



The Actuarial Profession

making financial sense of the future

The use of Econometric Time Series Modelling Techniques in ERM

36th Annual GIRO Convention

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Topics

- Introduction
- Spurious Relationship
- Stochastic Trends
- ARIMA Modelling
- Case Studies
- Conclusions
- Q&A

Introduction

Motivation

- Empirical analyses of many financial and business time series data sets reveals autoregressive nature of dependency structures over time e.g.
 - Annual RPI, Annual NAE, Annual FTSE All Share Return etc (see later)
 - Underwriting cycle in non-life insurance
- Fitting distributions to many years, months or days worth of data effectively loses any potentially valuable information that might be in such patterns over time
- ICA – Conditional Stress Tests
 - Equities – After a large stock market fall ~ 30 - 50%. Is an ICA Equity Stress Test of a further 40% price fall realistic ?
 - Credit Spreads – 2008 saw a large widening in credit spreads. Should an existing ICA Credit Spread Stress Test credit spread movement be reduced ?

Introduction

Objectives

- Two different methodologies:
 - Multivariate Methods – These methods seek relationships between the target and explanatory variable using linear or multiple regression techniques
 - Univariate Methods – These methods use only the time series of the target variable and exploit the non-independence of successive observations
- This presentation investigates the use of Univariate Methods only
- The following topics are outside the scope of this presentation:
 - Multivariate modelling or partial Univariate / Multivariate models
 - ARCH / GARCH modelling
 - Back-testing

Spurious Relationship

Two independent random variables X and Y

- Consider two independent random variables X_t and Y_t

$$X_t = X_{t-1} + \varepsilon_t$$

$$Y_t = Y_{t-1} + \delta_t$$

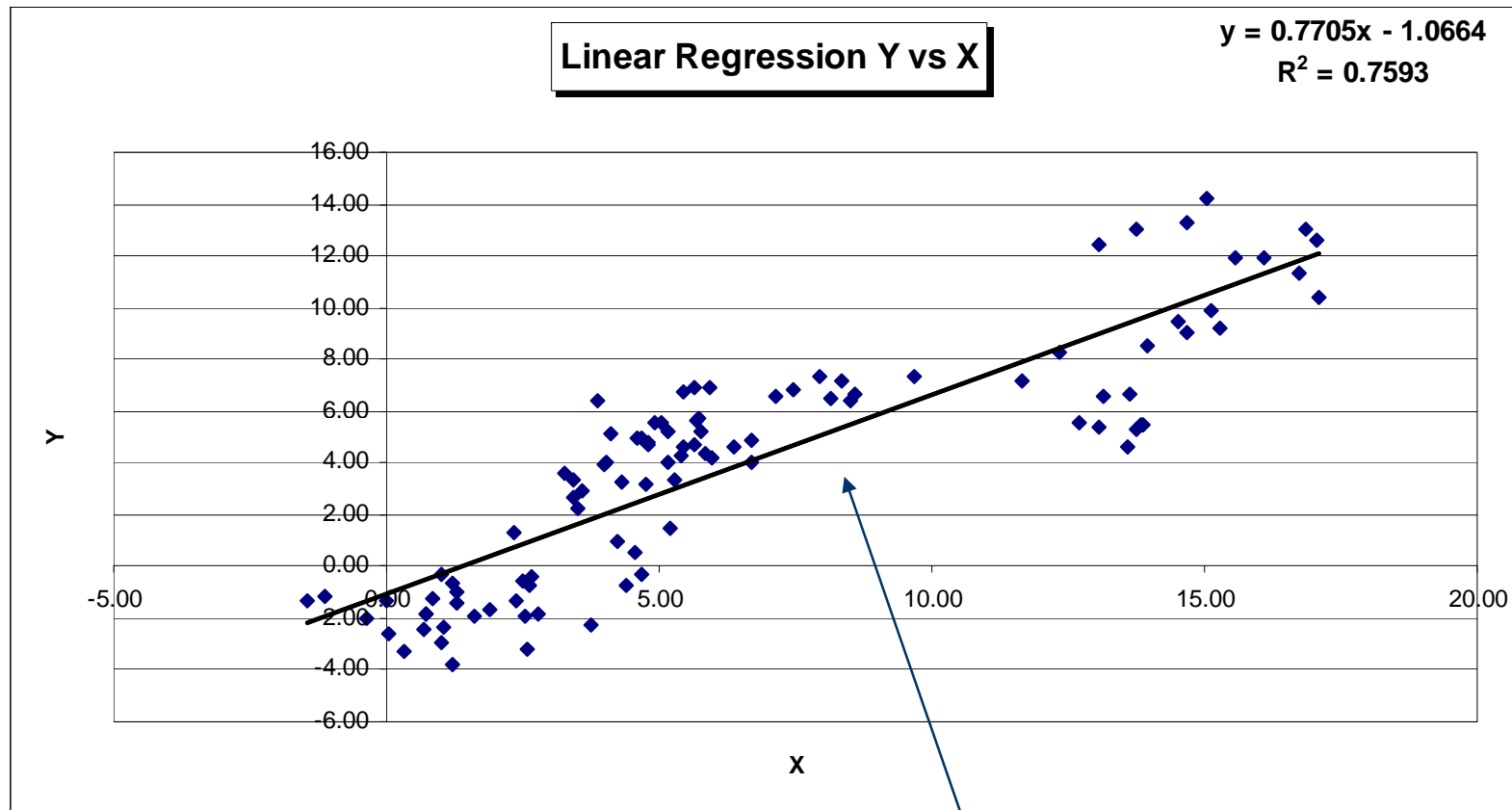
ε_t and δ_t are $N(0,1)$ distributed

$$X_0 = Y_0 = 5$$

- Generate a random sample of 100 values for X_t and Y_t for $t = 1$ to 100
- Using this output the linear correlation and R^2 have been calculated
- X and Y are not related and yet it is common, in repeated runs, to observe very high correlations far in excess of those expected from sampling error in the $N(0,1)$ values

Spurious Relationship

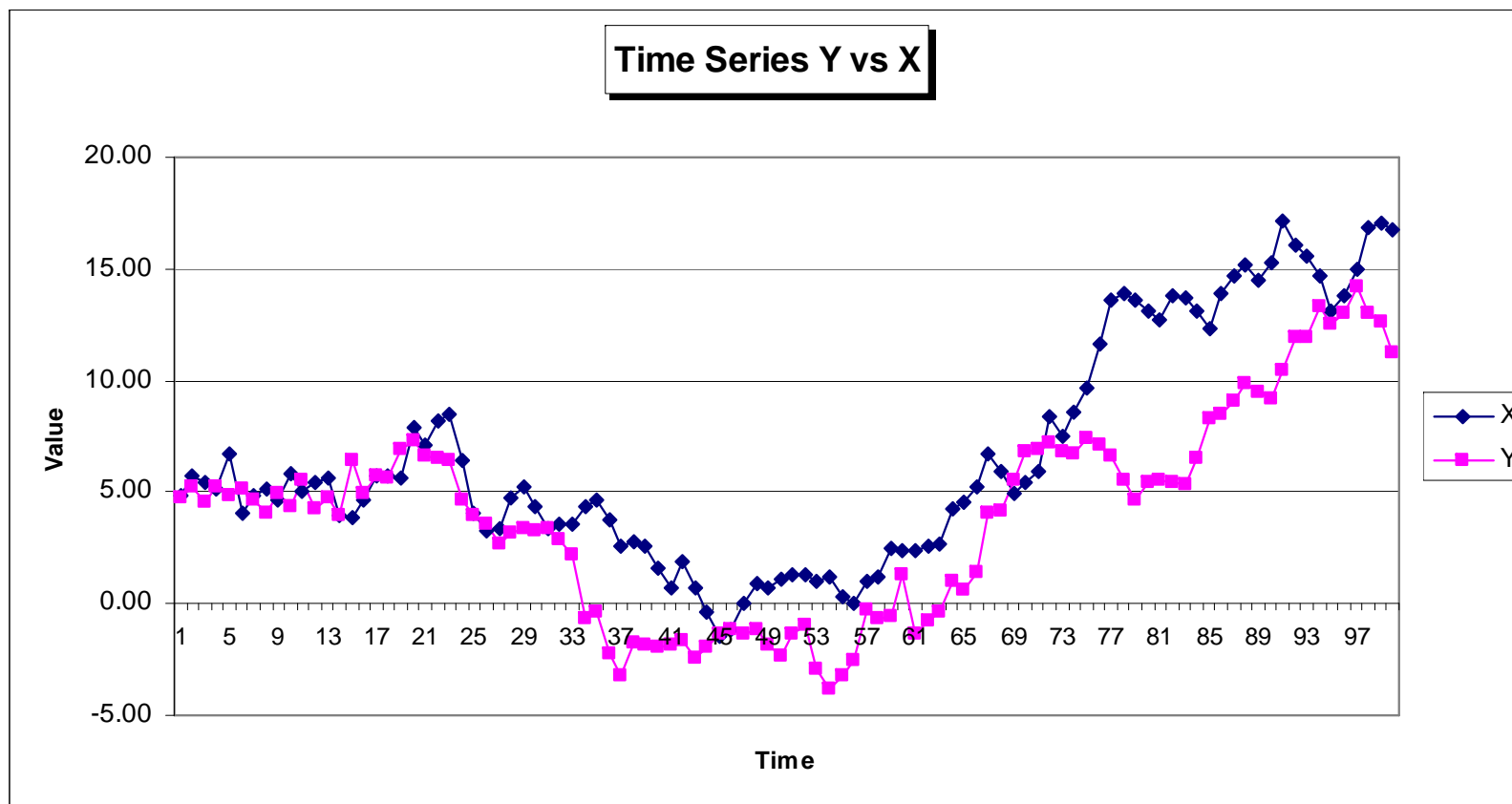
Scatter Diagram – Linear Regression



Linear Correlation = 87.1% ; DW = 0.283

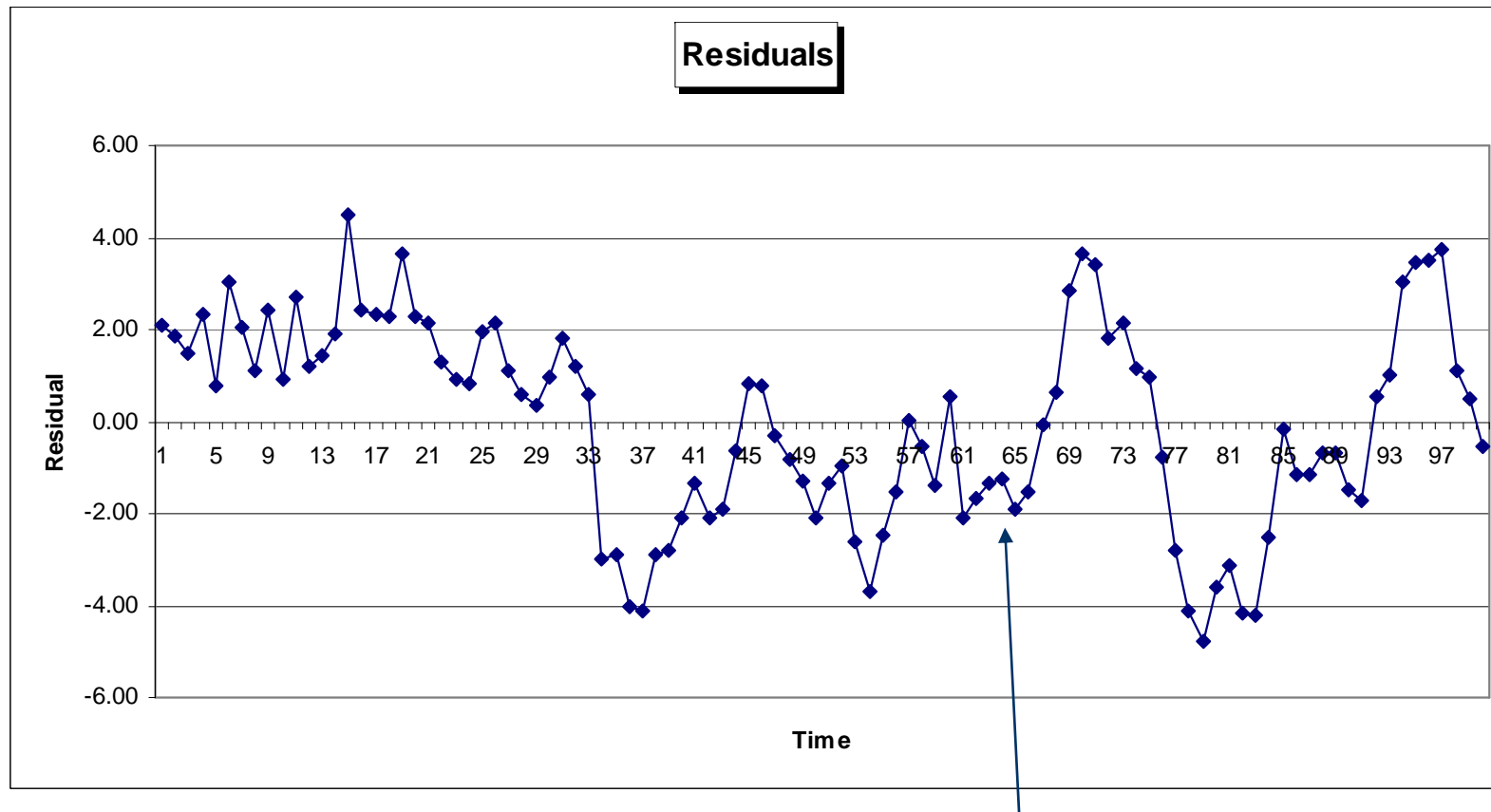
Spurious Relationship

Time Series Diagram



Spurious Relationship

Residuals Diagram



Significant autocorrelation in residuals

Spurious Relationship

Residual Assumptions

- $\text{Actual}_t - \text{Fitted}_t = \text{Residual } \varepsilon_t$
- Quality of parameter estimates and validity of significance tests rely upon the residuals $\varepsilon_t \sim N(0, \sigma)$
- Residuals must be
 - Normally distributed
 - Independent (no autocorrelation)
 - Same variance (no heteroscedasticity)
- Intuitively residuals should be simple randomness that remain after the deterministic part of the variation in a target variable has been modelled
 - Any systematic component in the error terms should really be in the model
 - If each residual is related to its predecessor they are described as **autocorrelated**

Spurious Relationship

Trending Variables

- The stochastic trends in X_t and Y_t are unrelated so linear regression cannot explain the variation of one with the other.
 - The residuals contain both stochastic trends – hence autocorrelation
- Establishing existence of trend is important for univariate modelling. Trend must first of all be removed. There are two types of trend:
 - Deterministic: e.g. $y_t = a + bt$
 - Stochastic: e.g. random walk $y_t = y_{t-1} + \varepsilon_t$
- Most trending series in economics and business are not deterministic but are stochastic i.e. they exhibit random walk type behaviour
 - The identification of stochastic trend is a test for stationarity
 - A stochastic trend is removed by differencing e.g. converting an RPI value at month t to an annual RPI return at month t is in effect differencing the variable.

Stochastic Trends

Autocorrelation

- Let the variable y at time $t = y_t$ and lagged variable y at time $t-k = y_{t-k}$
 - 1st order autocorrelation $r_1 = \text{corr}(y_t, y_{t-1})$
 - 2nd order autocorrelation $r_2 = \text{corr}(y_t, y_{t-2})$
 - **kth order autocorrelation $r_k = \text{corr}(y_t, y_{t-k})$**
- The Autocorrelation Function (“ACF”) measures the correlation between 2 variables y_t and y_{t-k} .
- The Partial Autocorrelation Function (“PACF”) measures the additional effect of y_{t-k} on y_t , once effects of $y_{t-1}, y_{t-2}, \dots, y_{t-(k-1)}$ have been accounted for
- Autocorrelation Plot (Correlogram)
 - This is very useful for analysing time series data and determining the most appropriate time series model
 - The correlogram displays 95% bounds at each lag that enable quick tests of whether each value is significantly different from zero.

Stochastic Trends

Uses for Autocorrelation

- y_t Random
 - All autocorrelations are small
- y_t Stationary
 - Autocorrelations rapidly decrease as lag increases
- y_t Trending
 - Many large autocorrelations
- Checking residuals are simple randomness.
 - It can be impossible to eliminate all autocorrelations from residuals
- ARIMA Modelling (see later)

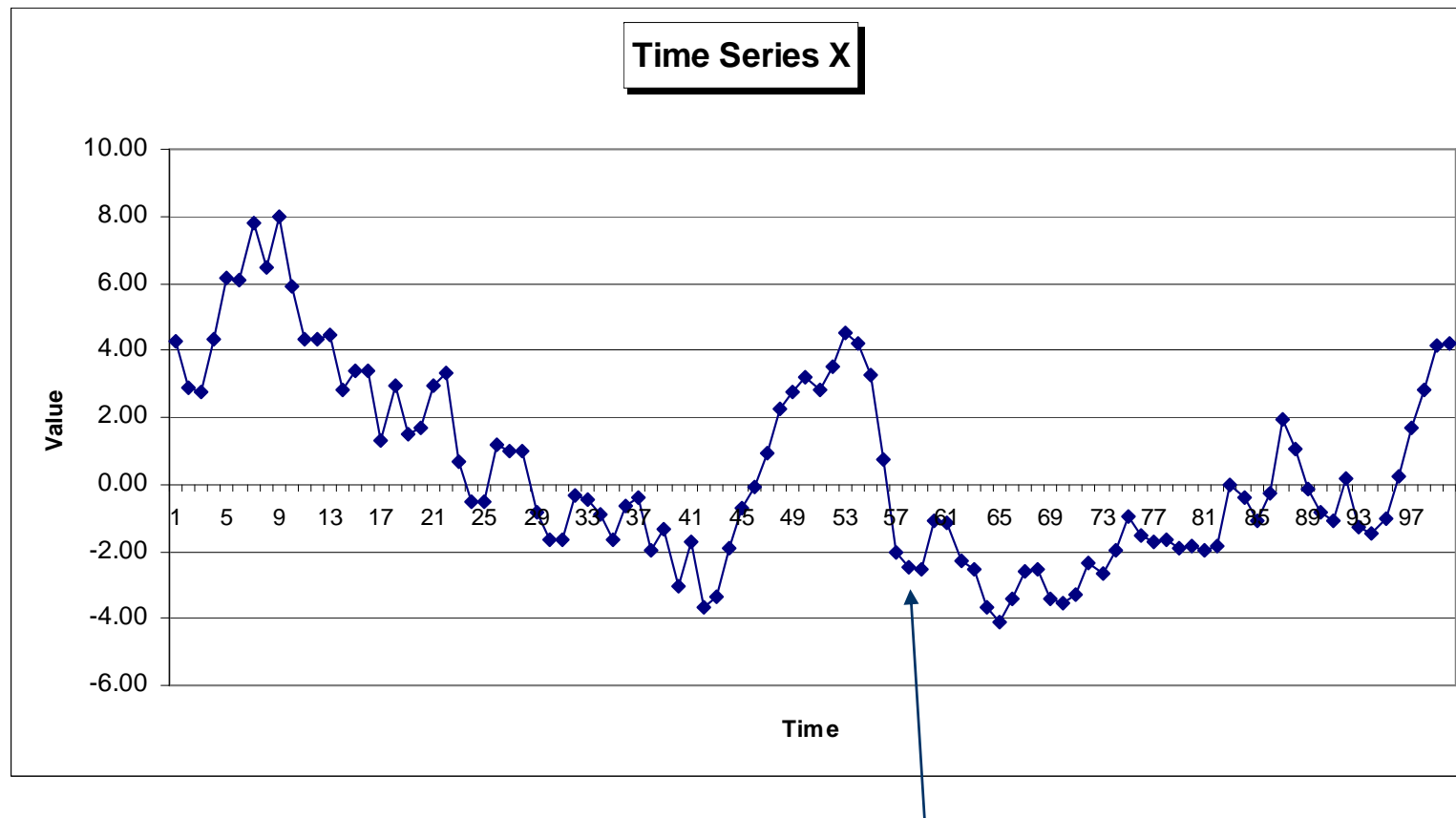
Stochastic Trends

Some Useful Time Series

- $y_t = \varepsilon_t$ Purely random process ('white noise')
 - ε_t has the same mean and variance and no autocorrelation
- y_t follows an autoregressive process if it depends linearly on past observations of y_t
 - $y_t = a_0 + a_1y_{t-1} + a_2y_{t-2} + a_3y_{t-3} \dots + a_py_{t-p} + \varepsilon_t$
 - ε_t is white noise as above
 - Simplest case is autoregression of order one $y_t = a_0 + a_1y_{t-1} + \varepsilon_t$
- Let $y_t = \phi y_{t-1} + \varepsilon_t$
 - If $|\text{Mod}(\phi)| > 1$ then y_t is said to be non-stationary – these are easy to spot
 - If $|\text{Mod}(\phi)| < 1$ then y_t is said to be stationary (mean reverting) – the forecast function converges to the mean
 - If $|\text{Mod}(\phi)| = 1$ then y_t is non-stationary – it meanders stochastically and is known as a random walk

Stochastic Trends

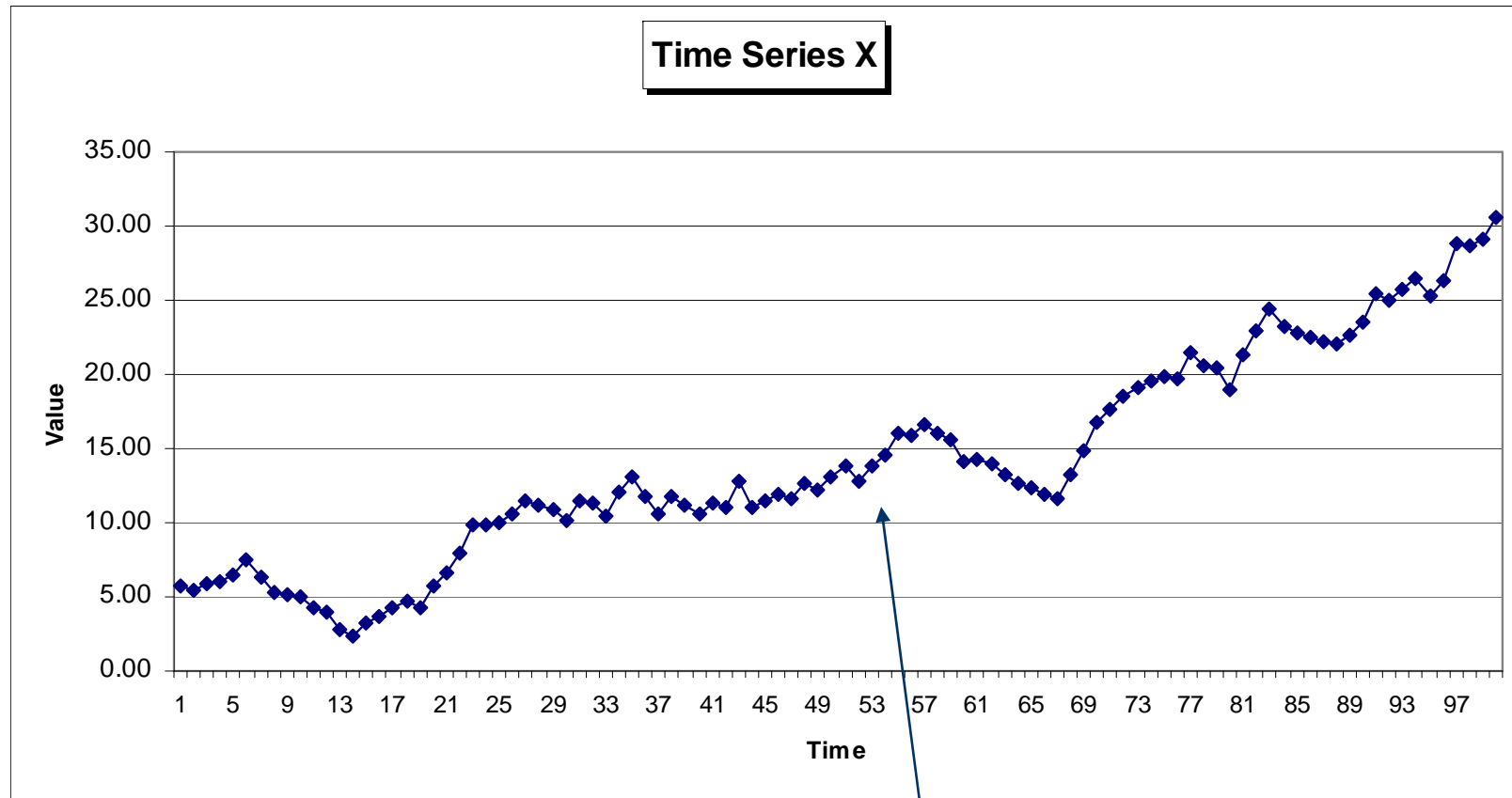
Some Useful Time Series – $y_t = 0.9y_{t-1} + \varepsilon_t$



Example of a mean reverting trend

Stochastic Trends

Some Useful Time Series – $y_t = 1.03y_{t-1} + \varepsilon_t$



Example of a non-stationary trend

ARIMA Modelling

Stationarity

- Time Series Modelling requires knowledge of the mean, variance and autocorrelations
- A series y_t is said to be stationary if it has constant mean, constant variance and constant autocorrelations at each lag
- If a series is stationary, modelling can proceed by estimating the mean, variance and autocorrelations from significantly long time averages of the series
- A stationary series is not necessarily completely random as it can have autocorrelation
- The most fundamental property is stationarity in the mean

ARIMA Modelling

ARIMA (p,d,q)

- Box-Jenkins is a univariate forecasting approach
 - It involves the careful examination of time series in order to identify the underlying data-generating process
 - The choice of best model can be systematically made using this approach
- It is useful to restrict the search for models to the class of **AutoRegressive Integrated Moving Average Models** – ARIMA(p,d,q)
- An ARMA(p,q) model for variable y_t is a combination of an autoregressive process of order p, AR(p) and a moving average process of order q, MA(q) where:

AR(p), ARMA(p,0) process $y_t = a_1y_{t-1} + a_2y_{t-2} + a_3y_{t-3} \dots + a_py_{t-p} + \varepsilon_t$

MA(q), ARMA(0,q) process $y_t = b_1e_{t-1} + b_2e_{t-2} + b_3e_{t-3} \dots + b_qe_{t-q} + \varepsilon_t$

ARIMA Modelling

ARIMA (p,d,q)

- An ARIMA(p,d,q) process:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + a_3 y_{t-3} \dots + a_p y_{t-p} + b_1 e_{t-1} + b_2 e_{t-2} + b_3 e_{t-3} \dots + b_q e_{t-q} + \varepsilon_t$$

- If a variable must be differenced d times in order to achieve stationarity it is said to be integrated or order d.
 - d = 1 would mean that the variable now being modelled = $\Delta y_t = y_t - y_{t-1}$
- An AR model of sufficiently high order can usually be found to model any business series
 - If a large number of parameters are required for a good fit, forecasts can be poor. This motivates working with a broader class of models
 - Since the amount of data is limited it is preferable to fit a model involving as few a parameters as possible
 - This is known as the “**Principle of Parsimony**”.
- Experience suggests that an ARMA(p,q) model may achieve as good a fit as an AR(p') model but with fewer parameters i.e. $p+q < p'$

ARIMA Modelling

Box-Jenkins Methodology

- Differencing a time series to achieve Stationarity
- Identification of a model to be tentatively used
 - Inspection of the Autocorrelation function (“ACF”) and
 - Partial autocorrelation function (“PACF”) at different lags
- Estimating the parameters of the model
 - Maximum Likelihood, Least Squares etc.
 - This amounts to the minimisation of a complicated non-linear function of parameters that involves iterative numerical procedures
- Diagnostic Evaluation – Is the model adequate
 - t-statistics (and p-values); Durbin-Watson (“DW”)
 - Residuals; Ljung-Box Q-statistic; AIC, SIC, Adj. R^2 etc.

ARIMA Modelling

Comparing the fit of different models

- Adjusted R^2 (“Adj. R^2 ”)

$$\text{Adj. } R^2 = 1/(n-k-1) \sum_{i=1} e_i^2 / 1/(n-1) \sum_{i=1} (y_i - E(y))^2$$

- Akaike Information Criterion (“AIC”)

$$\text{AIC} = 1 + \ln(2\pi) + \ln(\text{SSR}/n) + 2k / n$$

- Schwartz Bayesian Criterion (“SBC”)

$$\text{SBC} = 1 + \ln(2\pi) + \ln(\text{SSR}/n) + k \ln(n) / n$$

Sum of Squared Residuals (“SSR”)

$$\text{SSR} = \sum_{i=1} e_i^2$$

n = number of observations; k = number of explanatory variables

ARIMA Modelling

Durbin-Watson (“DW”) Statistic

- The DW Statistic evaluates autocorrelation for residuals placed in the same order as the data observations
- $DW = \sum_{i=2} (e_i - e_{i-1})^2 / \sum_{i=1} e_i^2$
- $DW \sim 2(1-r)$ where r = autocorrelation
 - $DW = 2$ – no autocorrelation
 - $DW > 2$ – negative autocorrelation
 - $DW < 2$ – positive autocorrelation
- The DW statistic is used instead of r because strict tests exist to examine whether DW is significantly different from 2

ARIMA Modelling

Autocorrelation diagnostic evaluation

- Residuals should be white noise
- The ACF of residuals should be investigated
- Can test for autocorrelation in residuals for several lags together
- Under null hypothesis of no autocorrelation in the first m lags, the Ljung-Box Q-statistic has a chi-squared distribution with d.f. = $(m-p-q)$

$$Q(m) = n(n+2) \sum_{i=1}^m r_i^2 / (n-i) \sim \chi^2_{m-p-q}$$

where $r_i = \text{corr}(e_t, e_{t-i})$

Case Studies

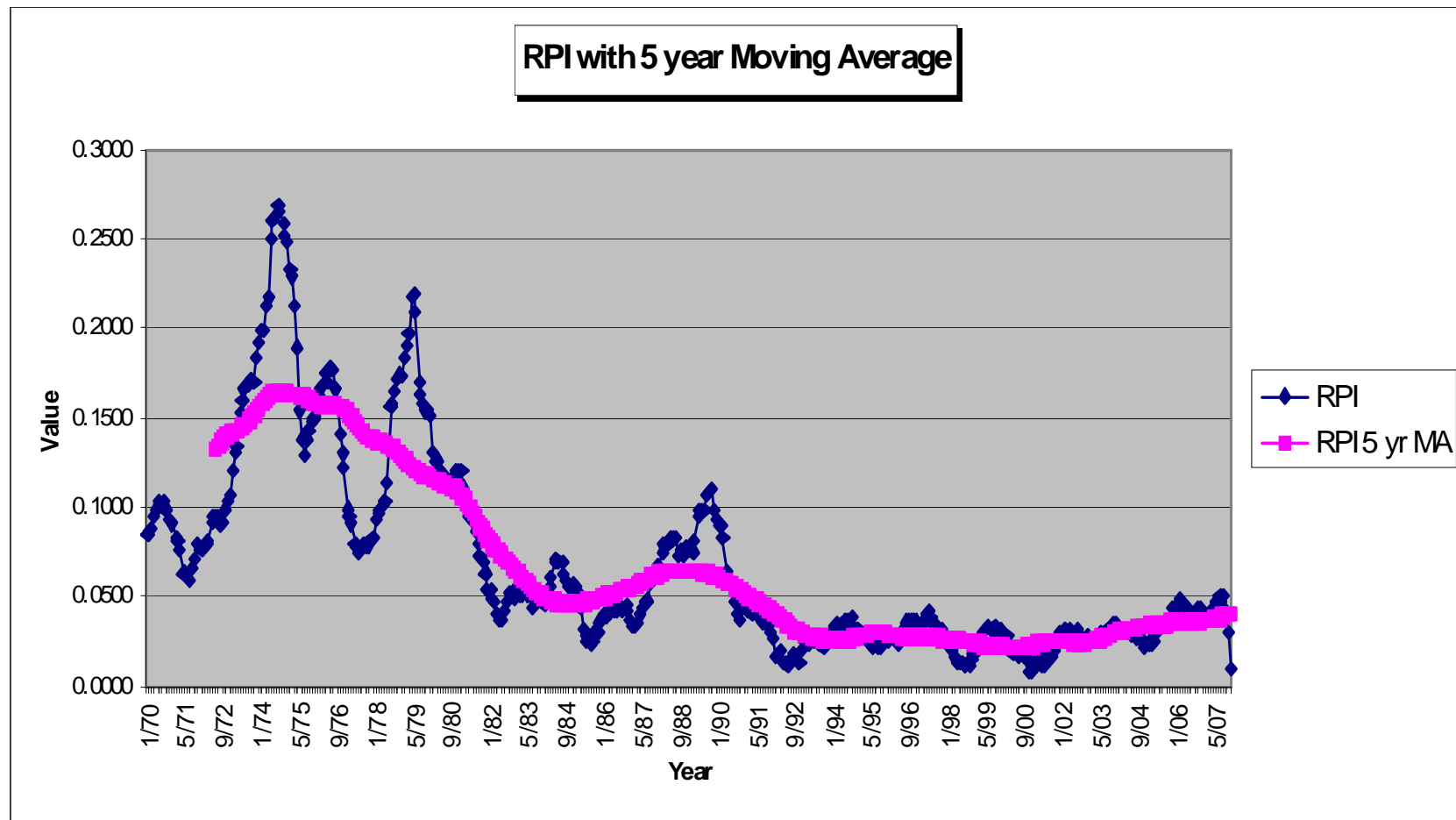
RPI Case Study – Data

- Data
 - Monthly data has been used
 - $RPI_Index(t)$ – RPI at the end of each month for the period Jan 1970 to Dec 2008 as provided by the Office of National Statistics (“ONS”).
 - Constructed an historical time series of a month rolling value of $RPI(t)$ at month t , where:
$$RPI(t) = \text{Annual RPI Change} = RPI_Index(t) / RPI_Index(t-12) - 1$$
- ARIMA(2,[12]) Model Fit
 - Monthly data Jan 1987 to Dec 2008
 - Box-Jenkins Diagnostic Evaluation tests OK
 - Large residuals in 2008
 - Simulation of 5,000 path-dependent scenarios of length 120 months

$$\begin{aligned} RPI(t) &= 0.02187 + Y(t) \\ Y(t) &= 1.37756 Y(t-1) - 0.38514 Y(t-2) - 0.7521 e(t-12) + e(t) \\ e(t) &\sim N(0.00000, 0.00296) \end{aligned}$$

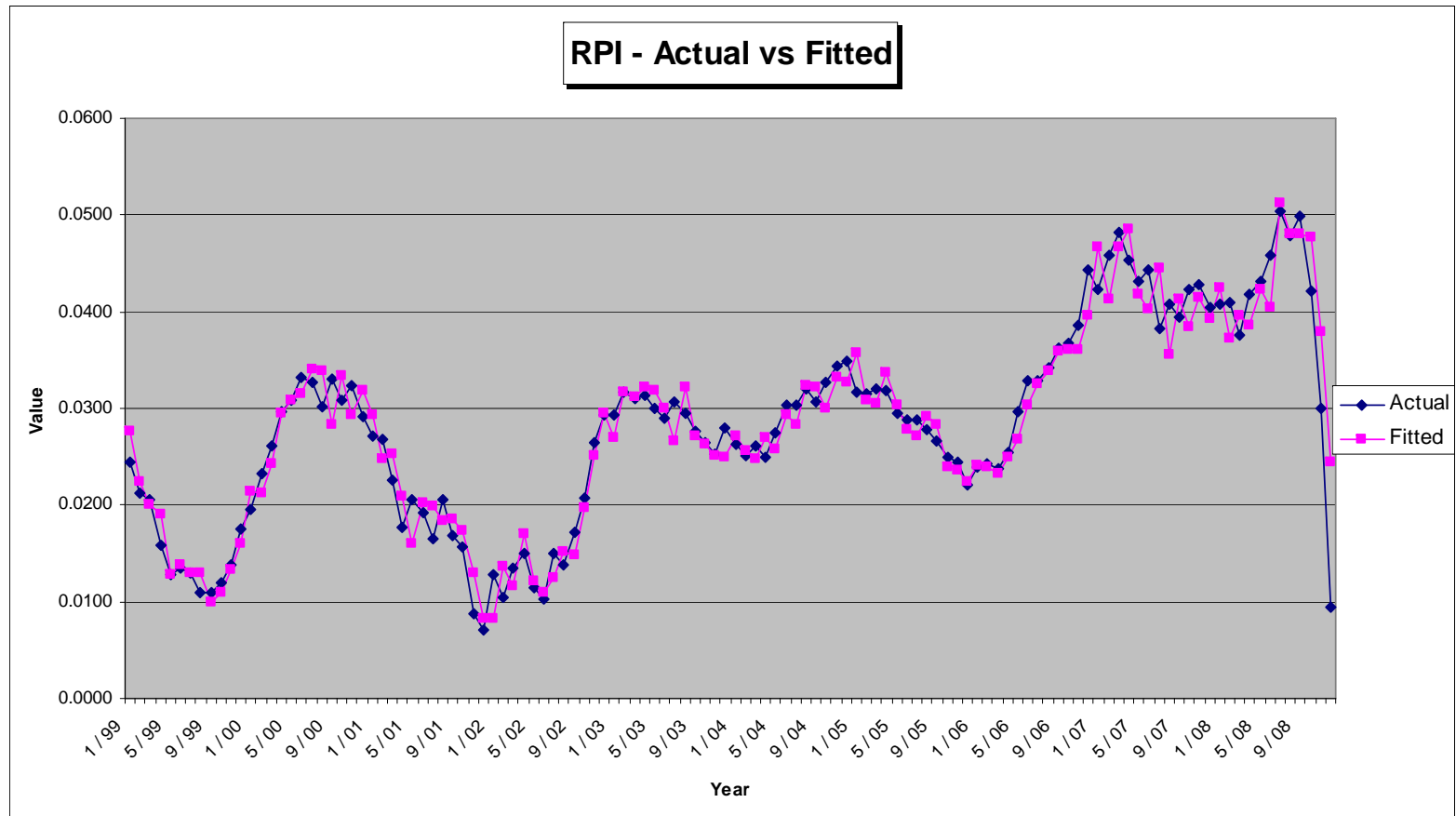
Case Studies

RPI Case Study – Annual RPI Data (1/70 to 12/08)



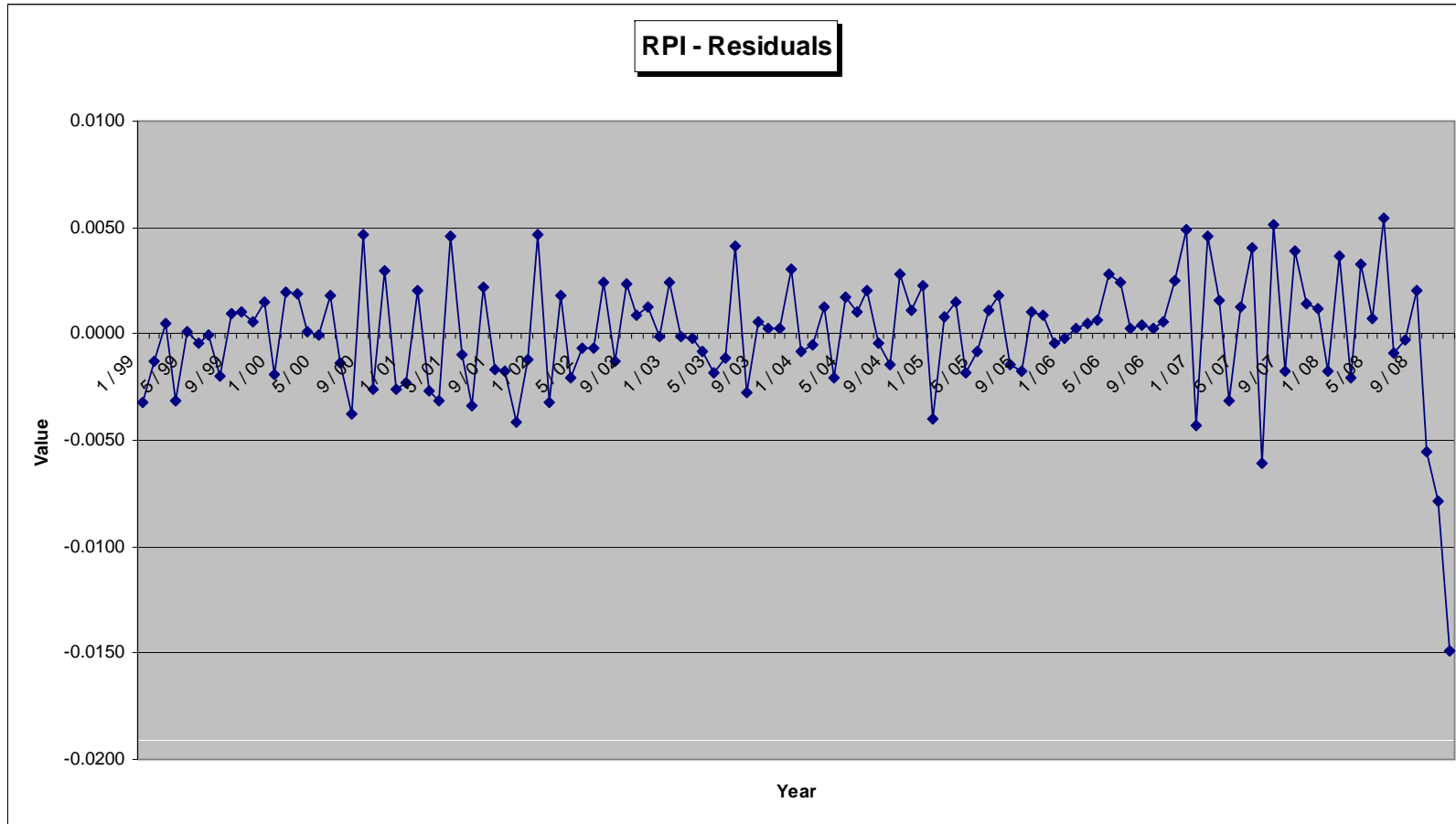
Case Studies

RPI Case Study – Actual vs Fitted (Last 10 years shown)



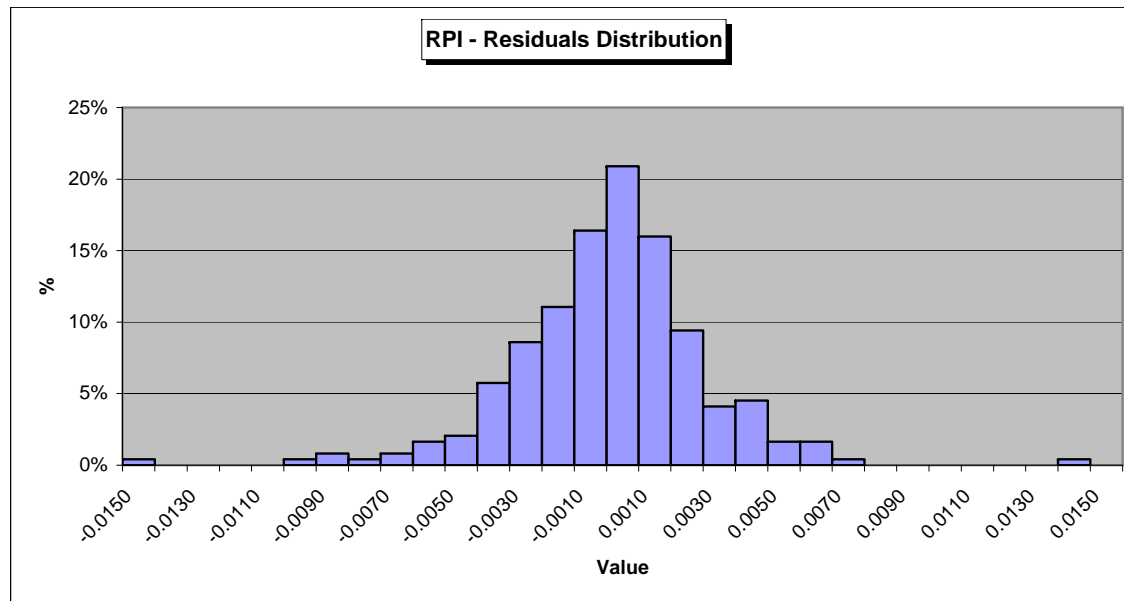
Case Studies

RPI Case Study – Residuals (Last 10 years shown)



Case Studies

RPI Case Study – Residuals Distribution (All years)



Residuals	
Sample No.	262
Mean	0.000094
Minimum	-0.014877
Maximum	0.014730
Std Dev	0.002958
Skewness	-0.285210
Kurtosis	7.529413

Case Studies

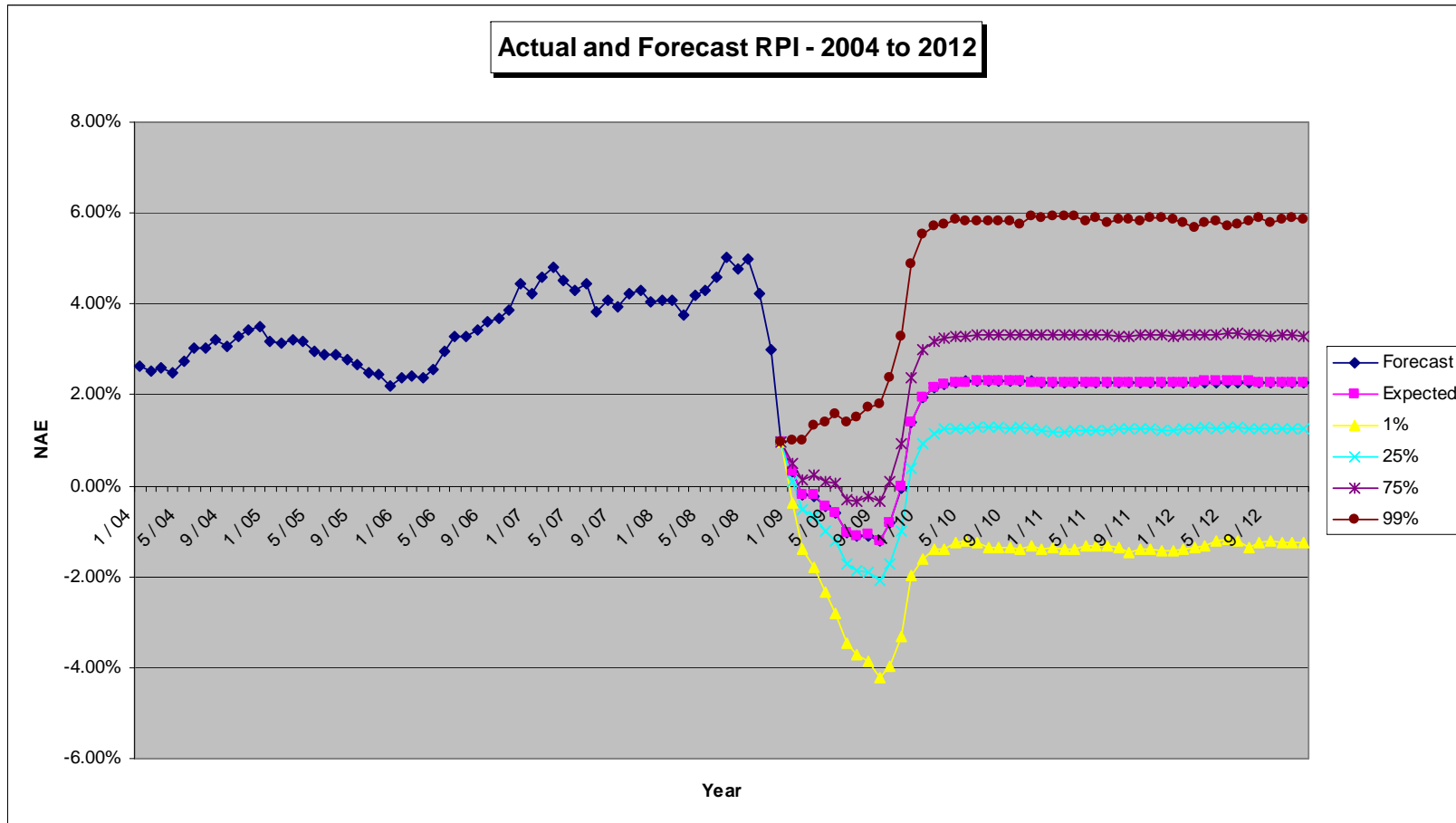
RPI Case Study – Model Fit and Future Projections

RPI			
Start	1987		
End	2008		
Variable	Coefficient	t-statistic	Probability
C	0.02187	2.112	3.56%
Y(t-1)	1.37756	22.084	0.00%
Y(t-2)	-0.38514	-6.204	0.00%
e (t-12)	-0.75210	0.041	0.00%
Adj R²	97.9%		
Durbin Watson	2.0022		
SSR	0.0023		
AIC	-8.7805		
SC	-8.7261		

	12 / 09	12 / 10	12 / 11	12 / 12	12 / 13	12 / 14	12 / 15	12 / 16	12 / 17	12 / 18
Forecast	1.39%	2.29%	2.27%	2.26%	2.25%	2.24%	2.23%	2.23%	2.22%	2.22%
Expected	1.40%	2.28%	2.27%	2.28%	2.23%	2.27%	2.24%	2.25%	2.25%	2.22%
Standard Deviation	1.50%	1.57%	1.55%	1.53%	1.54%	1.56%	1.55%	1.55%	1.55%	1.54%
Minimum	-3.79%	-3.94%	-2.82%	-2.81%	-3.72%	-3.20%	-3.62%	-3.48%	-3.45%	-5.17%
Maximum	6.57%	7.62%	7.65%	7.70%	7.62%	7.38%	7.84%	8.08%	8.49%	7.99%
Percentile										
0.5%	-2.50%	-1.78%	-1.87%	-1.58%	-1.68%	-1.70%	-1.64%	-1.86%	-1.70%	-1.78%
1.0%	-1.99%	-1.41%	-1.42%	-1.26%	-1.28%	-1.30%	-1.36%	-1.32%	-1.35%	-1.42%
5.0%	-1.04%	-0.29%	-0.36%	-0.24%	-0.30%	-0.28%	-0.36%	-0.31%	-0.29%	-0.34%
25.0%	0.39%	1.23%	1.23%	1.26%	1.19%	1.17%	1.21%	1.22%	1.22%	1.20%
50.0%	1.36%	2.31%	2.31%	2.25%	2.23%	2.31%	2.24%	2.26%	2.24%	2.22%
75.0%	2.39%	3.32%	3.30%	3.29%	3.25%	3.34%	3.28%	3.30%	3.29%	3.23%
95.0%	3.87%	4.89%	4.76%	4.80%	4.79%	4.82%	4.82%	4.82%	4.84%	4.76%
99.0%	4.88%	5.91%	5.86%	5.85%	5.77%	5.94%	5.87%	5.80%	5.92%	5.72%
99.5%	5.25%	6.34%	6.22%	6.30%	6.06%	6.28%	6.22%	6.18%	6.37%	5.95%

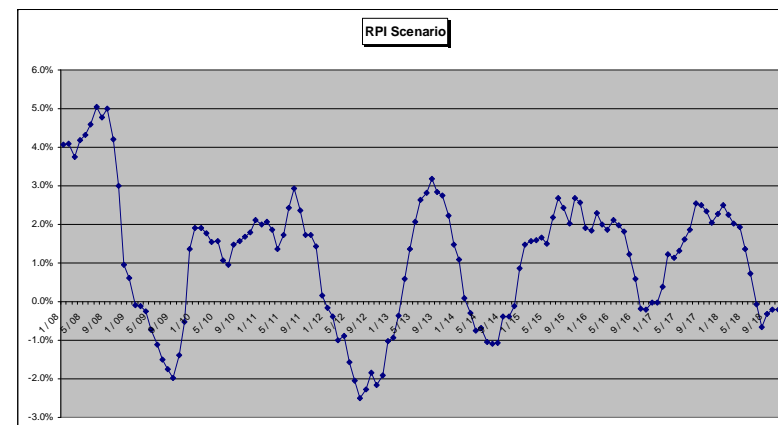
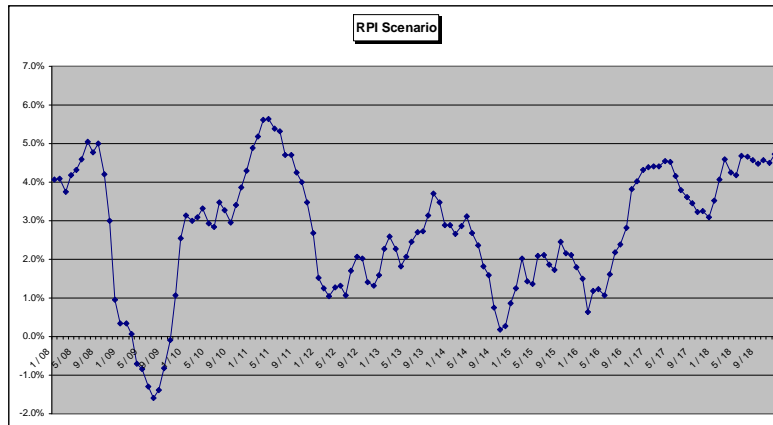
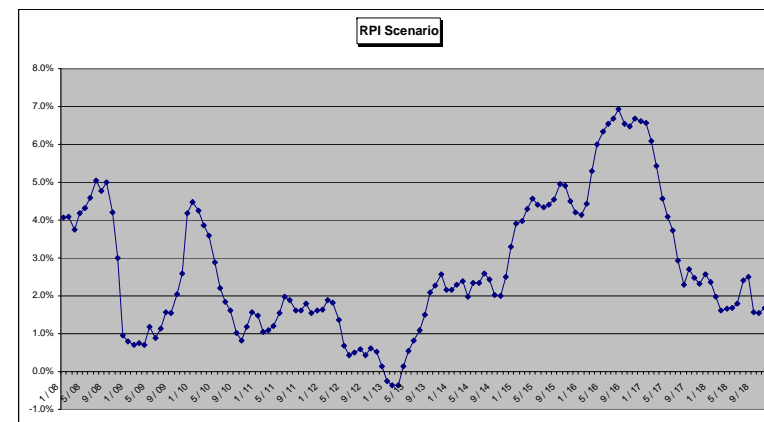
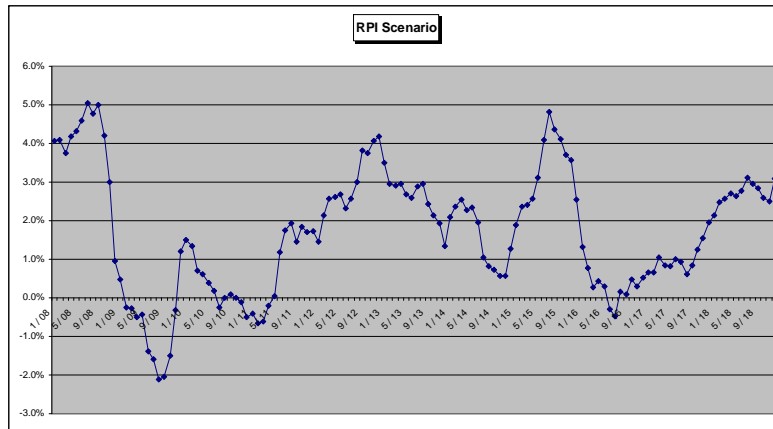
Case Studies

RPI Case Study – Future Projections



Case Studies

RPI Case Study – Four Random Scenarios (Press F9)



Case Studies

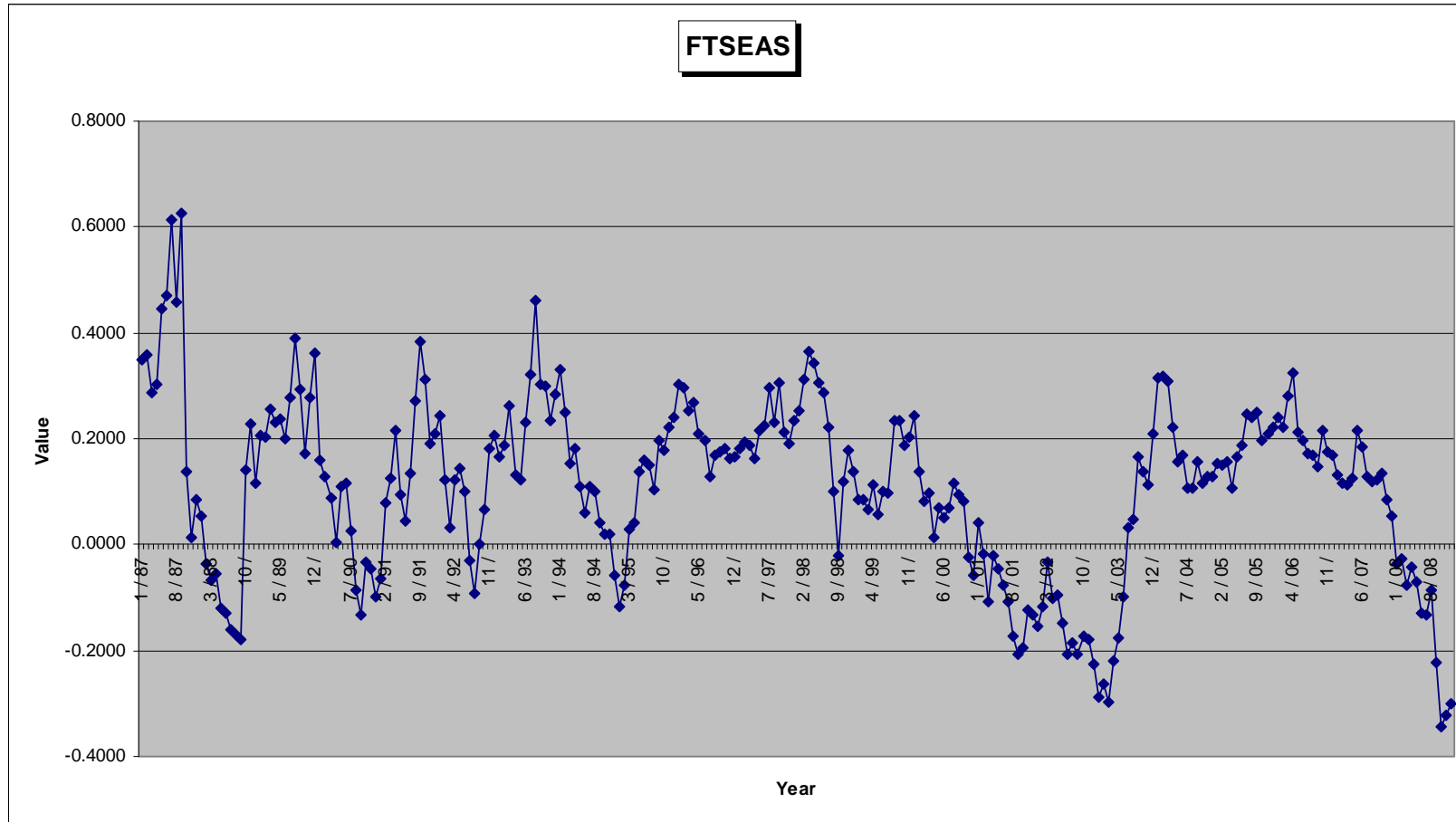
FTSE All Share Case Study – Data

- Data
 - Monthly data has been used
 - FTSEASTR(t) – FTSE All Share Total Return Index at the end of each month for the period Jan 1987 to Dec 2008 as provided by Bloomberg
 - Constructed an historical time series of a month rolling value of FTSEAS(t) at month t, where:
$$\text{FTSEAS}(t) = \text{FTSEAS Annual Return} = \text{FTSEASTR}(t) / \text{FTSEASTR}(t-12) - 1$$
- ARIMA(1,[12]) Model Fit
 - Monthly data Jan 1987 to Dec 2008
 - Box-Jenkins Diagnostic Evaluation tests OK
 - Relatively largish residuals but still random
 - Simulation of 5,000 path-dependent scenarios of length 120 months

$$\begin{aligned}\text{FTSEAS}(t) &= 0.07479 + Y(t) \\ Y(t) &= 0.97975 Y(t-1) - 0.93485 e(t-12) + e(t) \\ e(t) &\sim N(0.00000, 0.05191)\end{aligned}$$

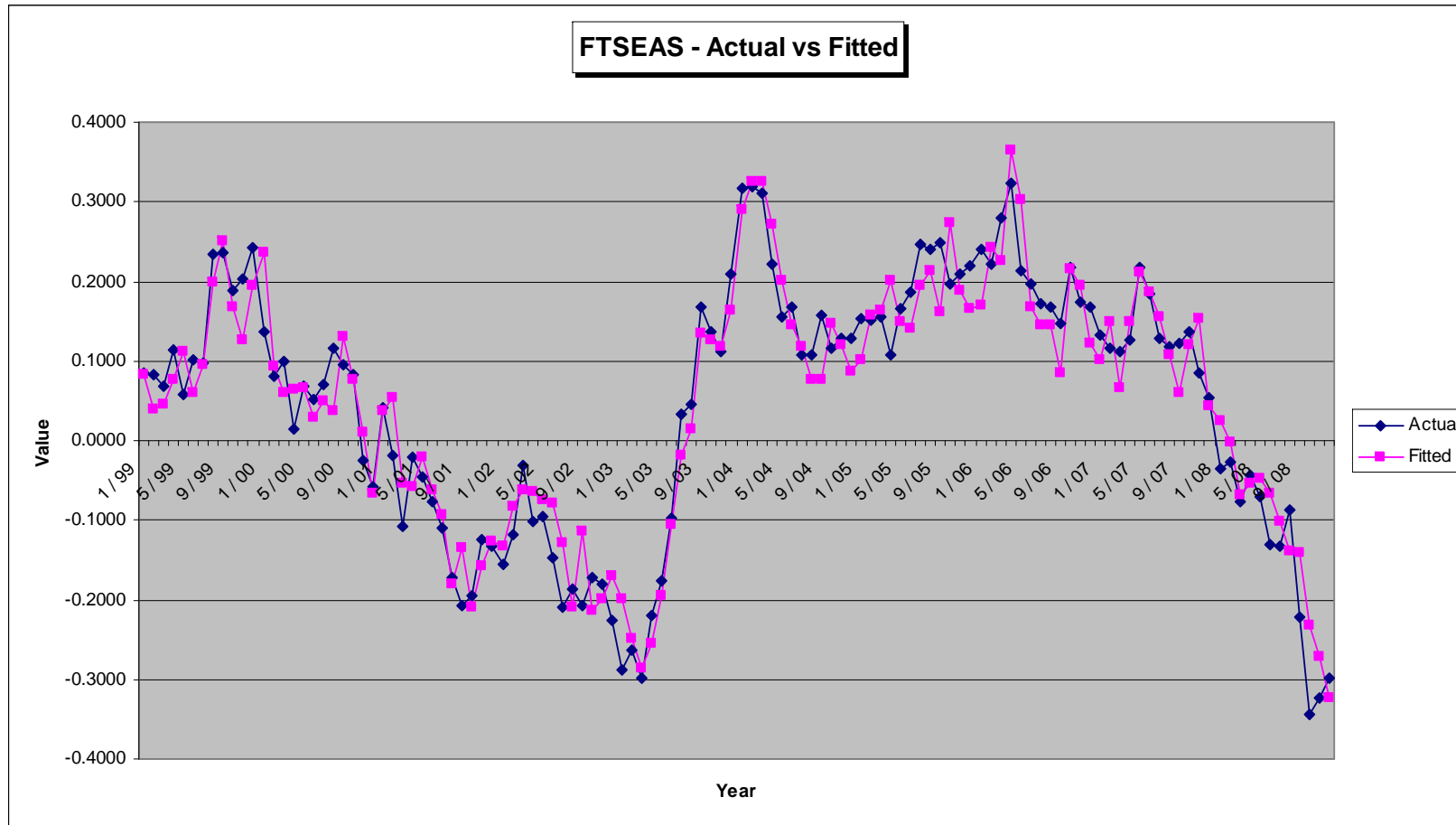
Case Studies

FTSE All Share Case Study – Annual FTSEAS Data (1/87 to 12/08)



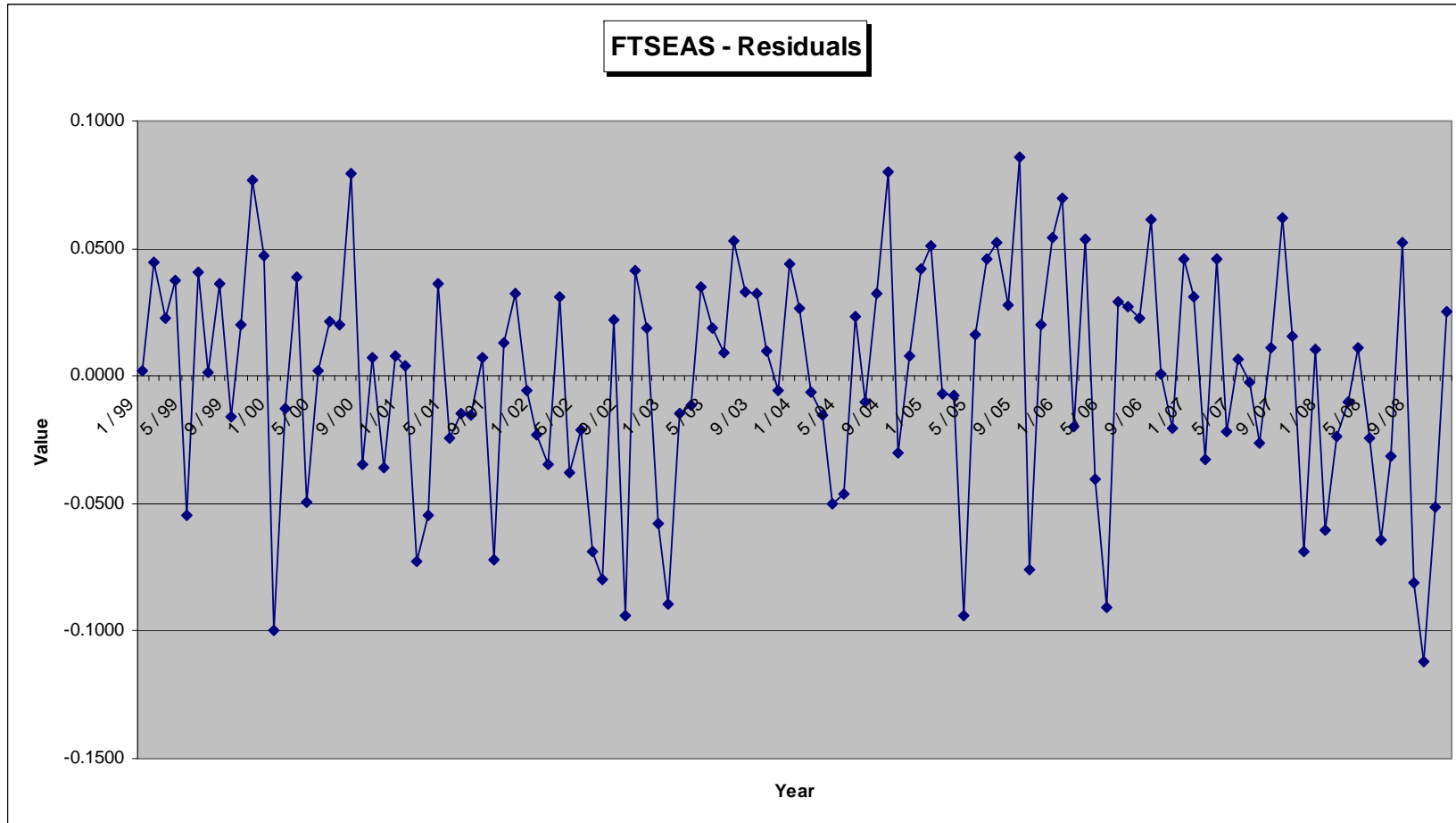
Case Studies

FTSE All Share Case Study – Actual vs Fitted (Last 10 years shown)



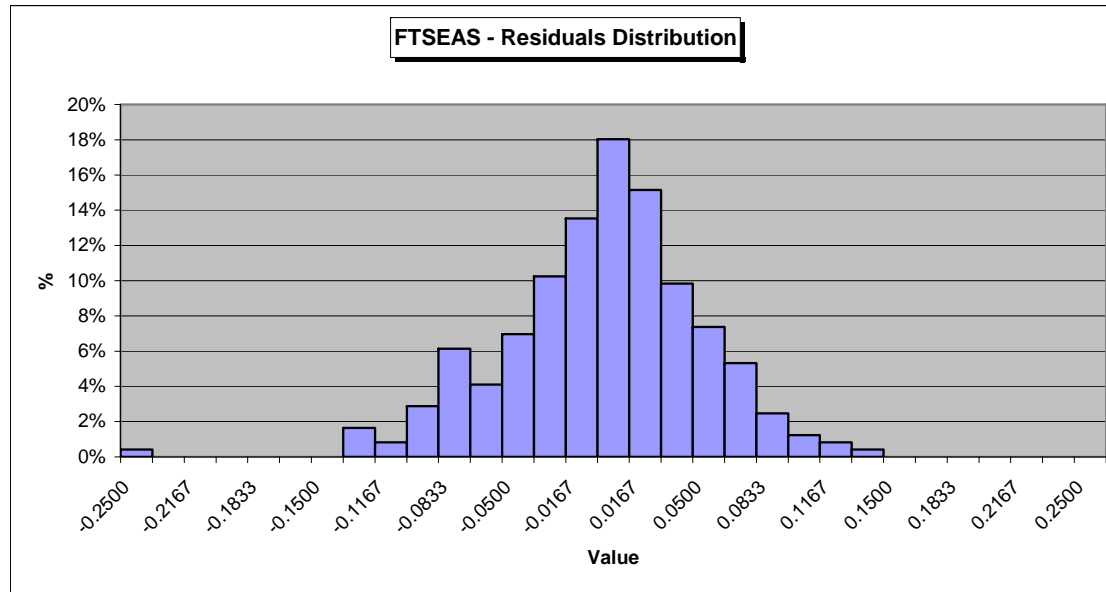
Case Studies

FTSE All Share Case Study – Residuals (Last 10 years shown)



Case Studies

FTSE All Share Case Study – Residuals Distribution (All years)



Residuals	
Sample No.	263
Mean	0.002808
Minimum	-0.246123
Maximum	0.137776
Std Dev	0.051908
Skewness	-0.562577
Kurtosis	4.535745

Case Studies

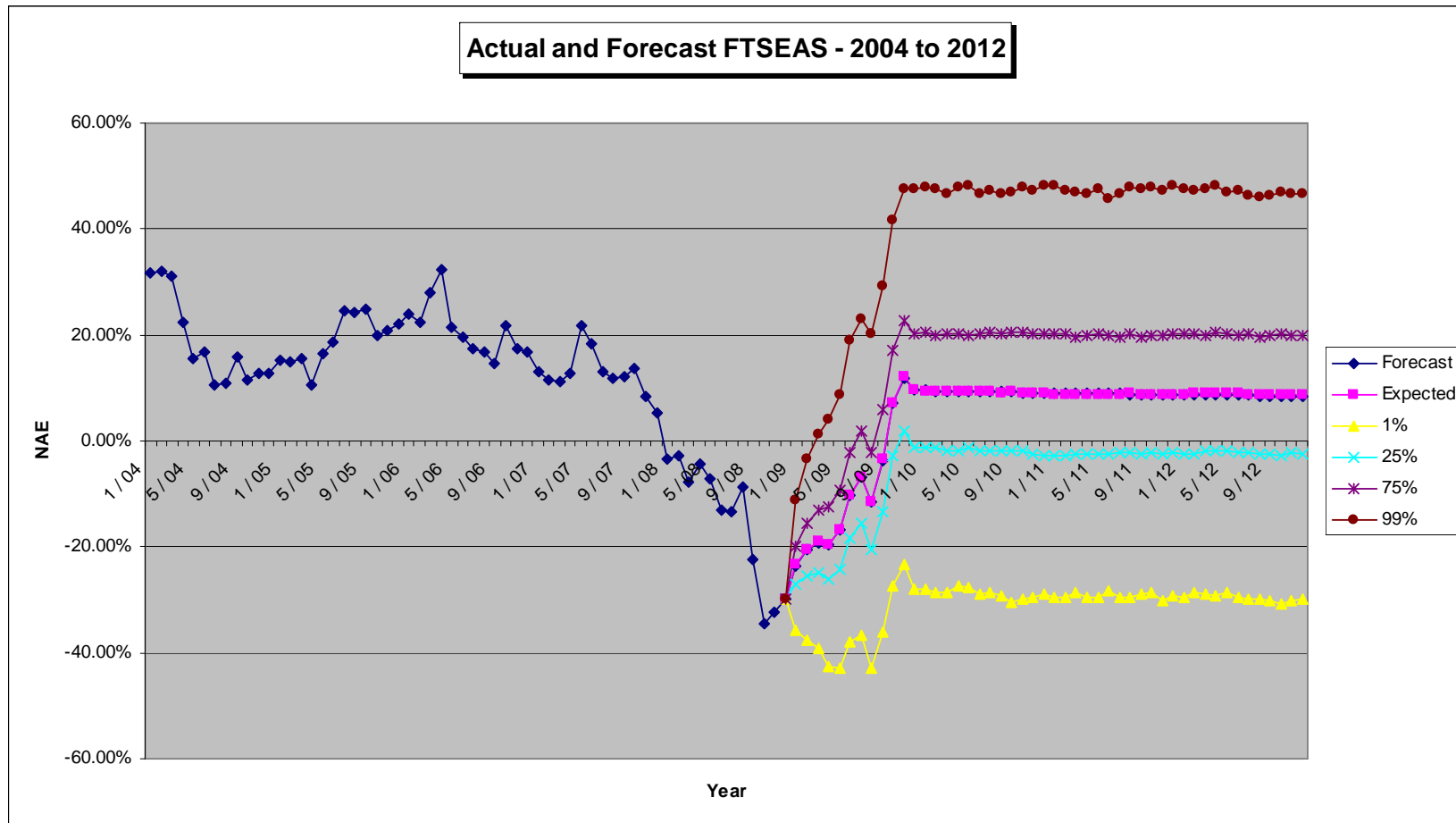
FTSE All Share Case Study – Model Fit and Future Projections

FTSEAS			
Start	1987		
End	2008		
Variable	Coefficient	t-statistic	Probability
C	0.07479	2.057	4.07%
Y(t-1)	0.97975	64.273	0.00%
e (t-12)	-0.93485	-78.133	0.00%
Adj R ²	90.1%		
Durbin Watson	1.8982		
SSR	0.7080		
AIC	-3.0567		
SC	-3.0160		

	12 / 09	12 / 10	12 / 11	12 / 12	12 / 13	12 / 14	12 / 15	12 / 16	12 / 17	12 / 18
Forecast	9.53%	9.08%	8.73%	8.46%	8.25%	8.08%	7.95%	7.85%	7.77%	7.70%
Expected	9.60%	8.97%	8.74%	8.69%	7.93%	8.47%	7.92%	8.11%	7.90%	7.47%
Standard Deviation	16.15%	16.95%	16.63%	16.65%	16.59%	16.78%	16.64%	16.71%	16.49%	16.71%
Minimum	-46.81%	-58.43%	-48.94%	-50.47%	-65.04%	-52.33%	-55.41%	-58.17%	-61.01%	-73.11%
Maximum	64.70%	71.63%	70.16%	64.54%	63.85%	74.63%	70.24%	79.36%	84.84%	68.42%
Percentile										
0.5%	-32.46%	-34.16%	-33.22%	-34.09%	-34.18%	-32.63%	-32.75%	-34.93%	-34.49%	-34.11%
1.0%	-27.88%	-28.91%	-29.30%	-29.93%	-30.46%	-29.02%	-29.77%	-30.07%	-30.88%	-31.18%
5.0%	-16.64%	-18.50%	-18.70%	-19.43%	-19.62%	-18.98%	-19.39%	-18.98%	-19.44%	-19.94%
25.0%	-1.24%	-2.82%	-2.29%	-2.36%	-3.15%	-3.38%	-3.46%	-3.00%	-3.18%	-3.88%
50.0%	9.28%	8.83%	9.09%	9.02%	7.93%	8.44%	7.86%	8.19%	7.89%	7.32%
75.0%	20.35%	20.27%	20.12%	19.87%	19.14%	19.97%	19.16%	18.97%	18.74%	18.97%
95.0%	36.12%	37.15%	35.83%	36.30%	35.22%	36.00%	35.28%	36.22%	35.42%	34.91%
99.0%	47.53%	48.27%	48.18%	46.62%	45.53%	48.05%	47.32%	47.12%	45.73%	45.82%
99.5%	51.84%	52.94%	52.45%	50.54%	49.29%	52.37%	51.51%	49.60%	50.70%	48.94%

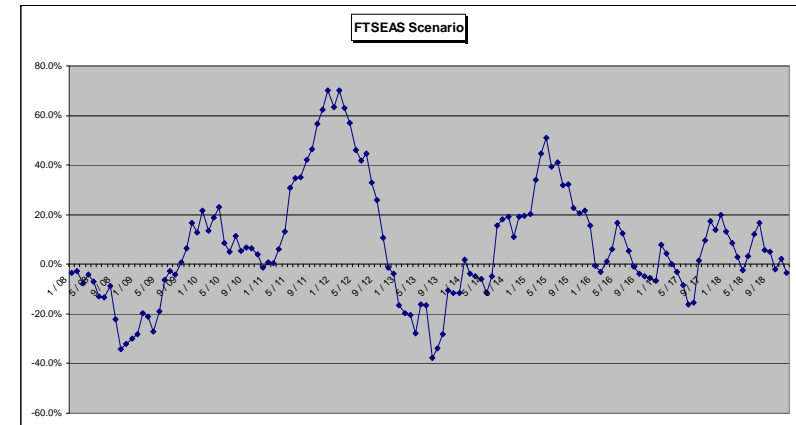
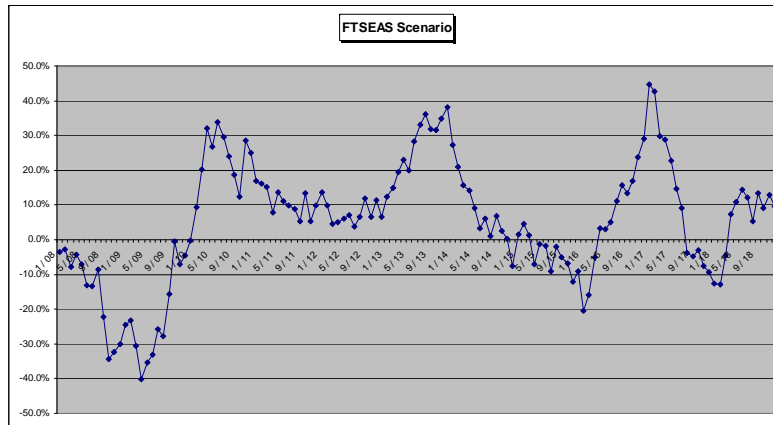
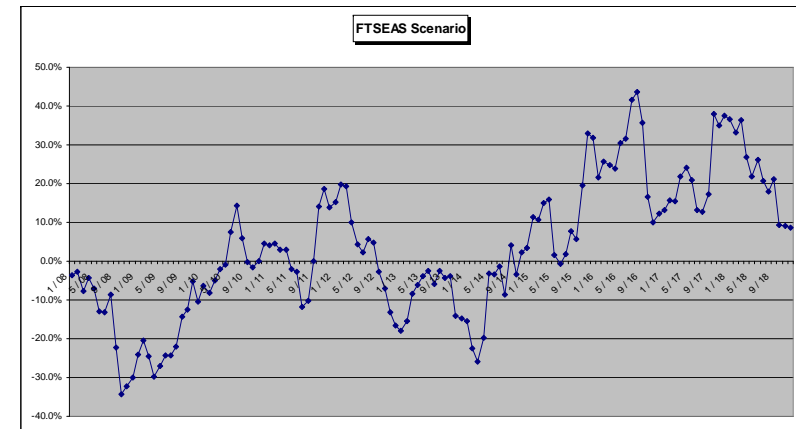
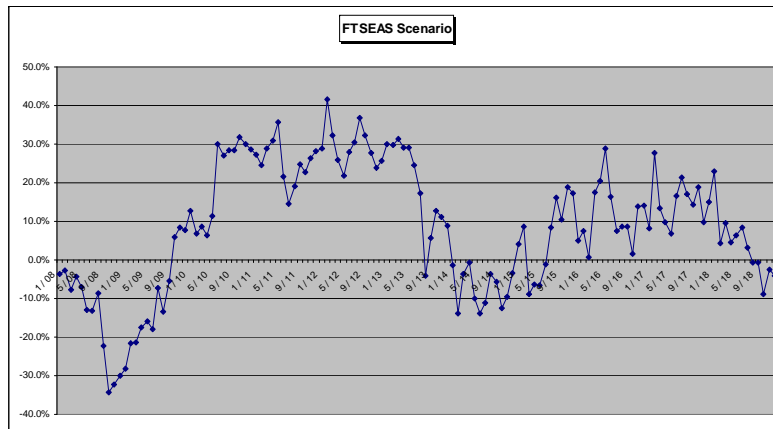
Case Studies

FTSE All Share Case Study – Future Projections



Case Studies

FTSE All Share Case Study – Four Random Scenarios (Press F9)



Case Studies

Underwriting (“UW”) Cycle Case Study – Risk Drivers *

- Target variable y_t
 - The concern here is price. If a company cannot compete at the prevailing price then it will lose money or business, yet price is multidimensional
 - Most analyses focus on some form of profitability measure such as the loss ratio or combined ratio with possible adjustments for the time value of money
- There are many potential explanatory variables:
 - Prior period values of profitability and its components
 - Other internal financial variables such as reserves, investment income, catastrophe losses, total capital and reinsurance
 - Regulatory / ratings variables – especially upgrades and downgrades
 - Reinsurance section financials
 - Economic variables such as inflation, unemployment and GNP
 - Financial market variables such as interest rates and stock market returns

* Enterprise Risk Analysis for Property & Liability Insurance Companies”; (2007); Guy Carpenter

Case Studies

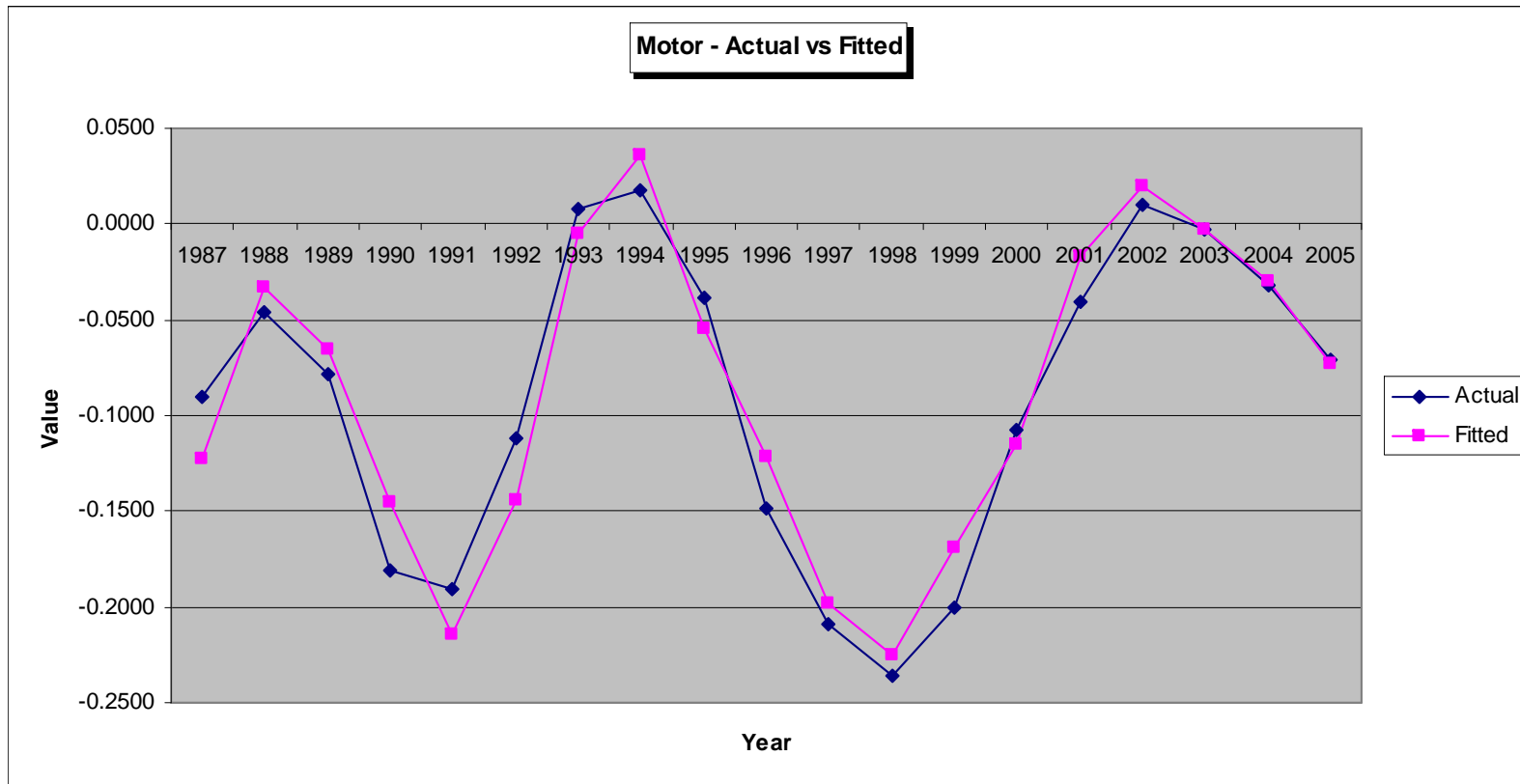
UW Cycle Case Study – Data

- Data
 - Annual data has been used
 - Annual Underwriting Profit as % of Net Written Premium for the FSA Motor insurance class grouping at an overall UK industry level.
 - *[Data by FSA insurance class grouping was provided to me. I have not been able to verify independently the data. The analysis therefore is more for illustration purposes only]*
- ARIMA(2,[3]) Model Fit
 - Annual data 1987 to Dec 2005
 - Box-Jenkins Diagnostic Evaluation tests OK
 - Not a large volume of data
 - Residuals OK but do not appear as random, more a data volume issue

$$\begin{aligned}\text{Motor}(t) &= -0.09598 + Y(t) \\ Y(t) &= 1.37739 Y(t-1) - 0.81563 Y(t-2) - 0.98131 e(t-3) + e(t) \\ e(t) &\sim N(-0.00370, 0.02046)\end{aligned}$$

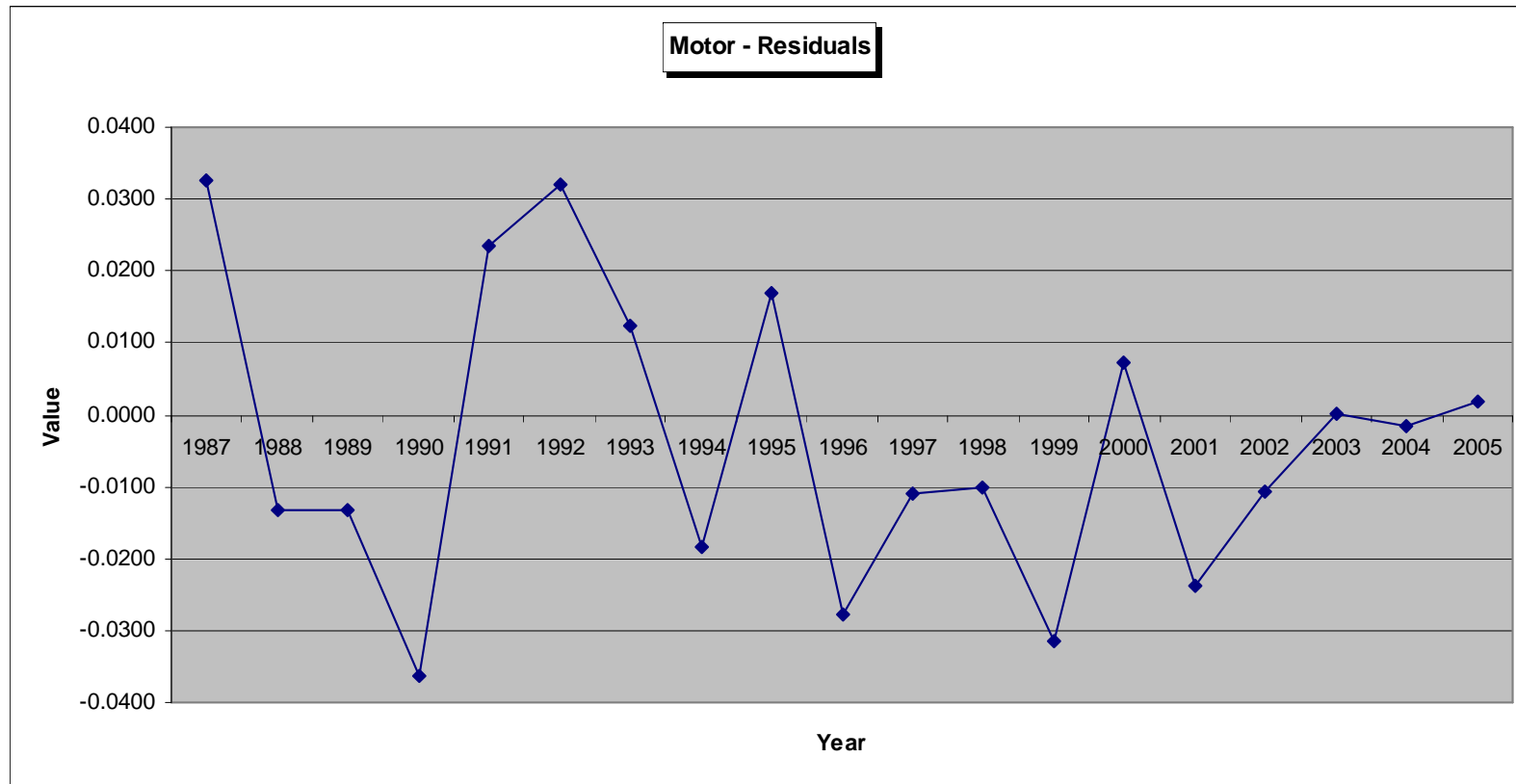
Case Studies

UW Cycle Case Study – Actual vs Fitted (All years)



Case Studies

UW Cycle Case Study – Residuals (All years)



Case Studies

UW Cycle Case Study – Model Fit

Motor			
Start	1987		
End	2005		
Variable	Coefficient	t-statistic	Probability
C	-0.09598	-20.794	0.00%
Y(t-1)	1.37739	9.340	0.00%
Y(t-2)	-0.81563	-5.325	0.01%
e(t-3)	-0.98131	-10.934	0.00%
Adj R ²	92.2%		
Durbin Watson	1.8374		
SSR	0.0228		
AIC	-4.5397		
SC	-4.3409		

Conclusions

Conclusions

- Time Series modelling techniques can provide an informative insight
 - It is helpful if target variables are functions of explanatory variables or prior values of itself that have economic or business rationale
 - Avoid over-parameterised models – in-sample vs out-of-sample testing
- A visual inspection of the data is key to any analysis
- Models fits need to be supported by rigorous statistical diagnostics:
 - It is far too easy to determine optimal models and parameters that fail basic statistical tests such as those for t-statistics and autocorrelation in residuals
 - If the Model fails these tests one needs to try a different model
- Test sensitivity of the model parameters and forecasts to different start and end periods

Q&A

Q&A

- Questions ?