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Price Bounds for the Swiss Re Mortality Bond 2003

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Quotation

"Nothing is certain in life except death and taxes."

— Benjamin Franklin

Agenda

- Introduction
- Historical Facts
- Design of the Swiss Re Bond
- A Model-independent Approach
- Lower Bound for the Swiss Re Bond
- Upper Bound for the Swiss Re Bond
- Numerical Results
- Conclusions
- What Lies Ahead?
- Further Research

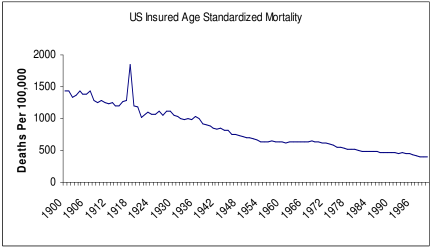
Motivation

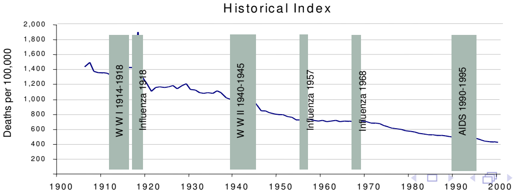
- In the present day world, financial institutions face the risk of unexpected fluctuations in human mortality
- This Risk has two aspects
  - *Mortality Risk*: Actual rates of mortality are in excess of those expected
  - *Longevity Risk*: People outlive their expected lifetimes

Possible Mortality Catastrophes

- Terrorist Attacks
- Wars
- Meteorite Crashes
- Influenza Epidemics
- Infectious diseases

Historical Facts





Historical Facts

Table 2: The change of death rates per 100,000 for each age group, from 1917 to 1919

Age groups	1917	1919	Ratio	Age groups	1917	1919	Ratio
All	1397.1	1810	1.296	35-44	900.8	1339.3	1.487
<1	10457.2	11167.2	1.068	45-54	1385.6	1524.1	1.100
1-4	1066.0	1573.5	1.476	55-64	2678.6	2648.1	0.989
5-14	256.0	412.8	1.613	65-74	5728.4	5505.0	0.961
15-24	468.9	1070.6	2.283	75-84	12386.2	11295.7	0.912
25-34	649.1	1643.5	2.532	>=85	24593.6	22213.5	0.903

- The 1918 influenza pandemic: Increase in mortality rate by 30% overall.
- Most affected age groups: 15-24 and 25-34
- For individuals aged 55 and over a little decrease in the death rate

Available Methodologies

- *Natural Hedging*: compensating longevity risk by mortality risk
  - Drawback: Cost prohibitive
- *Mortality-linked Securities (MLS'S)* or *Catastrophe (CAT) Mortality (CATM) Bonds*: Cash flows linked to a mortality index such that the bonds get triggered by a catastrophic evolution of death rates of a certain population
  - Swiss Re Bond 2003: The first mortality bond
- *Longevity Bonds*: Cash flows linked to a longevity index
  - EIB/BNP Longevity Bond 2004: The first longevity bond

Valuation approaches on MLS's

- *Risk-adjusted process/ No-arbitrage Pricing:*
  - Estimate the distribution of future mortality rates in the real world probability measure
  - Transform the real-world distribution to its risk-neutral counterpart
  - Calculate the price of MLS by discounting the expected payoff under the risk-neutral probability measure at the risk-free rate
- *The Wang Transform:*
  - Employs a distortion operator that transforms the underlying distribution into a risk-adjusted distribution
  - MLS price is the expected value under the risk-adjusted probability discounted by risk-free rate
- *Instantaneous Sharpe Ratio:* Expected return on the MLS equals the risk-free rate plus the Sharp ratio times its standard deviation
- *The utility-based valuation:* Maximisation of the agent's expected utility subject to wealth constraints to obtain the MLS equilibrium

History of Mortality Linked Securities

- Tontines: 17th and 18th century in France
- Annuities in Geneva: Payoffs directly linked to the survival of Genevan "mademoiselles"
- Speculations came to an end during French Revolution
- Detailed overview in [Bauer(2008)]

Recent Developments(1)

Blake & Burrows (2001)	derived the concept of longevity bond
Swiss Re. (2003)	issued the first mortality bond
European investment bank (2004)	issued the first longevity bond
Cowley & Cummins(2005)	show that securitization may increase a firm's value
Lin & Cox (2005)	study and price the mortality bonds and swaps
Cairns, Blake & Dowd(2006)	show how to price mortality-linked financial instruments such as the EIB bond
Blake et al. (2006)	Introduce five types of longevity bonds

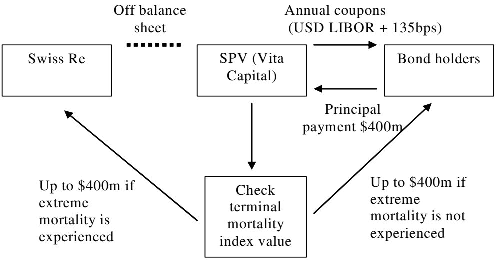
Recent Developments(2)

Pessler (2000)	Criticism of Wang Transform
Chen and Cox (2009)	Modelling mortality with Jumps
Cox et al (2010)	Mortality Risk Modelling
Shang et al (2011)	Recursive Approach to MLS
Cox et al (2013)	Mortality portfolio Risk Management
Lin et al (2013)	Pricing mortality securities with correlated indexes
Huang et al (2014)	Price jumps of MLS in incomplete markets

Prime Focus

- Why Swiss Re Bond...?
- An Innovative Security...one of its kind
- A kind of pioneer and path setter
- Shifted the risk exposure from the balance sheet to the capital markets
- Attracted lot of attention and was fully subscribed (Euroweek, 19 December 2003)
- Investors included a large number of pension funds
- Established a Special Purpose Vehicle (SPV) called VITA I for the securitization
- Strength: Extreme Transparency

The Bond Mechanism



Design of the Swiss Re Bond

- A 3-year bond issued in December 2003 with maturity on Jan 1, 2007
- Principal s.t. mortality risk defined in terms of an index q_i in yr t_i
- Mortality index constructed as a weighted average of mortality rates (deaths per 100,000) over age, sex (male 65%, female 35%) and nationality (US 70%, UK 15%, France 7.5%, Italy 5%, Switzerland 2.5%)

$$\text{Index} = \sum_j C_j \sum_i \left(G^m A_i q^m_{i,j,t} + G^f A_i q^f_{i,j,t} \right)$$

- $q^m_{i,j,t}$ and $q^f_{i,j,t}$ = mortality rates (deaths per 100,000) for males and females respectively in the age group i for country j
- C_j = weight attached to country j
- A_i = weight attributed to age group i (same for males and females)
- G^m and G^f = gender weights applied to males and females respectively
- q_0 = base index

Index Distribution

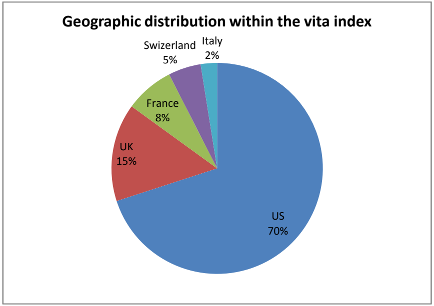


Table showing distribution by age within the VITA index

Age Group	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79
Weight	1%	5%	12.50%	20%	20%	16%	12%	7%	3%	2%	1%	0.50%

Design of the Swiss Re Bond(1)

Principal Loss Percentage

$$L_i = \begin{cases} 0 & \text{if } q_i \leq K_1 q_0 \\ \frac{(q_i - K_1 q_0)}{(K_2 - K_1) q_0} & \text{if } K_1 q_0 < q_i \leq K_2 q_0 \\ 1 & \text{if } q_i > K_2 q_0 \end{cases} \quad (1)$$

- For Swiss Re Bond $K_1 = 1.3$ $K_2 = 1.5$
- Proportion of the principal returned to the bondholders on the maturity date:

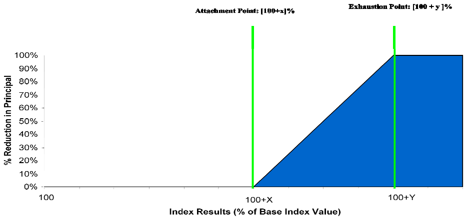
$$X = C \left(1 - \sum_{i=1}^{3} L_i\right)^+, \quad (2)$$

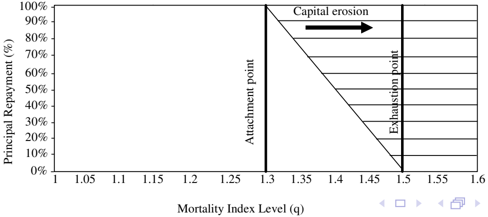
- $C = \$400$ million
- Risk-neutral price of the pay-off at time 0:

$$P = e^{-rT} \mathrm{E}_Q[X] \quad (3)$$

- Q is the EMM

Design of the Swiss Re Bond(2)





Our Approach for Bond Evaluation

- Adapt the payoff of the bond in terms of the payoff of an Asian put option
- Assume the existence of an Equivalent Martingale Measure (EMM)
- Present model-independent bounds
- Exploit comonotonic theory as illustrated in [Albrecher et al.(2008)Albrecher, Mayer, and Schoutens] for the pricing of Asian options
- Carry out Monte Carlo simulations to estimate the bond price under Black-Scholes Model
- Draw graphs of the bounds by varying the interest rate r and mortality rate q_0

Payoff as that of an Asian Put Option

Alternative form of writing Payoff

$$P = De^{-rT}\mathrm{E}\left[(q_0 - S)^+\right] \tag{4}$$

- $D = \frac{C}{q_0}$
- $S_i = 5\left(q_i - 1.3q_0\right)^+$
- $S = \sum_{i=1}^{3} S_i$

Call counterpart of the payoff

$$P_1 = De^{-rT}\mathrm{E}\left[(S - q_0)^+\right] \tag{5}$$

Put-call parity for the Swiss Re Bond

The relation

$$P_1 - P = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \tag{6}$$

- Define

$$G = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \tag{7}$$

- Bounding P_1 by bounds l_1 and u_1
- Corresponding bounds for the Swiss Re Mortality Bond:

$$l_1 - G \leq P \leq u_1 - G \tag{8}$$

Some Basic Concepts

Definition

Stop-loss Premium: The stop-loss premium with retention d of a random variable X is defined as $\mathbf{E}\left[(X-d)^{+}\right]$.

Definition

Stop-loss Order: Consider two random variables X and Y. Then X is said to precede Y in the stop-loss order sense, written as $X \leq_{sl} Y$, if and only if X has lower stop-loss premiums than Y:

$$\mathbf{E}\left[(X-d)^{+}\right] \leq \mathbf{E}\left[(Y-d)^{+}\right] \qquad -\infty < d < \infty \tag{9}$$

Definition

Convex Order: X is said to precede Y in terms of convex order, written as $X \leq_{cx} Y$, if and only if $X \leq_{sl} Y$ and $\mathbf{E}[X] = \mathbf{E}[Y]$.

Lower Bound for the Call Counterpart

Lower Bound using Jensen's Inequality

$$P_1 \geq De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - q_0\right)^+\right] \tag{10}$$

- We define: $Z_i = 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+; i = 1, 2, ..., n$ & $Z = \sum_{i=1}^{n} Z_i$
- $S \geq_{sl} Z$ or $\mathrm{E}\left[(S - q_0)^+\right] \geq \mathrm{E}\left[(Z - q_0)^+\right]$
- The conditioning variable Λ is chosen in such a way that $\mathbf{E}\left[q_i|\Lambda\right]$ is either increasing or decreasing for every i
- This implies the vector: $\mathbf{Z}^{\mathbf{l}} = (Z_1, \ldots, Z_n)$ is comonotonic & yields

Stop-loss lower bound for the call-counterpart

$$P_1 \geq De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^+\right] \tag{11}$$

The Trivial Lower Bound

- if the random variable Λ is independent of the mortality evolution $\{q_t\}_{t\geq 0}$ we get

The Trivial Lower Bound

$$P_1 \geq Ce^{-rT}\left(\sum_{i=1}^{n} 5\left(\exp\left(rt_i\right) - 1.3\right)^{+} - 1\right)^{+} =: \ \mathrm{lb}_0 \qquad (12)$$

- Using

$$G = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \qquad (13)$$

- Corresponding bound for the Swiss Re Mortality Bond:

$$P \geq \ \mathrm{lb}_0 - G =: \ \mathrm{LB}_0 \qquad (14)$$

The Lower Bound LB_1

- We choose $\Lambda = q_1$ in (11)
- Use the martingale argument for the discounted mortality process

$$\mathbf{E}\left[q_i | q_1\right] = \mathbf{E}\left[e^{rt_i} e^{-rt_i} q_i | q_1\right] = e^{r(t_i - t_1)} q_1 .$$

The Lower Bound LB_1

$$P_1 \geq 5D \sum_{i=1}^{n} e^{-r(T-t_i)} C\left(q_0 \left(\frac{1.3}{e^{r(t_i-t_1)}} + \left(x - \frac{1.3}{e^{r(t_i-t_1)}}\right)^+\right), t_1\right) =: \text{lb}_1 \tag{15}$$

- where x is the solution of $\sum_{i=1}^{n} \left(e^{r(t_j - t_1)} x - 1.3\right)^+ = 0.2$
- $C(K, t_1)$ is the price of a European call on the mortality index with strike K, maturity t_1 and current mortality index q_0

The Lower Bound $\text{LB}_t^{(1)}$

- Further improvement using additional assumptions
- The following inequality holds for every random variable Y and every constant c

$$\Rightarrow \mathbf{E}\left[a^+\right] \geq \mathbf{E}\left[a\mathbb{1}_{\{Y \geq c\}}\right] \tag{16}$$

- Utilizing the above inequality twice
- and further assume: q_i and $\mathbb{1}_{\{q_t \geq c\}}$ are non-negatively correlated for $t > t_i$

The Lower Bound $\text{LB}_t^{(1)}$

$$P_1 \geq 5De^{-rT} \max_{0 \leq t \leq T} C\left(\tilde{c}_t, t\right) \sum_{i=j}^{n} e^{rt_i} =: \text{lb}_t^{(1)} \tag{17}$$

- where $j = \min\{i \ : \ t_i \geq t\}$ and

$$\tilde{c}_t = q_0 \left(\frac{(0.2 + 1.3n) - \sum_{i=1}^{j-1} e^{rt_i}}{\sum_{i=j}^{n} e^{r(t_i - t)}} \right) \tag{18}$$

A Model-independent Lower Bound(1)

- Additional assumption that holds good for stationary exponential Lèvy models

$$\sum_{i=1}^{n} q_i \geq_{sl} \left(\sum_{i=1}^{j-1} q_0^{(1-t_i/t)} q_t^{t_i/t} + \sum_{i=j}^{n} e^{r(t_i-t)} q_t \right) \tag{19}$$

- for $0 \leq t \leq T$ and $j = \min\{i \,:\, t_i \geq t\}$
- We then use the following two results

Proposition

Let $(X,\ Y) \sim BVN\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, where BVN stands for bivariate normal distribution. The conditional distribution function of X, given the event $Y = y$, is given as

$$F_{X|Y=y}(x) = \Phi\left(\frac{x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}} \right) \tag{20}$$

A Model-independent Lower Bound(2)

Proposition

Let $W = (W_t)\,, t \geq 0$ be a standard Brownian motion. Then the conditional expectation of W_{t_i} given W_t is given as

$$\boldsymbol{E}\left[W_{t_i} | W_t\right] = \frac{t_i}{t} W_t \quad \text{for any } t_i < t$$

- The above proposition then leads to the following proposition

Proposition

The additional assumption (19) holds for stationary exponential Lèvy models with mortality evolution $q_t = q_0 \exp\left(U_t\right)$, where $\left(U_t\right)_{t \geq 0}$ is a Lèvy process

A Model-independent Lower Bound(3)

- We use this result to achieve the lower bound for the Asian-type call option

$$\begin{aligned}\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|q_t\right) - 1.3q_0\right)^{+} &= \sum_{i=1}^{j-1} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^{+} \\ &\quad + \sum_{i=j}^{n} 5q_0\left(\frac{q_t}{q_0}e^{r(t_i-t)} - 1.3\right)^{+} \\ &=: S^{l_2}. \qquad (21)\end{aligned}$$

- S^{l_2} is the same as Z with Λ being replaced by q_t
- So we have $S \geq_{sl} S^{l_2}$

A Model-independent Lower Bound(4)

- Define $\mathbf{Y} = (Y_1, \ldots, Y_n)$ with

$$Y_i = \begin{cases} 5q_0 \left(\left(\frac{q_t}{q_0} \right)^{t_i/t} - 1.3 \right)^+ & i < j \\ 5q_0 \left(\left(\frac{q_t}{q_0} \right) e^{r(t_i - t)} - 1.3 \right)^+ & i \geq j \end{cases}$$

- $i = 1, 2, ..., n$
- $\mathbf{Y}$ is comonotonic:-components are strictly increasing functions of q_t
- By the comonotonic theory

$$\mathbf{E}\left[\left(S^{l_2} - q_0 \right)^+ \right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_2}}\left(q_0 \right) \right) \right)^+ \right] \tag{22}$$

- where $F_{S^{l_2}}(q_0)$ is the distribution function of S^{l_2} evaluated at q_0

- such that for an arbitrary t, we have:

$$F_{S^{l_2}}(q_0) = \mathbf{P}\left[S^{l_2} \leq q_0\right]$$
$$= \mathbf{P}\left(\sum_{i=1}^{j-1}\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^+ + \sum_{i=j}^{n}\left(\left(\frac{q_t}{q_0}\right)e^{r(t_i-t)} - 1.3\right)^+ \leq 0.2\right) \quad (23)$$

- Substitute x for q_t/q_0 in (23)
- where x solves

$$\sum_{i=1}^{j-1}\left(x^{t_i/t} - 1.3\right)^+ + \sum_{i=j}^{n}\left(xe^{r(t_i-t)} - 1.3\right)^+ = 0.2 \quad (24)$$

- Then $S^{l_2} \leq q_0$ if and only if $q_t \leq xq_0$

A Model-independent Lower Bound(6)

- This yields

$$F_{S^{l_2}}(q_0) = F_{q_t}(xq_0) = \begin{cases} F_{Y_i}\left(5q_0\left(x^{t_i/t} - 1.3\right)^+\right) & i < j \\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)} - 1.3\right)^+\right) & i \geq j \end{cases}$$

The Lower Bound $\mathrm{lb}_t^{(2)}$

$$\begin{aligned} P_1 \quad &\geq \quad 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0^{1-t_i/t}\mathbf{E}\left[\left(q_t^{t_i/t} - q_0^{t_i/t}\left(1.3 + \left(x^{t_i/t} - 1.3\right)^+\right)\right)^+\right]\right. \\ &\qquad \left. + \sum_{i=j}^{n} e^{rt_i} C\left(q_0\left(\frac{1.3}{e^{r(t_i-t)}} + \left(x - \frac{1.3}{e^{r(t_i-t)}}\right)^+\right), t\right)\right) \\ &=: \quad \mathrm{lb}_t^{(2)} \end{aligned} \qquad (25)$$

A Model-independent Lower Bound(7)

- $\mathrm{lb}_t^{(2)}$ is a lower bound for all t and can be maximized w.r.t. t to yield the optimal lower bound:

$$P_1 \geq \max_{0 \leq t \leq T} \mathrm{lb}_t^{(2)} \tag{26}$$

- As before, we have on using the put-call parity

$$P \geq \mathrm{lb}_t^{(2)} - G =: \mathrm{LB}_t^{(2)} \tag{27}$$

A Lower Bound under Black-Scholes Model(1)

- Assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ follows the Black-Scholes model written as $q_t = e^{U_t}$
- where

$$U_t = \log_e(q_0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^* \tag{28}$$

and $\{W_t^*\}_{t\geq 0}$ denotes a standard Brownian motion

- $$U_t \sim N\left(\log_e q_0 + \left(r - \frac{\sigma^2}{2}\right)t,\ \sigma^2 t\right) \tag{29}$$

Proposition

If $(X,\ Y) \sim BVN\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, the conditional distribution of the lognormal random variable e^X, given the event $e^Y = y$ is

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}\left(\log_e y - \mu_Y\right)\right)}{\sigma_X\sqrt{1-\rho^2}}\right) \tag{30}$$

A Lower Bound under Black-Scholes Model(2)

- Given the time points t_i, t for each i
- let ρ be the correlation between U_{t_i} and U_t
- Then, $(U_{t_i}, U_t) \sim \text{BVN}\left(\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}, \sigma^2_{U_t}, \rho\right)$
- where $\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}$ and $\sigma^2_{U_t}$ are given by (46)
- Now $q_t = e^{U_t}$
- The distribution function of q_i conditional on the event $q_t = s_t$ is given as

$$F_{q_i|q_t=s_t}(x) = \Phi(a(x))$$

- where $a(x)$ is given by

$$a(x) = \frac{\log_e x - \left(\log\left(q_0\left(\frac{s_t}{q_0}\right)^{\rho\sqrt{\frac{t_i}{t}}}\right) + \left(r - \frac{\sigma^2}{2}\right)\left(t_i - \rho\sqrt{t_i t}\right)\right)}{\sigma\sqrt{t_i\left(1-\rho^2\right)}}. \tag{31}$$

A Lower Bound under Black-Scholes Model(3)

- For the mortality evolution process $\{q_t\}_{t\geq 0}$ defined as $q_t = e^{U_t}$

$$\mathbf{E}\left(q_i|q_t\right) = \begin{cases} q_0 \left(\frac{q_t}{q_0}\right)^{\frac{t_i}{t}} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} & t_i < t, \\ q_t e^{r(t_i-t)} & t_i \geq t. \end{cases} \tag{32}$$

- Use this result to achieve the lower bound for the Asian-type call option
-
- Define $\mathbf{Y} = (Y_1, \ldots, Y_n)$
- where

$$Y_i = \begin{cases} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ & i < j \\ 5q_0\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ & i \geq j \end{cases}$$

- $i = 1, 2, ..., n$
- $\mathbf{Y}$ is comonotonic

A Lower Bound under Black-Scholes Model(4)

- Define $S^{l_3} = \sum_{i=1}^{n} Y_i$
- By the comonotonic theory

$$\mathbf{E}\left[\left(S^{l_3} - q_0\right)^+\right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_3}}\left(q_0\right)\right)\right)^+\right] \tag{33}$$

- where $F_{S^{l_3}}\left(q_0\right)$ is the distribution function of S^{l_3} evaluated at q_0
- such that for an arbitrary t, we have:

$$\begin{aligned} F_{S^{l_3}}\left(q_0\right) &= \mathbf{P}\left[S^{l_3} \leq q_0\right] \\ &= \mathbf{P}\left(\sum_{i=1}^{j-1}\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ \right. \\ &\qquad \left. + \sum_{i=j}^{n}\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ \leq 0.2\right) \end{aligned} \tag{34}$$

A Lower Bound under Black-Scholes Model(5)

- Substitute x for q_t/q_0 in (34)
- where x solves

$$\sum_{i=1}^{j-1}\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}-1.3\right)^{+}+\sum_{i=j}^{n}\left(xe^{r(t_i-t)}-1.3\right)^{+}=0.2 \qquad (35)$$

- Then $S^{l_3}\leq q_0$ if and only if $q_t\leq xq_0$
- This yields

$$F_{S^{l_3}}(q_0)=F_{q_t}(xq_0)=\begin{cases}F_{Y_i}\left(5q_0\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}-1.3\right)^{+}\right) & i<j,\\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)}-1.3\right)^{+}\right) & i\geq j\end{cases}$$

A Lower Bound under Black-Scholes Model(6)

- As a result we have:

$$
\begin{aligned}
P_1 \quad \geq \quad & 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0^{1-t_i/t}\mathbf{E}\left(\left(q_t^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}\right.\right.\right. \\
& \left.\left.- q_0^{t_i/t}\left(1.3+\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}-1.3\right)^+\right)\right)^+\right) \\
& \left.+\sum_{i=j}^{n} e^{rt_i} C\left(q_0\left(\frac{1.3}{e^{r(t_i-t)}}+\left(x-\frac{1.3}{e^{r(t_i-t)}}\right)^+\right), t\right)\right)
\end{aligned}
$$

- Denote the term within the first summation as E_1 and its value is given below.

$$\mathbf{E}_1 = 5q_0 \left(e^{rt_i}\Phi\left(d_{1ai}\right) - \left(1.3 + \left(x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+\right)\Phi\left(d_{2ai}\right)\right) \tag{36}$$

- where d_{2ai} and d_{1ai} are given respectively as

$$d_{2ai} = \frac{-\log_e\left(\frac{da_i}{q_0}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{37}$$

$$d_{1ai} = d_{2ai} + \sigma\frac{t_i}{\sqrt{t}} \tag{38}$$

- and da_i is given as

$$da_i = q_0\left(\frac{1.3}{e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}} + \left(x^{t_i/t} - \frac{1.3}{e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}}\right)^+\right)^{t/t_i} \tag{39}$$

A Lower Bound under Black-Scholes Model(8)

- As a result we have

The Lower Bound $\text{lb}_t^{(3)}$

$$
\begin{aligned}
P_1 \quad &\geq \quad 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0 \left(e^{rt_i}\Phi\left(d_{1ai}\right) - \left(1.3 + \left(x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+\right)\right)\right. \\
&\qquad \left. + \sum_{i=j}^{n} e^{rt_i} C\left(q_0\left(\frac{1.3}{e^{r(t_i-t)}} + \left(x - \frac{1.3}{e^{r(t_i-t)}}\right)^+\right), t\right)\right) \\
&=: \quad \text{lb}_t^{(3)}
\end{aligned}
$$

- The bound $\text{lb}_t^{(3)}$ can undergo treatment similar to $\text{lb}_t^{(2)}$ in sense of maximization with respect to t yielding

$$P_1 \geq \max_{0 \leq t \leq T} \text{lb}_t^{(3)} \tag{41}$$

An Upper Bound for the Swiss Re Bond(1)

Proposition

The payoff of the call option is a convex function[a] of the strike price, i.e., $\boldsymbol{E}\left[(X-x)^{+}\right]$ is convex in x.

[a] A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is convex if and only if $f\left(ax+(1-a)\,y\right) \leq af\left(x\right)+(1-a)\,f\left(y\right) \quad \forall a \in [0,1]$ and any pair of elements $x, y \in I$.

- Define a vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$
- With the help of $\boldsymbol{\lambda}$ we can write the payoff of the Asian-type call option as

$$P_1 = Ce^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{n}\left(5\left(\frac{q_i}{q_0}-1.3\right)^{+}-\lambda_i\right)\right)^{+}\right]. \qquad (42)$$

- The above result for the call option implies

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} e^{rt_i}\,C\left(q_0\left(1.3+\frac{\lambda_i}{5}\right), t_i\right) \qquad (43)$$

An Upper Bound for the Swiss Re Bond(2)

- Employing the Lagrangian with ϕ as the Lagrange's multiplier, we have

$$L(\boldsymbol{\lambda}, \phi) = \frac{5}{q_0} \sum_{i=1}^{n} e^{rt_i} C\left(q_0\left(1.3 + \frac{\lambda_i}{5}\right), t_i\right) + \phi\left(\sum_{i=1}^{n} \lambda_i - 1\right) \quad (44)$$

The Upper Bound ub_1

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(x), t_i\right) =: \mathrm{ub}_1 \quad (45)$$

- where $x \in (0,1)$ solves $\sum_{i=1}^{n} F_{q_i}^{-1}(x) = \frac{q_0}{5}(1 + 6.5n)$
- Put-Call parity yields: $P \leq \mathrm{ub}_1 - G =: \mathrm{UB}_1$

Numerical Results(1)

- Assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ obeys the Black-Scholes model, specified by the following stochastic differential equation (SDE)

$$dq_t = rq_t dt + \sigma q_t dW_t.$$

- In order to simulate a path, we will consider the price of the asset on a finite set of $n = 3$ evenly spaced dates $t_1, ..., t_n$.

The Brownian Simulation

$$q_{t_j} = q_{t_{j-1}} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}U_j\right] \quad U_j \sim N(0,1), \quad j = 1, 2, \ldots, n \tag{46}$$

Parameter choices in accordance with [Lin and Cox(2008)]

$$q_0 = 0.008453, \; r = 0.0, \; T = 3, \; t_0 = 0, \; n = 3, \; \sigma = 0.0388$$

Numerical Results(2)

Table 1: Table showing the various lower bounds, upper bound and the Monte Carlo estimate for varying values of r

r	LB0	LB1	LBt_(1)	LBt_(2)	LBt_(3)	UB1	MC
0.035	0.899130889131400	0.899130889153152	0.899130889163207	0.899131563852078	0.899131577418890	0.899131637780299	0.899130939228525
0.03	0.913324024542464	0.913324024546338	0.913324024548259	0.913324251738880	0.913324256505855	0.913324320930395	0.913324120543246
0.025	0.927447505802074	0.927447505802722	0.927447505803066	0.927447578831809	0.927447580428344	0.927447619324390	0.927447582073642
0.02	0.941626342686440	0.941626342686542	0.941626342686600	0.941626365090140	0.941626365599735	0.941626384748977	0.941626356704134
0.015	0.955935721003105	0.955935721003120	0.955935721003129	0.955935727561107	0.955935727716106	0.955935736078305	0.955935715488521
0.01	0.970419124545862	0.970419124545864	0.970419124545865	0.970419126377220	0.970419126422140	0.970419129771609	0.970419112046475
0.005	0.985101139986133	0.985101139986134	0.985101139986134	0.985101140473942	0.985101140486345	0.985101141738075	0.985101142704466
0	0.999995778015617	0.999995778015617	0.999995778015617	0.999995778139535	0.999995778142797	0.999995778583618	0.999995730678518

Table 2: Table showing the various lower bounds, upper bound and the Monte Carlo estimate for varying values of q0 when r=0.0

q0	Lb0	Lb1	Lbt(1)	Lbt(2)	Lbt(3)	UB1	MC
0.007	0.999999999999517	0.999999999999517	0.999999999999517	0.999999999999517	0.999999999999517	0.999999999999517	1.000000000000000
0.008	0.999999915251651	0.999999915251651	0.999999915251651	0.999999915252160	0.999999915252175	0.999999915253115	0.999999935586330
0.008453	0.999995778015617	0.999995778015617	0.999995778015617	0.999995778139535	0.999995778142797	0.999995778583618	0.999995730678518
0.009	0.999821987943444	0.999821987949893	0.999821987949893	0.999822025862818	0.999822025862818	0.999822875816246	0.999816103328680
0.01	0.978292691034648	0.978310383929407	0.978310383929037	0.978503560221413	0.978503560221499	0.986262918346612	0.978738658827918
0.011	0.572750782003669	0.610962124257773	0.610962123857399	0.610962123857399	0.610962123857400	0.877336305501968	0.652440509314875
0.012	0.029980287407555	0.040209774144029	0.040209770810356	0.040209770810359	0.040209770810359	0.395672911251278	0.094615386163640
0.013	0.001068265288866	0.000791137238546	0.000791141242590	0.000791141242584	0.000791141242578	0.083466184427206	0.001662471990070
0.014	0.000019422582024	0.000019358292710	0.000019362301765	0.000019362301763	0.000019362301756	0.008942985848261	0.000003376858132

Numerical Results(3)

Figure1: Rel. Diff. of LBt(2), LBt(3) and UB1 w.r.t. MC estimate under Black-Scholes model

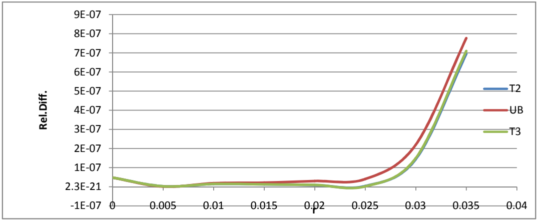
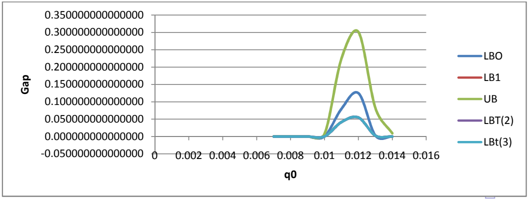
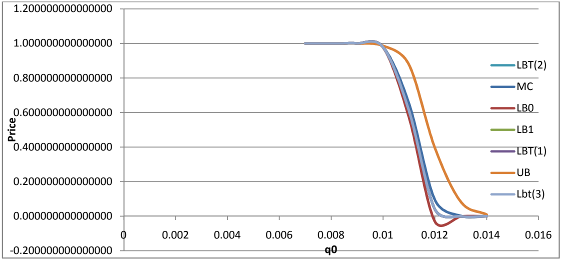


Figure2: Comparison of different bounds under B-S model in terms of difference from MC estimate for r=0



Numerical Results(4)

Figure3: Price Bounds under Black-Scholes model for the parameter choice of Lin and Cox(2008) Model



Conclusions

- Swiss Re thrives from Life Insurance Business
- It achieved Mortality Risk Transfer
- Main purpose of Swiss Re:- Protection against extreme mortality events
- Profitability negatively correlated to mortality rates
- Needed counter parties to offload mortality risk
- No dependence on retrocessionaire
- Methodology: Catastrophic bond with loss measurement based on a parametric index
- Investors in the bond took opposite position
- Received an enhanced return if an extreme mortality event doesn't occur

What Lies Ahead...?

Extreme Mortality Securitizations

Company	Year	Principal Amount	No. of tranches
Swiss Re – Vita Capital 1	2003	$400 million	1
Swiss Re – Vita Capital 2	2005	$362 million	3
Scottish Re –Tartan capital	2006	$155 million	2
AXA-Osiris Capital	2006	$250 million	4
Swiss Re -Vita Capital 3	2007	$390 million	2
Munich Re – Nathan Ltd	2008	$100 million	1
Swiss Re –Vita Capital 4	2009	$75 million	1

Further Research

- Using the model-independent bounds for mortality jump models
- Deriving even more tighter upper bound
- Drawing correspondence between these bounds and the bounds in literature

"If there will be one day such a severe world-wide pandemic that one of the bonds I bought will be triggered, there will be more important things to look after than an investment portfolio."

— ANONYMOUS CATM INVESTOR

Thanks!

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