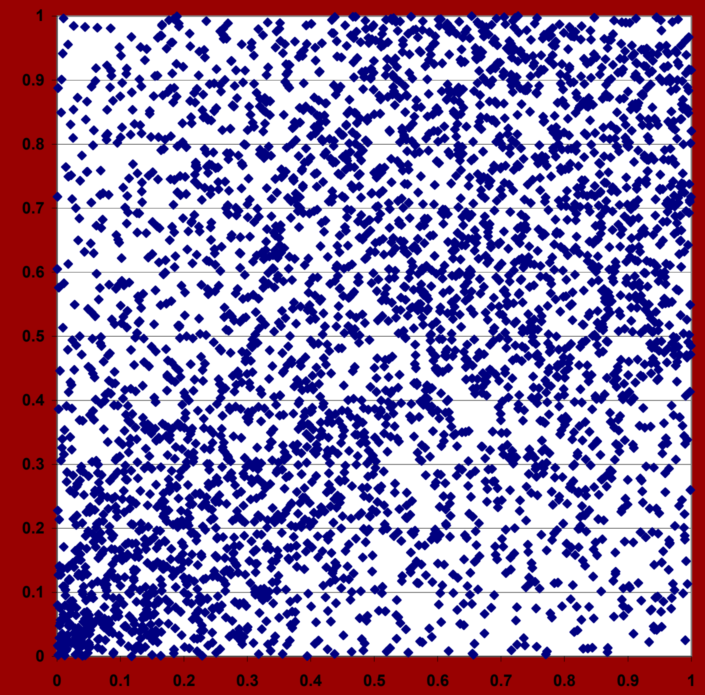
Attritional	
mean	**60**
stdev	**15**
Catastrophe	
Poisson lamda	**0.3**
Pareto	
minimum	**50**
maximum	**250**
a	**1.1**
b	**0**
Reinsurance	
proportional a	**50%**
XL	
limit	**250**
deductible	**50**
# Simulations	**2000**



- Asymmetric Tail
- Rank Correlation = 60%

Effect of Inflation





Independent Data

Rank correlation: 0%

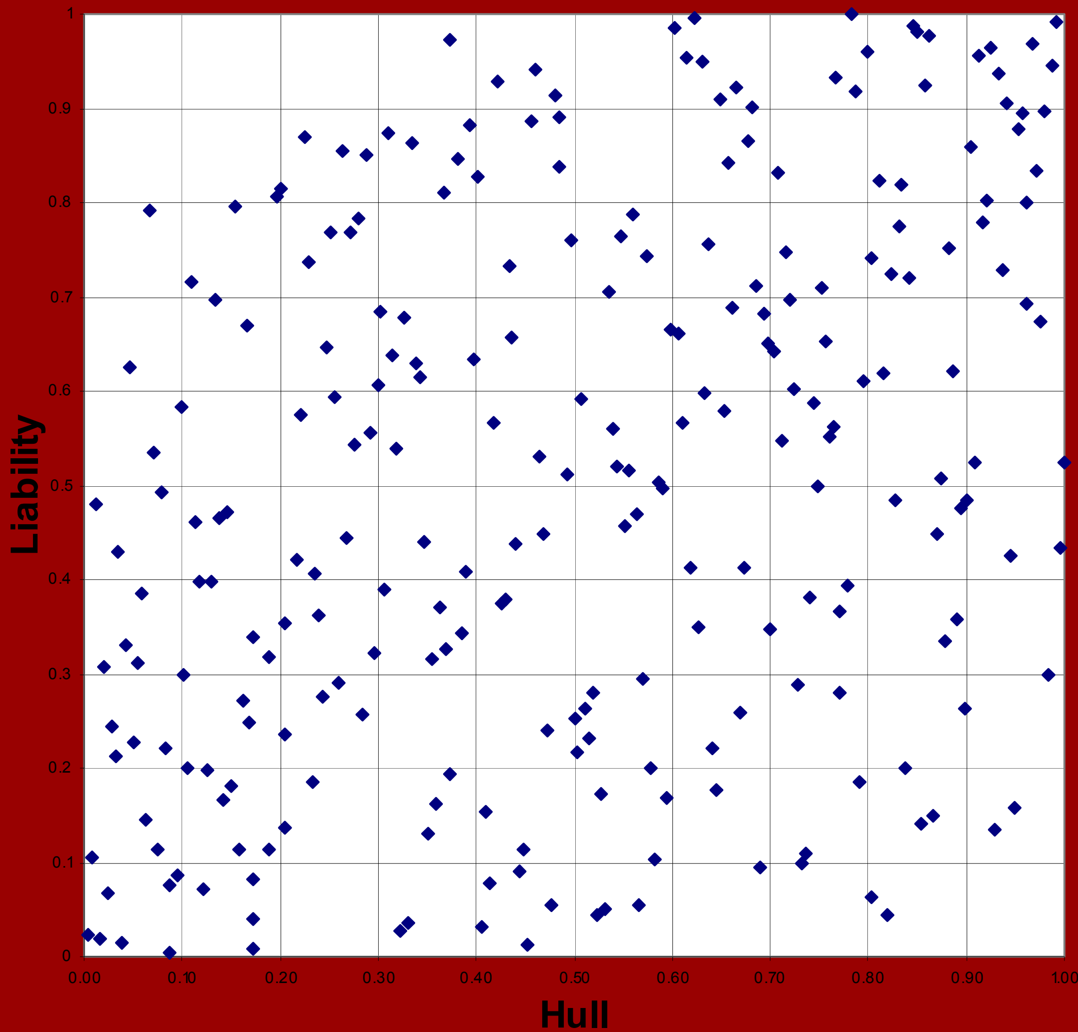
Data on left Graph Inflated at 15%p.a. over 10 years

Rank correlation: 31%

Educated Guesses
Quiz: Aviation Hull and Liability Size of Loss

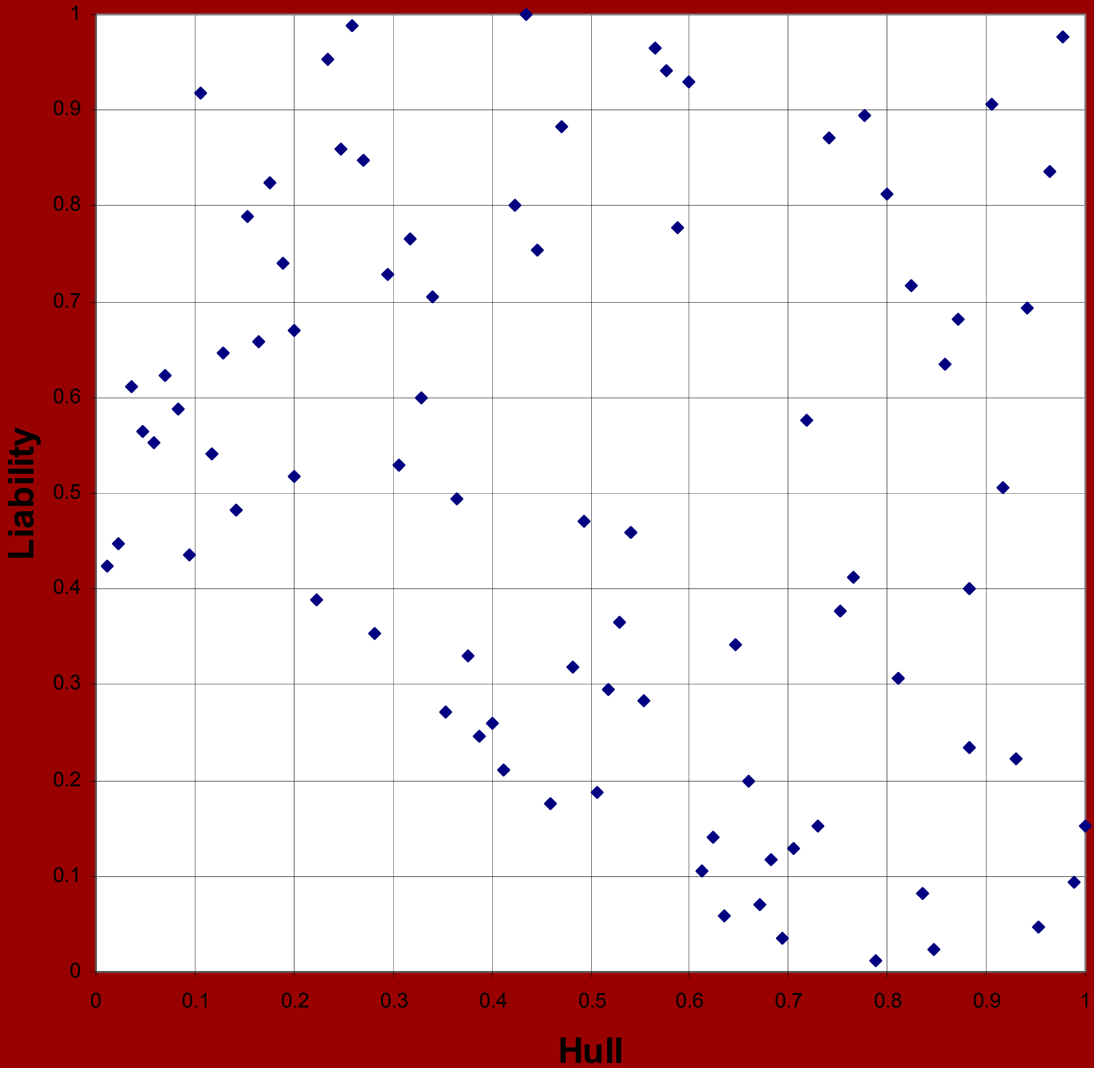
- Highly positively related
- Positively related
- Unrelated
- Negatively Related
- Highly Negatively Related

Empirical Dependence Structure Aviation Losses



Rank correlation 36%

Aviation Losses > 50m
Effect of threshold



Rank correlation -27%

Judge Dependence

	Disease Present	Disease not Present
Symptoms A	**180**	**45**
Lack of Symptoms of A	**36**	**9**

Limiting Theorems

- Sparse data is often an issue in determining tail dependence
- In the univariate case
  - We use the Normal distribution because of the Central Limit Theorem
  - We use Extreme Value distributions because of Extreme Value Theorem
- Copulas ?

Definitions: Conditional Copula

- Copula is the dependence function of (X,Y)
- Copula is the joint distribution of U=F(X), V=F(Y)
- Conditional copula of U and V given U>u and V>v

Definition: Archimedian Copulas

- An Archimedian copula is defined by

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

- φ is a convex decreasing function in [0,1], with $\varphi(1) = 0$
- φ is called the generator of the Archimedian copula
- Examples are Gumbel, Clayton,
- Archimedian copulas have some nice properties

Example of Archimedian Copula: Gumbel

- Gumbel Copula (one of them)

$$C(u,v;\delta) = \exp\left\{-((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta}\right\}, \ 0 \le u, v \le 1$$

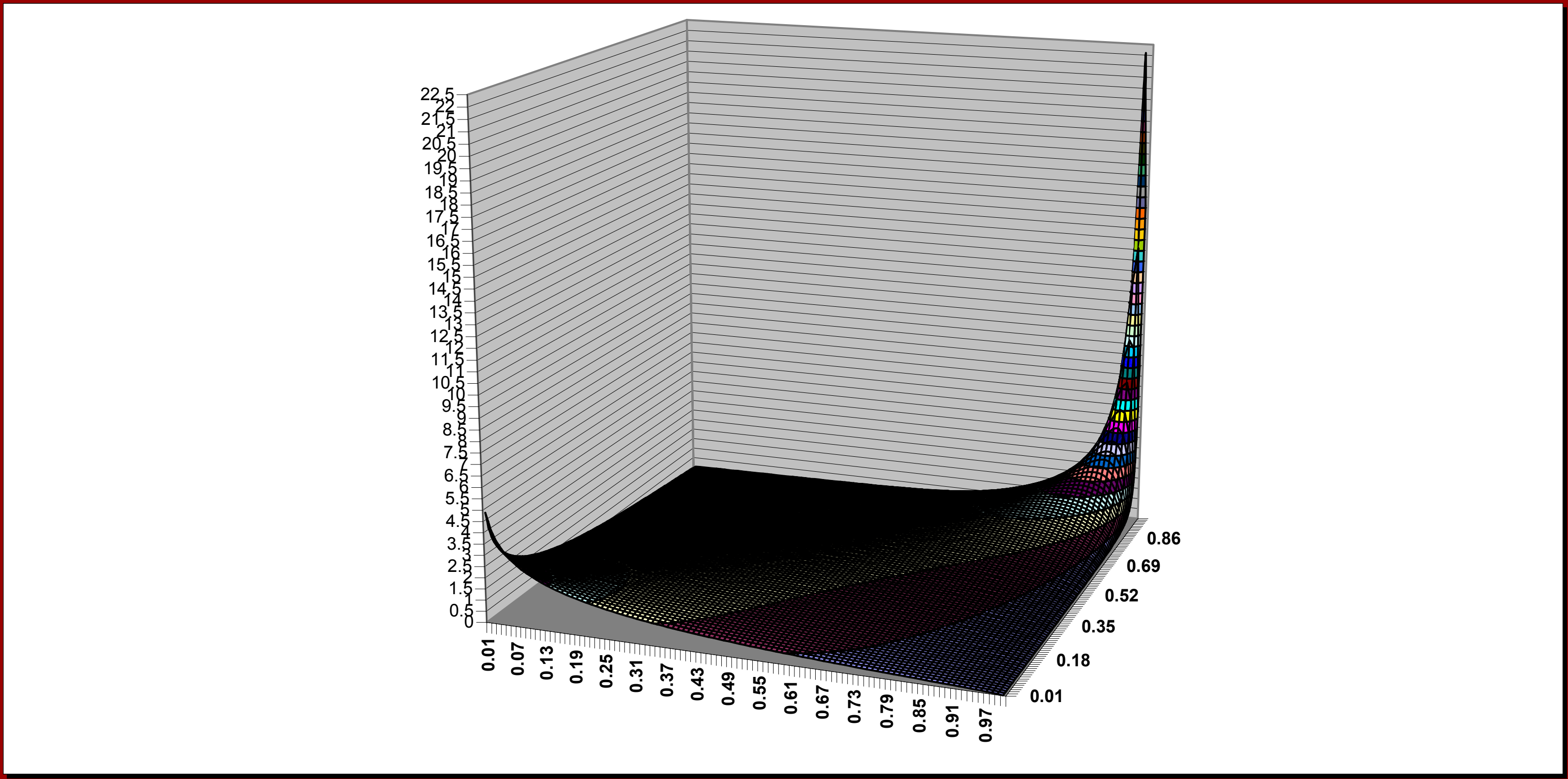
  - The generator function is

$$\varphi(t) = (-\ln t)^{\delta}$$

  - It is just the copula of the bivariate distribution

$$\exp(-(e^{-\delta x} + e^{-\delta y})^{1/\delta})$$

Example of Archimedian Copula Gumbel



Example of Archimedian Copula: Clayton

- Clayton Copula

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \;\; \theta \geq 0$$

  - The generator function is:

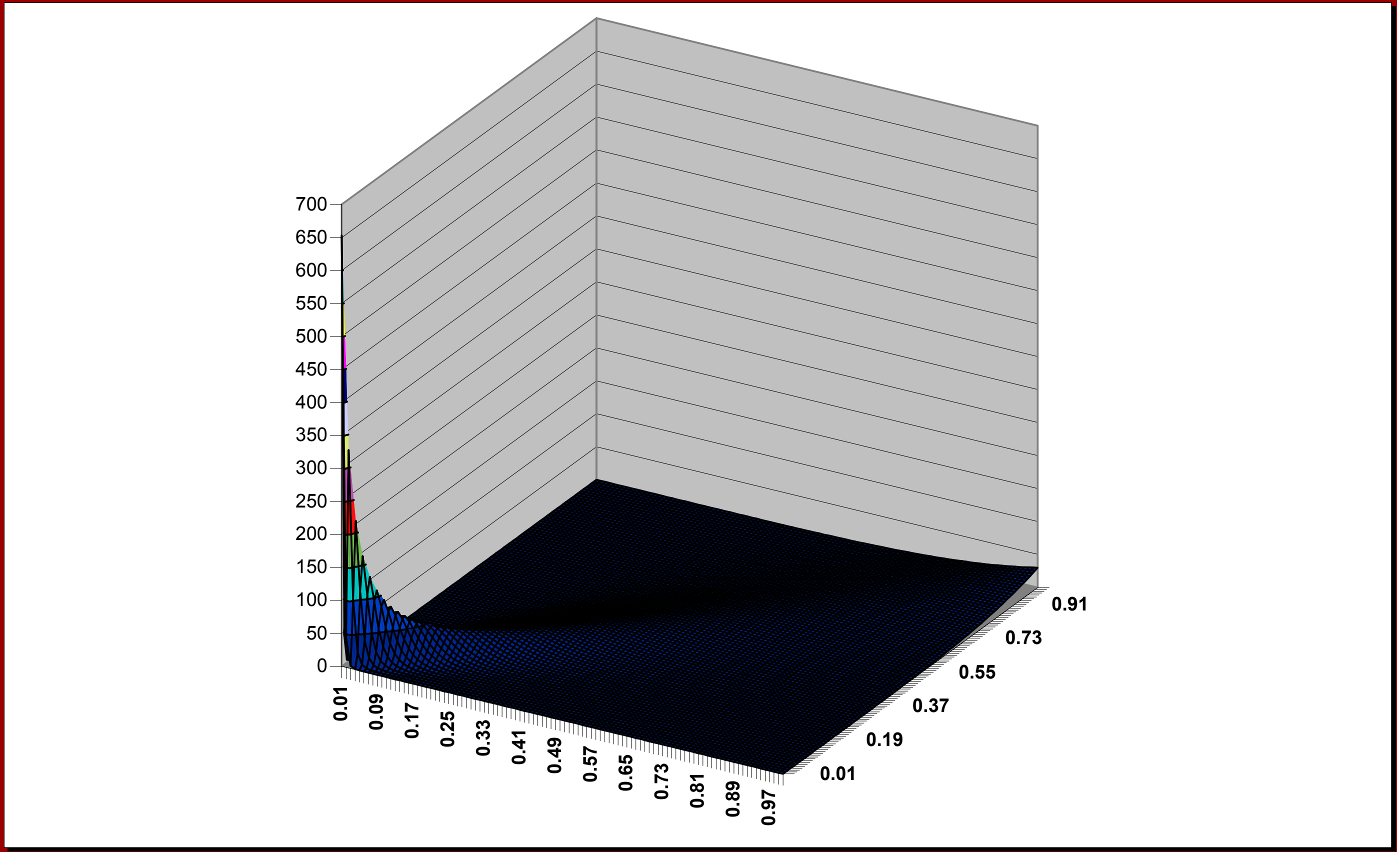
$$\varphi(t) = (t^{-\theta} - 1)$$

- Clayton has heavy lower tail. The survival Clayton copula has heavy upper tail

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

- Both Gumbel and Clayton copulas allow only positive dependence

Example of Archimedian Copula Clayton



Limiting Theorems - Current Research

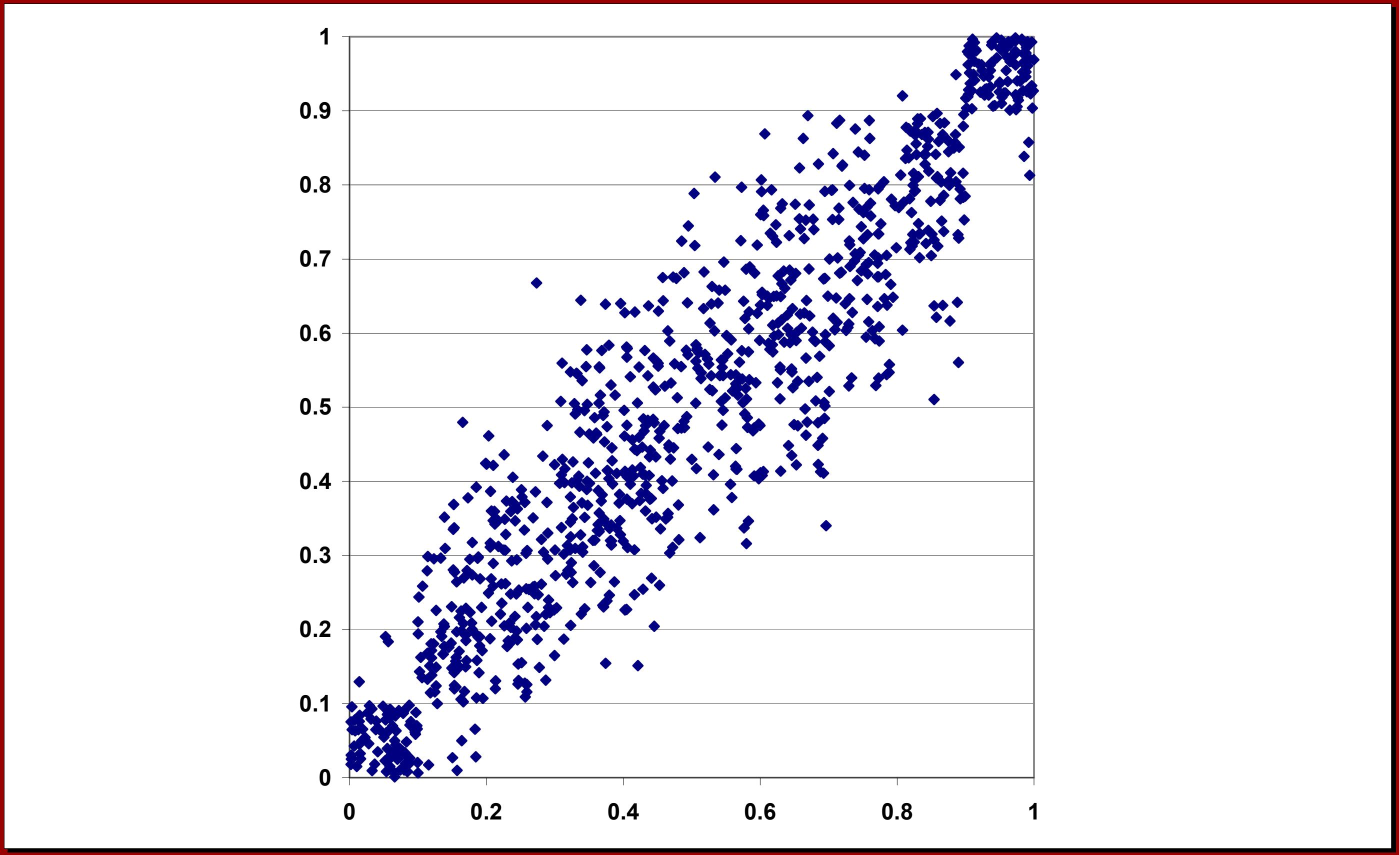
- Research on limiting theorems by Charpentier, Juri and Wuthrich
- The conditional Copula of an Archimedian copula is also an Archimedian copula, but with a different generator
- Clayton is an invariant copula: The conditional copula of a Clayton Copula is also a Clayton copula. Only Clayton copula has this property.
- As u and v tend to 0 the conditional copula of an Archimedian copula tends to a Clayton copula!
- Similar result holds as u and v tend to 1, but for u=v.

Simulation of a copula
General Method

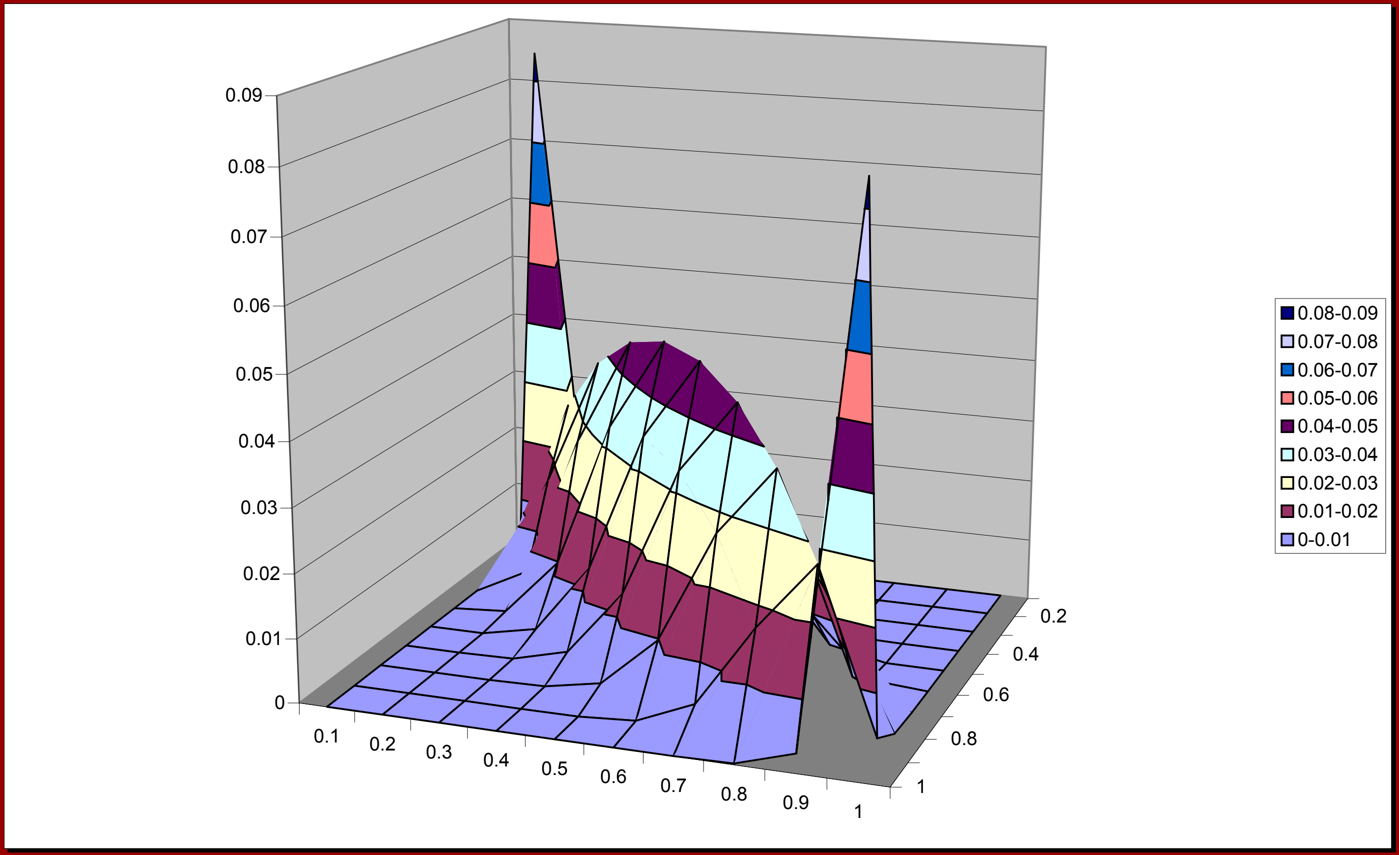
- simulate a value u_1 from U (0,1)
- simulate a value u_2 from $C_2 (u_2 | u_1)$
- simulate a value u_n from $C_n (u_n | u_1 \ldots u_{n-1})$

where $C_i = C(u_1, \ldots, u_i, 1, \ldots, 1)$ for $i = 2, \ldots, n$

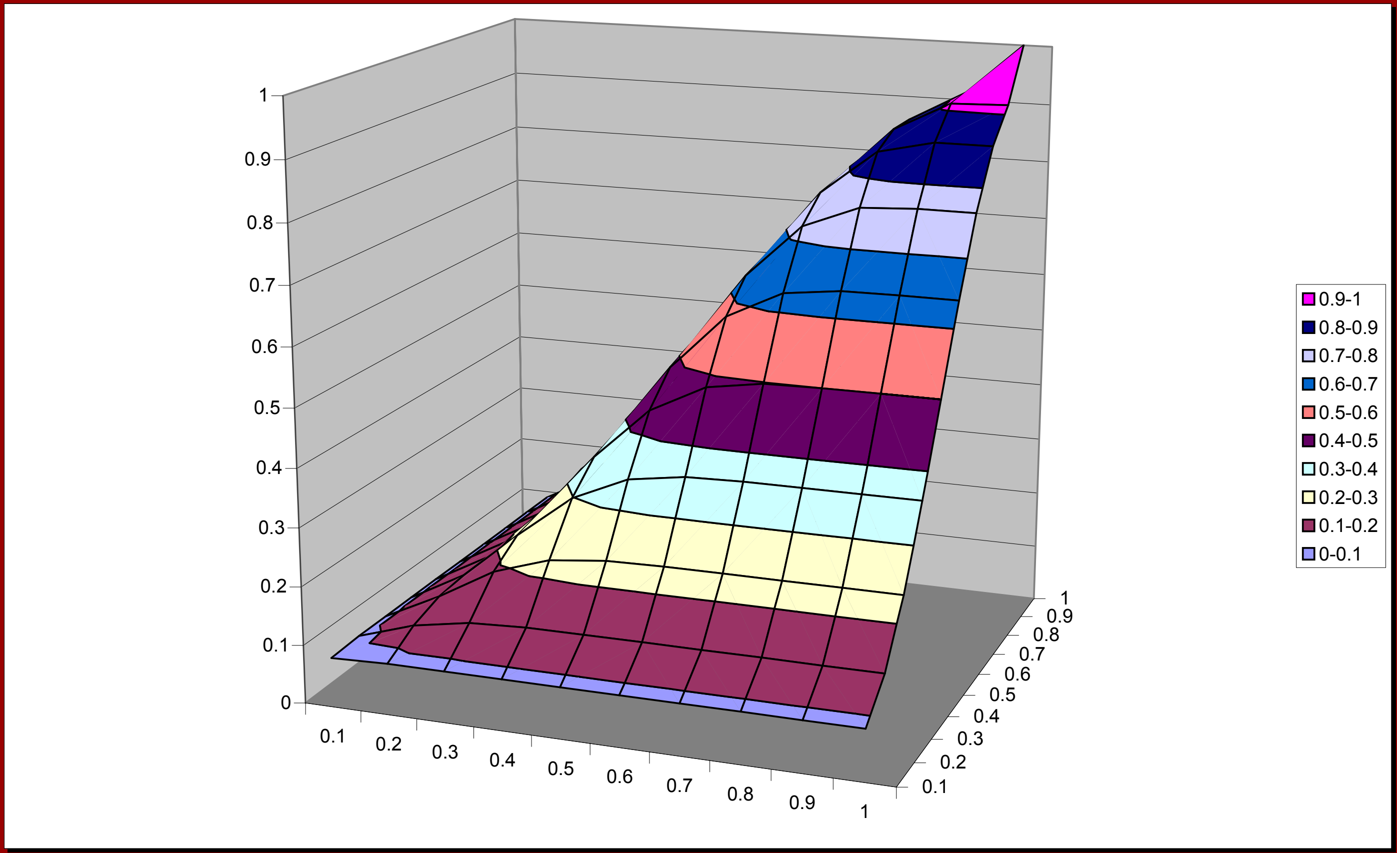
Example Simulation: Empirical Copula



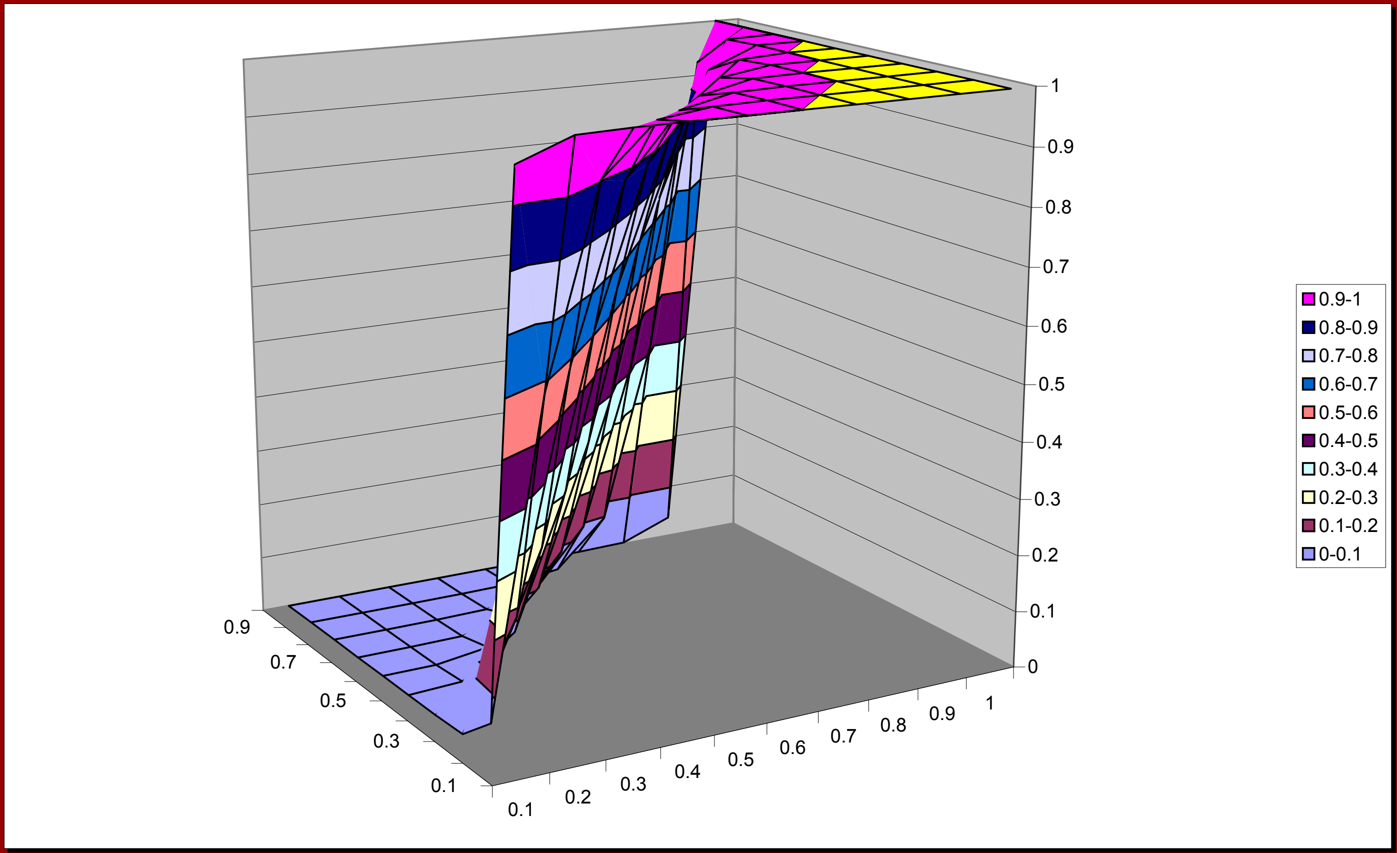
Example Simulation: Customised Density Copula



Example Simulation: Cumulative Copula



Example Simulation: Conditional Copula (partial derivative for continuous case)



Comments on General Method for Simulating Copulas

- If you have many dependent variables the computer time may be prohibitive
- Other methods exist
  - Simulation of univariate distributions can be done in other ways than inverting the cumulative distribution

Spherical / Elliptical Distributions

- Multivariate extensions of the univariate spherical and elliptical distributions
- Spherical Distributions: Density has the form:

$$f(\underset{\sim}{x}) = g(\underset{\sim}{x}^t \underset{\sim}{x})$$

- If $\underset{\sim}{X}$ has a spherical distribution, then $\underset{\sim}{X}$ can be written as

$$\underset{\sim}{X} = R \cdot \underset{\sim}{U}$$

- where $\underset{\sim}{U}$ is a uniform distribution on the hypersphere and R a positive random variable independent of $\underset{\sim}{U}$
- Examples of Elliptical Distributions:
  - For n-variate Normal
  - For n-variate t

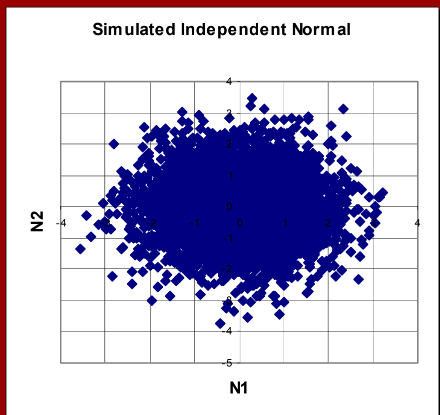
Comments on Elliptical Copulas

- Easy to simulate large number of variables
- Can be simulated by transforming simulated Multivariate Normal data
- Independent variables are multiplied by an appropriate matrix
- Different dependences between any pairs of variables possible
- Overall correlation determined by one parameter (matrix).

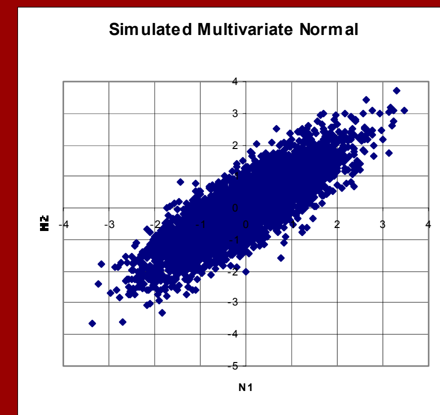
Simulating Multivariate Normal Copula

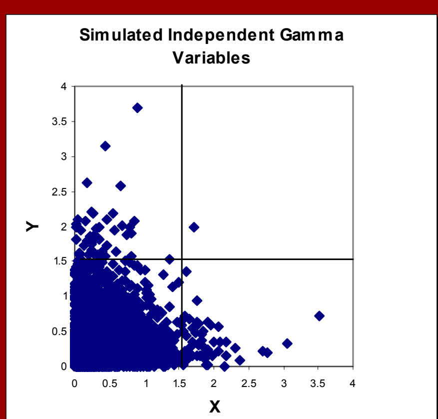
- **Step 1**
- perform Cholesky decomposition of *nxn* correlation matrix ρ
- **Step 2**
- generate *n* series of random normal distributions and put into a matrix
- **Step 3**
- apply Cholesky decomposition to matrix
- **Step 4**
- generate *n* independent series of the appropriate marginal distributions and put into a matrix
- **Step 5**
- re order simulations in step 4 to have the same ranks as in step 3

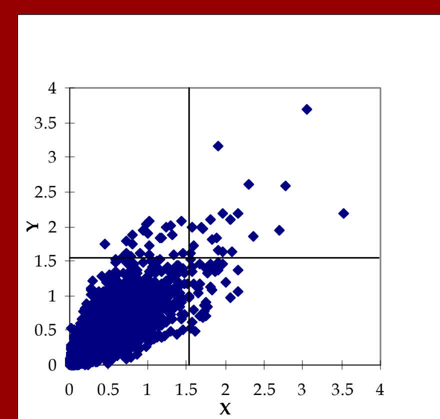
Simulating Multivariate Normal Copula



Choleski







Comments on Normal Copula

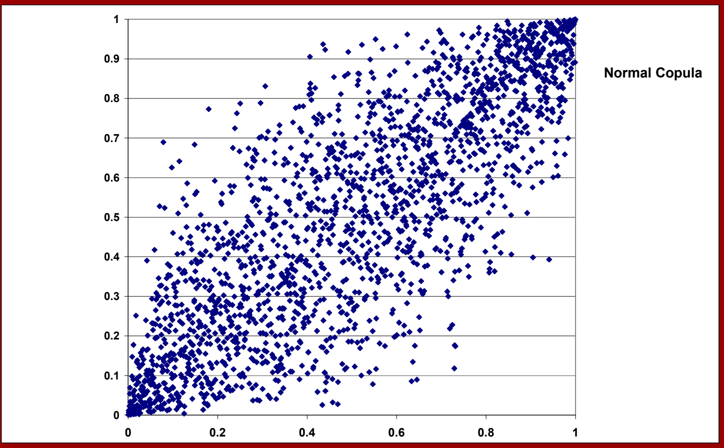
- Dependence structure may not be appropriate
- The tails are asymptotically independent
- For any correlation $-1<r<1$ if you move far enough at the tails the variables behave as independent
- Is this a problem?
  - It depends on the problem

Student T-Copula

- The tails are asymptotically dependent

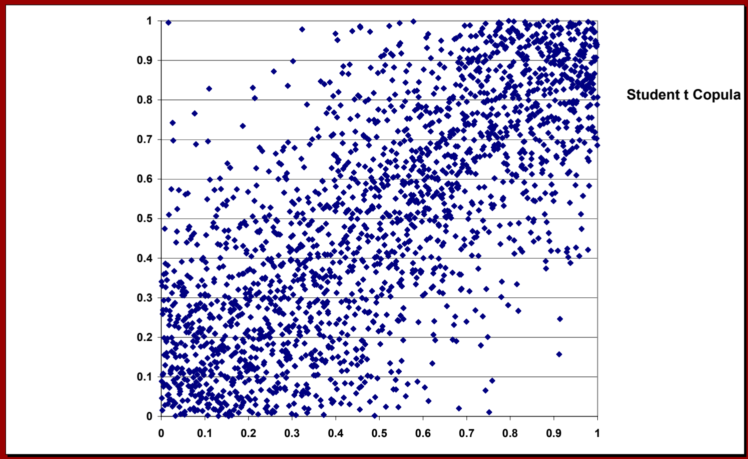
- Easy to simulate: Simulate Multivariate Normal with Correlation table Σ and then multiply each normal variable by a simulated $\sqrt{\nu / \chi^2_\nu}$, where ν is the degrees of freedom

Simulating Student t-copula



Multiply by

$$\sqrt{\nu / \chi^2_\nu}$$



Student T-Copula

- Student T-copula has two parameters. One (Σ) determines the overall correlation and the other (ν) determines the dependence at the tail.

- Kendall's tau and linear correlation rho are related by the following:

$$\rho = \sin(\frac{\pi\tau}{2})$$

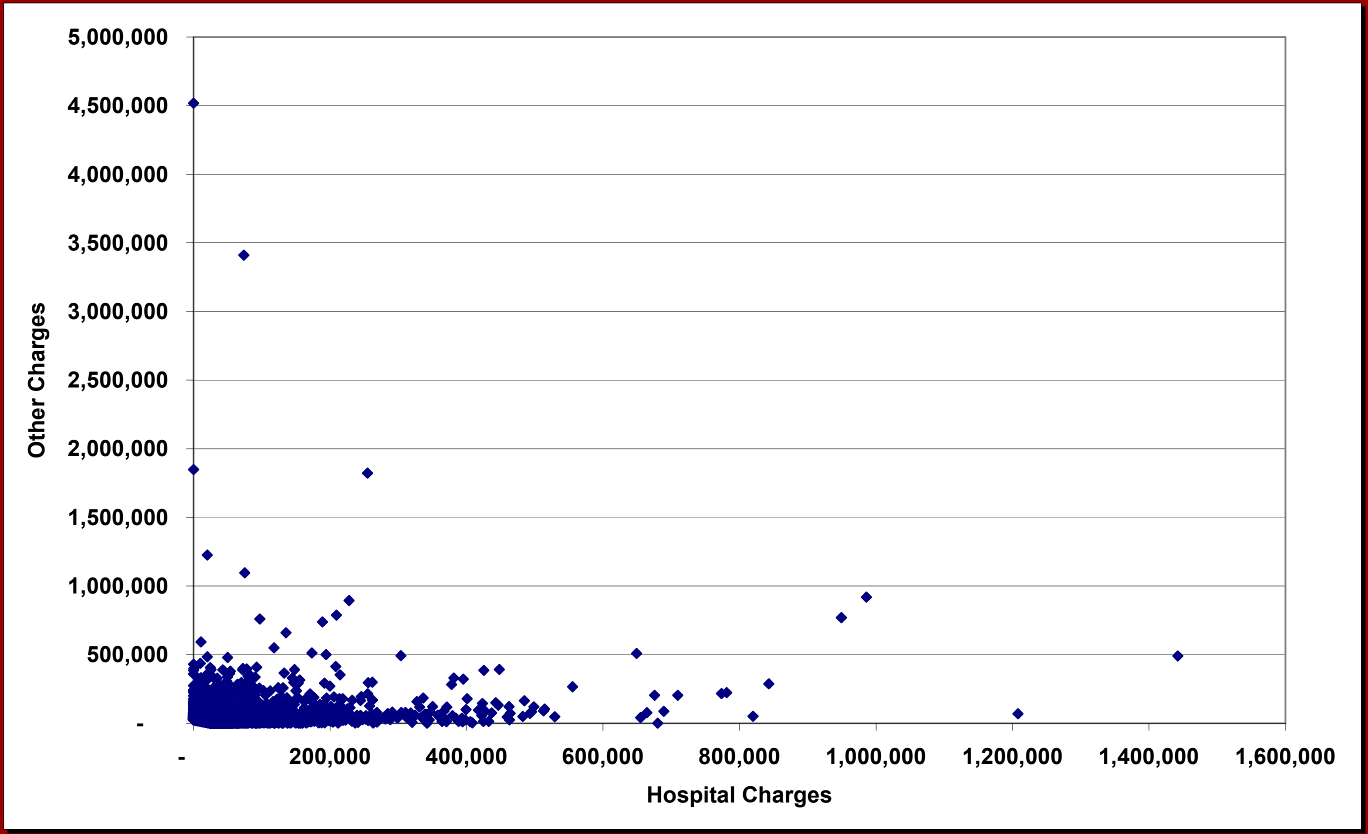
- The smaller the value of ν, the higher the tail dependence

- The relation $\rho = \sin(\frac{\pi\tau}{2})$ holds for all elliptical distributions and can be used for robust estimation of the parameters

Student T-Copula

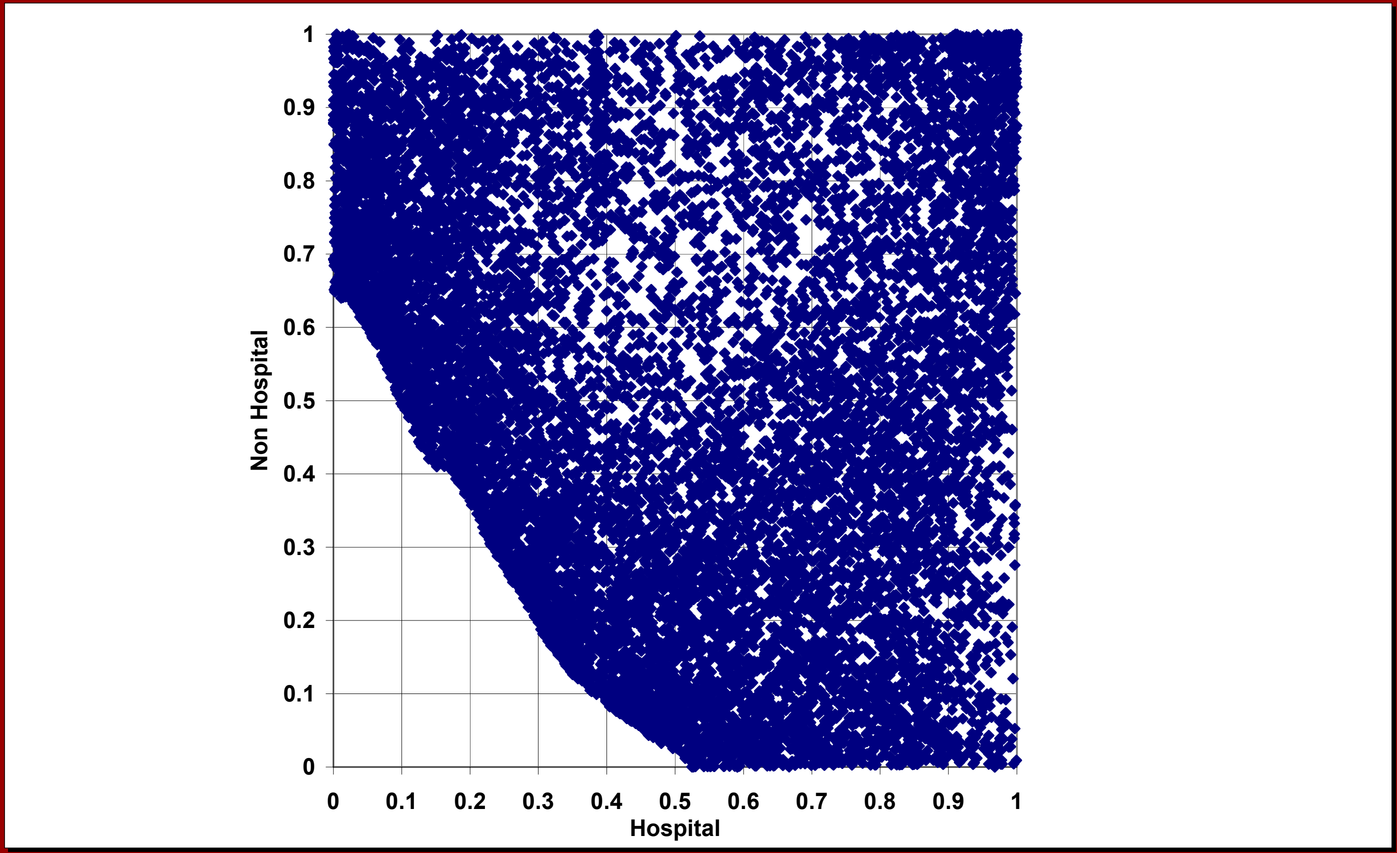
- t-copula is used extensively to model dependences in financial data
- There is evidence that the Clayton copula fits well financial data at the extremes
- Lots of financial data available

Medical Insurance: Hospital v Non Hospital Charges
US 1991 experience Group 2: Losses in excess of $25K



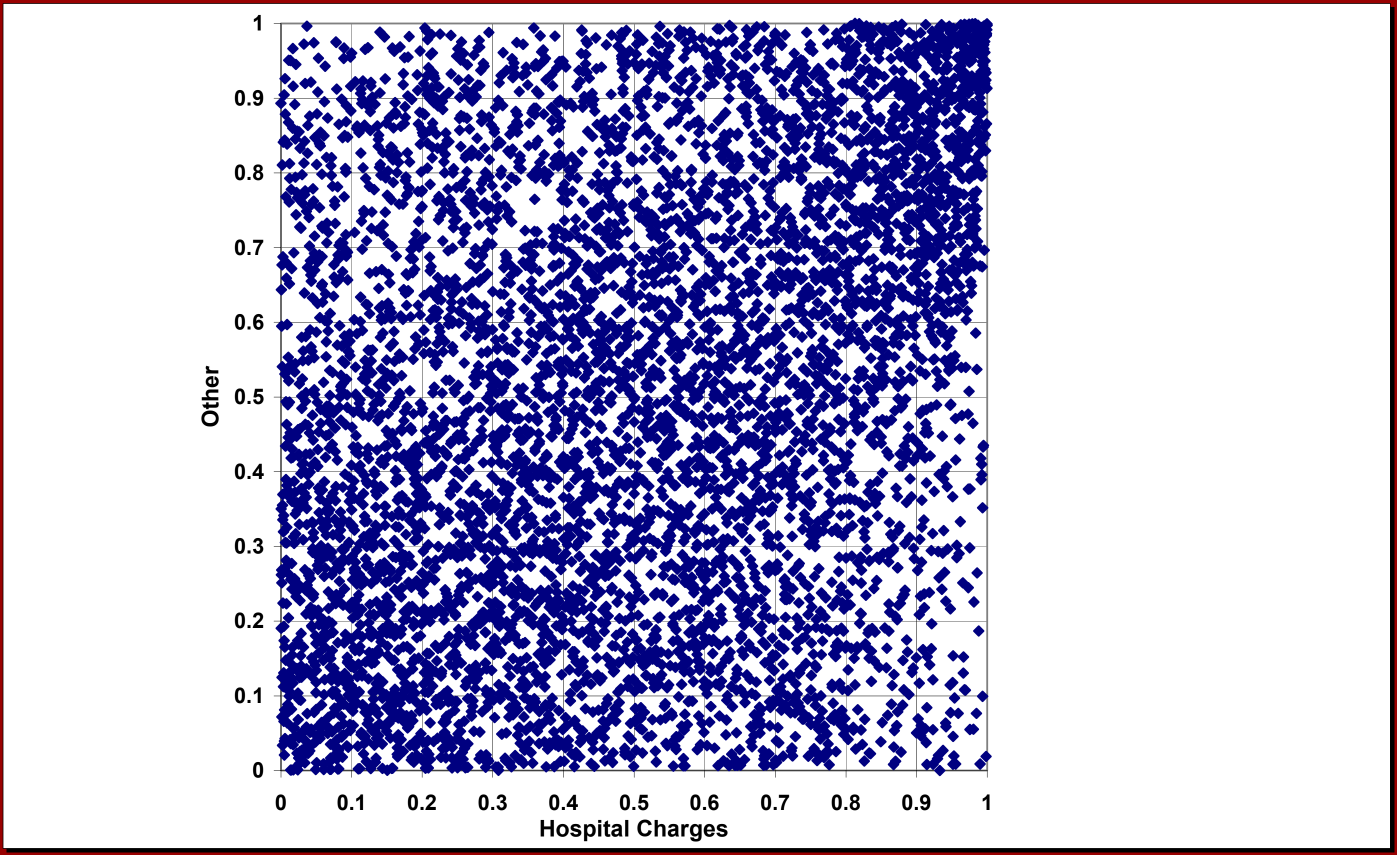
- Pearson Correlation = 16%
- What dependence structure can you see?

Medical Insurance: Hospital v Non Hospital Charges US 1991 experience Group 2: Losses in excess of $25K



- Effect of threshold
- Rank Correlation = -20%
- Tail Dependence

Medical Insurance: Hospital v Non Hospital Charges
Losses in excess of $25K AND Hospital in excess of 25K



- Rank Correlation =30%
- Tail Dependence

Fitting Copula

- Maximum Likelihood Estimates
- Minimising Distance
- Conditional Correlation
- Other Methods Specific to Classes of Copulas, e.g. Archimedian, elliptical

MLE

- Random Vector Y (m dimension) has cdf

$$F(y;a_1,\ldots,a_m,\theta)=C(F_1(y_1;a_1),\ldots,F_m(y_m;a_m);\theta)$$

- a's are the marginal distributions parameters, θ are the dependence structure parameters
- and pdf

$$f(y;a_1,\ldots,a_m,\theta)=c(F_1(y_1;a_1),\ldots,F_m(y_m;a_m);\theta)\cdot\prod_{j=1}^{m}f_j(y_j;a_j)$$

- maximize Log Likelihood for sample size n

$$\sum_{i=1}^{n}\log f(y_i;a_1,\ldots,a_m,\theta)$$

MLE

- In practice it may be easier to fit the marginal distributions first.

- Then given the parameters for the marginal distributions maximise the dependence structure part of the likelihood:

$$\sum_{i=1}^{n} c(F_1(y_{i1};a_1),\ldots,F_m(y_{im};a_m);\theta)$$

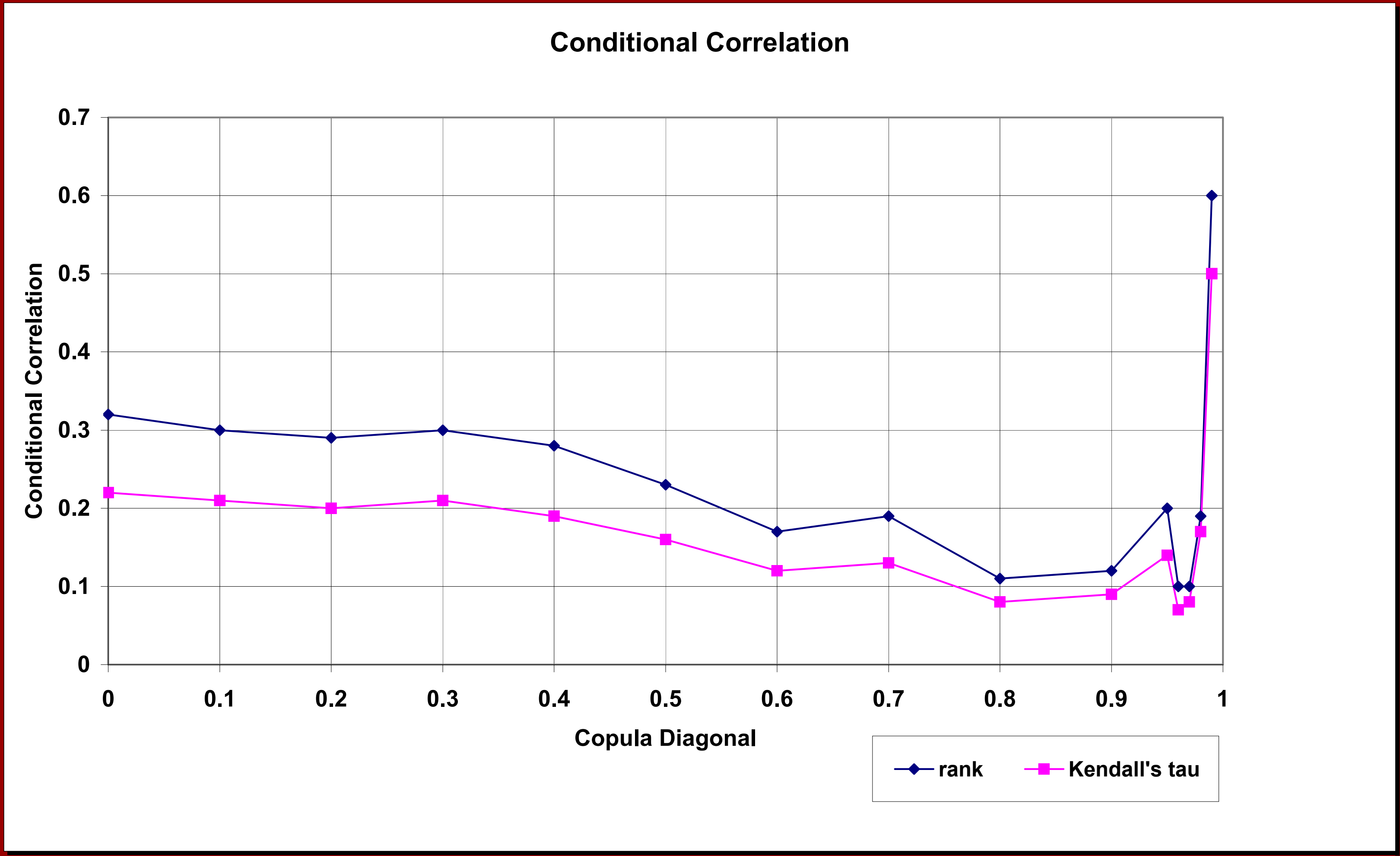
- Answers are not generally the same as for MLE, but they are usually close to those of MLE.

Fitting

- Four Copulas were tested
- By Comparing AIC

1. Clayton
2. Student t
3. Gumbel
4. Bivariate Normal

Data Conditional Correlation



Conditional Correlation

- Student t can have heavy tails even when overall correlation is 0!
- Clayton conditional correlation remains constant
- Gumbel not as heavy as the two above
- Multivariate Normal tail dependence tends to 0.

Conclusion

- Correlation may not be a sufficient measure of dependence. We may need to consider the whole dependence structure

- The method of modelling the dependence structure depends on the purpose of the analysis

- There are many methods of selecting an appropriate dependence structure
  - Historical Data: Lack of data, consider error in estimations
  - Educated Guesses: People are not very good at guessing dependences
  - Limiting Theorems: Clayton as a limiting Archimedian Copula. Research in progress

Conclusion (continued)

- Usual mathematical methods for fitting and testing the fit apply, as well as other methods more specific to certain classes of copulas

- Simulation of copulas can be computationally intense. Elliptical copulas are easier to simulate and have other desirable properties as well

References (1)

- Some of the people who attended the workshop asked for references. Here is a short, non exhaustive list of references:

- Books:
  - Joe H (1997) *Multivariate Models and Dependence Concepts*, Chapman and Hall
  - Nelsen R.B.(1999) *An Introduction to Copulas*, Springer NY

- Background Papers with a lot of explanation as well as mathematics:
  - Embrechts P., McNeill A., Straummann D., *Correlation and Dependence in Risk Management: Properties and Pitfalls*, ETH RiskLab papers
  - Frees E., Valdez E. (1998) *Understanding Relationships Using Copulas*, North American Actuarial Journal

References (2)

- Papers on more specific topics:
  - Klugman SA., Parsa R. (1999), *Fitting Bivariate Loss Distributions with Copulas*, Insurance Mathematics and Economics
  - Juri A., Wurthrich M.V. (2002) *Copula Convergence Theorems for Tail Events*, Insurance: Mathematics and Economics
  - Charpentier A. (2003) *Tail Distribution and Dependence Measures*, ASTIN 2003
  - Venter G. (2003) *Quantifying Correlated Reinsurance Exposures with Copulas*, CAS 2003 Spring Forum
- GIRO Workshops
  - Czernuszewicz A., Papachristou D. (1999) *Correlation and Dependency Structures*, GIRO 1999
  - Smith A. (2002) *Dependent Tails*, GIRO 2002