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Stochastic Loss Reserving with the Collective Risk Model

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Outline of Presentation

- General Approach to Stochastic Modeling
  - Allows for better estimate of the mean
  - Quantify uncertainty in estimate
- The Paper - “Stochastic Loss Reserving with the Collective Risk Model”

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Introduce Stochastic Modeling with an Example

- X ~ lognormal with $\mu = 5$ and $\sigma = 2$
- Two ways to estimate E[X] (= 1,097)
- Straight Average – $\hat{\text{E}}_N[X] = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Lognormal Average – $\hat{\text{E}}_L[X] = e^{\hat{\mu}+\hat{\sigma}^2/2}$

where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\log(X_i),\ \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\log(X_i)-\hat{\mu}\right)^2}$

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Which Estimator is Better? $E_N[X]$ or $E_L[x]$?

- Straight Average, $E_N[X]$, is simple.
- Lognormal Average, $E_L[X]$ is complicated.
  - But derived from the maximum likelihood estimator for the lognormal distribution
- Evaluate by a simulation
  - Sample size of 500
  - 2,000 samples
- Look at the variability of each estimator

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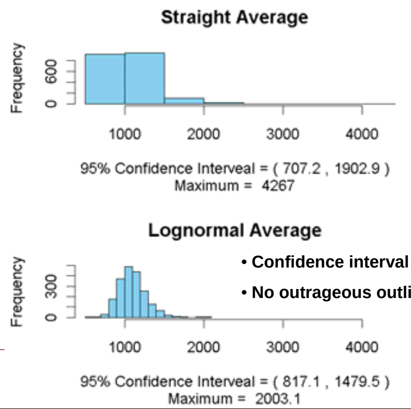
Results of Simulation



- **Confidence interval is narrower**
- **No outrageous outliers**

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Lesson from Example

- ***Knowing the distribution of the observations can lead to a better estimate of the mean!***
- Actuaries have long recognized this.
  - Longtime users of robust statistics
    - Calculate basic limit average severity
    - Fit distributions to get excess severity
- More recently recognized in the growing use of the Generalized Linear Model.

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Parameter Uncertainty and the Gibbs Sampler

- Gibbs sampler is often used for Bayesian analyses.
- It randomly generates parameters in proportion to posterior probabilities.
- Parameters randomly fed into the sampler in proportion to prior probabilities.
- Accepted in proportion to $\dfrac{\text{Likelihood}}{\text{Maximum Likelihood}}$
- Results in the posterior distribution of parameters.

Gibbs Sampler on a Lognormal

Example from February 2008 Actuarial Review

- Simulate μ and σ from a prior distribution of parameters.
- Calculate the likelihood of each simulated μ and σ.
- Select a random uniform number U.
- Accept μ and σ into the posterior distribution if

$$\frac{\text{Likelihood}}{\text{Maximum Likelihood}} < U$$

Figure 1

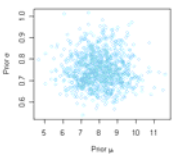
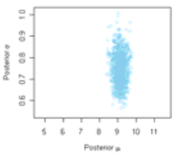


Figure 2



Posterior Distribution of μ and σ is Only of Temporary Interest!

- Most often we are interested in functions of μ and σ.
- For example: Mean

$$e^{\mu+\sigma^2/2}$$

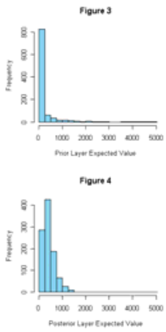
Limited Expected Value

$$e^{\mu+\sigma^2/2}\cdot\Phi\left(\frac{\log(L)-\mu-\sigma^2}{\sigma}\right)+L\cdot\left(1-\Phi\left(\frac{\log(L)-\mu}{\sigma}\right)\right)$$

Layer Expected Value 25,000 to 30,000

- Some posterior parameters generated by Gibbs sampler

□	□	LEV
9.194	0.723	392
9.206	0.708	383
8.817	0.707	119
8.944	0.644	120
9.461	0.785	836
9.150	0.651	252
9.043	0.739	280
9.240	0.773	514
9.392	0.863	845
9.018	0.781	311



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Evolving Strategy for Modeling Uncertainty

- Point Estimates
  - Based on MLE or (Bayesian) Predictive Mean
- Ranges - Bayesian
  - "Quantities of Interest" weighted by posterior probabilities of the parameters
    - Discrete prior or Gibbs Sampler
- Some Applications
  - Claim severity models – COTOR Challenge
  - Loss reserve models – Today's topic

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S&P Report, November 2003

Insurance Actuaries – A Crisis in Credibility

"Actuaries are signing off on reserves that turn out to be wildly inaccurate."

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Prior Work on Loss Reserve Models

- Estimating Predictive Distributions for Loss Reserve Models – 2006 CLRS and ***Variance***
  - Initial application of the strategy to loss reserves
  - Tested results on subsequent loss payments
    - **Set a standard for evaluating loss reserve formulas**
- Thinking Outside the Triangle – 2007 ASTIN Colloquium
  - Tested a formula based on simulated outcomes
  - Provided an example
    - **Model parameters from MLE understated range**
    - **Bayesian mixing (spreading out) provided accurate range**

Stochastic Loss Reserving with the Collective Risk Model

- Focuses mainly on "How to do it"
  - "Data" is simulated from collective risk model
  - Code for implementing algorithms included
- Secondary Objective
  - Use Gibbs sampler (as does Verrall in ***Variance***)

Method Illustrated on Data

Incremental Paid Losses

AY	Premium	*Lag* 1	*Lag* 2	*Lag* 3	*Lag* 4	*Lag* 5	*Lag* 6	*Lag* 7	*Lag* 8	*Lag* 9	*Lag* 10
1	50,000	7,168	11,190	12,432	7,856	3,502	1,286	334	216	190	0
2	50,000	4,770	8,726	9,150	5,728	2,459	2,864	715	219	0	$X_{2,10}$
3	50,000	5,821	9,467	7,741	3,736	1,402	972	720	50	$X_{3,9}$	$X_{3.10}$
4	50,000	5,228	7,050	6,577	2,890	1,600	2,156	592	$X_{4,8}$	$X_{4,9}$	$X_{4,10}$
5	50,000	4,185	6,573	5,196	2,869	3,609	1,283	$X_{5,7}$	$X_{5,8}$	$X_{5,9}$	$X_{5,10}$
6	50,000	4,930	8,034	5,315	5,549	1,891	$X_{6,6}$	$X_{6,7}$	$X_{6,8}$	$X_{6,9}$	$X_{6,10}$
7	50,000	4,936	7,357	5,817	5,278	$X_{7,5}$	$X_{7,6}$	$X_{7,7}$	$X_{7,8}$	$X_{7,9}$	$X_{7,10}$
8	50,000	4,762	8,383	6,568	$X_{8,4}$	$X_{8,5}$	$X_{8,6}$	$X_{8,7}$	$X_{8,8}$	$X_{8,9}$	$X_{8,10}$
9	50,000	5,025	8,898	$X_{9,3}$	$X_{9,4}$	$X_{9,5}$	$X_{9,6}$	$X_{9,7}$	$X_{9,8}$	$X_{9,9}$	$X_{9,10}$
10	50,000	4,824	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$	$X_{10,5}$	$X_{10,6}$	$X_{10,7}$	$X_{10,8}$	$X_{10,9}$	$X_{10,10}$

Plan of Attack

- Specify stochastic model needed to calculate likelihood of the data
- Calculate MLE and parameters for Gibbs sample
- Quantity of Interest = Percentiles of OS Loss

$$R = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} X_{AY,Lag}$$

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Model for Expected Losses

- Two models for expected loss
  - Cape Cod Model
    $E\left[Loss_{AY,Lag}\right] = Premium_{AY} \cdot ELR_{AY} \cdot Dev_{Lag}$
  - Beta Model
    $Dev_{Lag} = \beta\left(Lag/10 \mid a,b\right) - \beta\left((Lag-1)/10 \mid a,b\right)$
- $\{ELR_{AY}\}$ and $\{Dev_{Lag}\}$ and/or $\{a,b\}$ parameters estimated from data

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Need a Stochastic Model to Calculate Likelihoods

Use the collective risk model.

- Select a random claim count, $N_{AY,Lag}$ from a Poisson distribution with mean λ.
- For i = 1, 2, …, $N_{AY,Lag}$, select a random claim amount, $Z_{Lag,i}$.
- Set, $X_{AY,Lag} = \sum_{i=1}^{N_{AY,Lag}} Z_{Lag,i}$

or if $N_{AY,Lag}$ = 0, then $X_{AY,Lag}$ = 0..

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Details of Distributions

- Pareto severity distribution $$F(z) = 1 - \left(\frac{\theta}{z+\theta}\right)^{\alpha}$$
- for all lags – $\alpha = 2$
- Table of θ's

Lag	1	2	3	4	5	6	7-10
θ (000)	10	25	50	75	100	125	150

- Severity increases with lag
- Approximate likelihood calculated by matching moments with an overdispersed negative binomial distribution (for now).

Maximum Likelihood Parameter Estimates for the Two Models

	Cape Cod		Beta	
	ELR	*Dev*	*ELR*	*Dev*
AY/Lag 1	0.89090	0.16948	0.89205	0.15991
2	0.65285	0.26864	0.65670	0.27295
3	0.64448	0.23763	0.69949	0.24156
4	0.55233	0.15539	0.51727	0.16661
5	0.48569	0.07865	0.51696	0.09488
6	0.57259	0.05524	0.53697	0.04410
7	0.56411	0.01771	0.60935	0.01576
8	0.58207	0.00581	0.53487	0.00378
9	0.61922	0.00654	0.68940	0.00044
10	0.52190	0.00491	0.63902	0.00001
			a =	1.90742
			b =	5.78613

Bayesian Analyses
Specify Prior Distributions

$$ELR_{AY} \sim \Gamma(\alpha,\theta) \text{ with } \alpha = 100 \text{ and } \theta = 0.07$$

Beta Model

$$a \sim \Gamma(\alpha,\theta) \text{ with } \alpha = 75 \text{ and } \theta = 0.02$$
$$b \sim \Gamma(\alpha,\theta) \text{ with } \alpha = 25 \text{ and } \theta = 0.20$$

Cape Cod Model

Γ*Lag*	1	2	3	4	5	6	7	8	9	10
α	11.1010	64.6654	190.1538	34.9314	10.7284	4.4957	2.1298	1.0295	0.4574	0.1556
θ	0.0206	0.0041	0.0011	0.0040	0.0079	0.0101	0.0097	0.0073	0.0039	0.0009

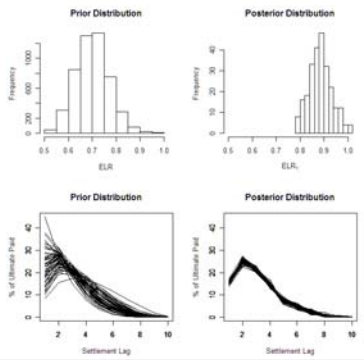
- Prior parameters were derived from looking at "Estimating Predictive Distributions … " paper.

Sample Output from Gibbs Sampler Beta Model

Iteration	ELR_1	ELR_2	ELR_3	ELR_4	ELR_5	ELR_6	ELR_7	ELR_8	ELR_9	ELR_{10}
251	0.75403	0.69942	0.62441	0.56447	0.51833	0.60362	0.61284	0.60487	0.66776	0.63434
252	0.84815	0.73598	0.64300	0.59662	0.50991	0.66961	0.63046	0.65861	0.78867	0.62436
253	0.82959	0.65535	0.62372	0.58285	0.54446	0.65939	0.62067	0.66337	0.77458	0.68514
254	0.82214	0.67833	0.72494	0.58108	0.59692	0.64186	0.64096	0.69660	0.63273	0.71884
255	0.85885	0.70338	0.65320	0.60643	0.57145	0.65768	0.74067	0.64207	0.61538	0.56273
256	0.82655	0.68402	0.71207	0.57685	0.50899	0.62291	0.68097	0.60459	0.73825	0.62867
257	0.86339	0.71486	0.62554	0.55949	0.54898	0.57494	0.63603	0.66952	0.68241	0.61616
258	0.81831	0.64761	0.73752	0.61186	0.63983	0.62646	0.61374	0.67133	0.64861	0.62245
259	0.80801	0.66089	0.70570	0.61823	0.57213	0.62688	0.58704	0.69212	0.62392	0.67231
260	0.81955	0.65917	0.61623	0.64292	0.56440	0.61969	0.61458	0.67270	0.74439	0.59132

Iteration	Dev_1	Dev_2	Dev_3	Dev_4	Dev_5	Dev_6	Dev_7	Dev_8	Dev_9	Dev_{10}
251	0.17353	0.26609	0.23075	0.16171	0.09592	0.04754	0.01863	0.00509	0.00072	0.00002
252	0.17373	0.26219	0.22815	0.16179	0.09773	0.04965	0.02012	0.00576	0.00087	0.00003
253	0.15662	0.25141	0.22857	0.16863	0.10601	0.05625	0.02396	0.00730	0.00119	0.00004
254	0.15514	0.24770	0.22656	0.16906	0.10796	0.05847	0.02559	0.00808	0.00139	0.00005
255	0.16275	0.25121	0.22557	0.16608	0.10487	0.05622	0.02435	0.00760	0.00130	0.00005
256	0.16274	0.24870	0.22378	0.16595	0.10596	0.05768	0.02550	0.00819	0.00145	0.00006
257	0.16549	0.25142	0.22449	0.16497	0.10422	0.05600	0.02436	0.00766	0.00132	0.00005
258	0.15983	0.24720	0.22401	0.16705	0.10721	0.05865	0.02607	0.00842	0.00151	0.00006
259	0.17049	0.25879	0.22734	0.16312	0.09993	0.05165	0.02138	0.00629	0.00099	0.00003
260	0.16584	0.26100	0.23092	0.16494	0.09979	0.05056	0.02034	0.00574	0.00085	0.00003

Graphical Representation of Gibbs Sample – Cape Cod Model



Note that the posteriors are tighter, showing how the data narrows the range of results.

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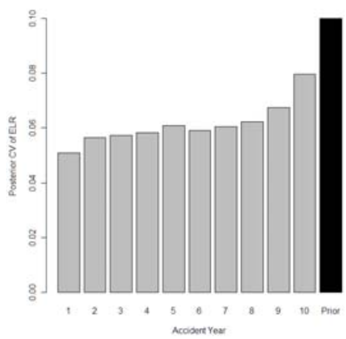
Graphical Representation of Gibbs Sample – Beta Model



Note that the coefficient of variation decreases as we get more paid data.

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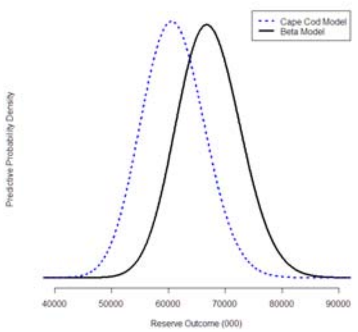
Quantity of Interest
Predictive Distributions of Reserve Outcomes

- Collective risk model
- Simulation
  - Randomly select $\{ELR_i\}$ and $\{Dev_j\}$
  - Simulate $R = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} X_{AY,Lag}$ as done above.
- Use a Fast Fourier Transform
  - Faster, more accurate, but uses some math
  - Used in the paper

Quantity of Interest
Predictive Distributions of Reserve Outcomes



Cape Cod Model
Mean = 60,871
StDev = 5,487

Beta Model
Mean = 67,183
StDev = 5,605

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