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THE VALUATION OF WHOLE-LIFE ASSURANCES BY THE USE OF MOMENTS

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W. Perks in his recent paper (see p. 377 of this issue of the *Journal*) has shown how we may avoid tabulating net premiums or other special functions applicable to a particular valuation basis on individual valuation cards, and how a change may thus be made easily and rapidly from one valuation basis to another. The primary purpose of the present note is to develop Perks's principles and to test the methods on the actual valuation of the whole-life assurances of a life office.

Machinery of the new principles

If x denotes the age at entry of an assured person and t denotes the duration of a policy, the sum assured being S_{xt}, as many of the following moments as may be required for the particular formulae used are obtained from the valuation data:

$$\Sigma S_{xt};\quad \Sigma x S_{xt},\ \Sigma t S_{xt};\quad \Sigma x_{(2)} S_{xt},\ \Sigma xt S_{xt},\ \Sigma t_{(2)} S_{xt};$$
$$\Sigma x_{(3)} S_{xt},\ \Sigma x_{(2)} t S_{xt},\ \Sigma x t_{(2)} S_{xt},\ \Sigma t_{(3)} S_{xt};\quad \text{etc.}$$

Various ways of obtaining the moments

(A) The products, xS_x, $x_{(2)} S_x$, $x_{(3)} S_x$, etc., to the order required for the particular formulae used, are inscribed on the cards. All information on the cards is tabulated according to year of entry.

(B) Two products only, xS_x, $x_{(2)} S_x$, are inscribed on the cards. All information is tabulated both according to year of entry and year of birth. The moments of order 0, 1 and 2 are derived from the year of entry classification. So also are the following moments of order 3 and 4, viz. $\Sigma x_{(2)} t S_{xt}$, $\Sigma x t_{(2)} S_{xt}$, $\Sigma t_{(3)} S_{xt}$, $\Sigma x_{(2)} t_{(2)} S_{xt}$, $\Sigma x t_{(3)} S_{xt}$ and $\Sigma t_{(4)} S_{xt}$. $\Sigma x_{(3)} S_{xt}$ is obtained from the relation

$$\Sigma x_{(3)} S_{xt} = \Sigma \{\tfrac{1}{3} x_{(2)} (x+t) - \tfrac{1}{3} x_{(2)} t - \tfrac{2}{3} x_{(2)}\} S_{xt},$$

which uses one moment derived from the year of birth classification. Similar but more complicated expressions can be found for $\Sigma x_{(4)} S_{xt}$ and $\Sigma x_{(3)} t S_{xt}$. All the fifth moments can also be computed.

(C) The product xS_x is inscribed on the cards, and all information is tabulated both according to year of entry and year of birth. All the moments up to the third order may be computed. The moments which require use of the year of birth classification are:

$$\Sigma x_{(2)} S_{xt} = \Sigma \{(x+t)_{(2)} - xt - t_{(2)}\} S_{xt},$$
$$\Sigma x_{(3)} S_{xt} = \Sigma \{x (x+t)_{(2)} - 2 (x+t)_{(3)} - 2 (x+t)_{(2)} + xt + xt_{(2)} + 2t_{(2)} + 2t_{(3)}\} S_{xt},$$
$$\Sigma x_{(2)} t S_{xt} = \Sigma \{3 (x+t)_{(3)} - x (x+t)_{(2)} + 2 (x+t)_{(2)} - xt - 2xt_{(2)} - 2t_{(2)} - 3t_{(3)}\} S_{xt}.$$

There are two checks on the tabulations besides the obvious dual computations of ΣS_{xt} and $\Sigma x S_{xt}$, viz.

$$\Sigma (x+t) S_{xt} = \Sigma x S_{xt} + \Sigma t S_{xt},$$
$$2\Sigma (x+t)_{(2)} S_{xt} + \Sigma x S_{xt} = \Sigma x (x+t) S_{xt} + \Sigma xt S_{xt} + 2\Sigma t_{(2)} S_{xt}.$$

(D) No products are inscribed on the cards but three classification books are kept, one according to year of entry, one according to year of birth, and one according to age at entry.

All the moments up to the second can be obtained:

$\Sigma x S_{xt}$ and $\Sigma x_{(2)} S_{xt}$ from the age at entry classification,

$\Sigma t S_{xt}$ and $\Sigma t_{(2)} S_{xt}$ from the year of entry classification,

$\Sigma xt S_{xt}$ from the relation $\Sigma xt S_{xt} = \Sigma\{(x+t)_{(2)} - x_{(2)} - t_{(2)}\} S_{xt}$,

which involves the year of birth classification.

A check on the classifications follows from

$$\Sigma (x+t) S_{xt} = \Sigma x S_{xt} + \Sigma t S_{xt}.$$

Bonuses and office premiums need only be tabulated according to year of birth.

The three classifications cannot be made independently of one another. Two only of the classifications are entirely at our disposal and the third is a consequence of the other two. For example, it would be wrong to obtain the t moments by classifying according to calendar year of entry, the x moments by classifying according to nearest age at entry, and the $x+t$ moments by classifying according to an office year of birth based on nearest attained age at the end of a calendar year.

It should be noted that if it is desired to employ method (A), (B) or (C) it is not necessary to inscribe products on existing valuation cards. The cards can be sorted first according to year of entry and subdivided according to age at entry. This will enable the totals of the respective products $\Sigma x S$, $\Sigma x_{(2)} S$, etc., to be computed and tabulated for each year of entry. For method (B) or (C) a similar process may also be applied, the primary group being year of birth.

Policies with special net premiums

There are several ways of bringing such special cases into the valuation.

(1) A special valuation year of entry may be calculated.

(2) The sum assured may be considered to consist of two parts—a normal sum assured corresponding to the net premium and a free-policy sum assured. The former would be recorded as the normal sum assured of the policy, and the latter would be recorded specially in the year of birth tabulations and valued separately.

(3) The net premium may be regarded as being the normal net premium reduced by a certain fixed amount, which would be recorded in the year of birth tabulations and valued specially.

The three methods would give slightly different results on a change of valuation basis, but it is difficult to maintain that any one of them is more correct than any other.

How the moments are used for the valuation of whole-life assurances

There are two methods:

Perks's Method. The valuation factor $V(x, t)$ over the whole range is expressed in terms of its values at particular points, and hence an expression for the value of $\Sigma S_{xt} V(x, t)$ can be obtained. Numerous formulae will be found in Perks's paper. Three more formulae are given in section 2 of the Appendix to this note, viz. a five-point formula correct to the second order, a six-point formula correct to the second order, and a twelve-point formula correct to the third order.

An extension to two variables of Henry's method for one variable. Instead of making $V(x, t)$ over the range depend on the values of $V(x, t)$ at a number of well-chosen points, we may find the best values of the constants in the expansion of $V(x, t)$ as a polynomial in x and t. If $V(x, t)$ may be represented by the second-degree curve $a+bx+ct+dx_{(2)}+ext+ft_{(2)}$, we find a, b, c, d, e and f by taking moments up to the second order. In matrix notation we have

$$\begin{bmatrix} \Sigma 1 & \Sigma x & \Sigma t & \Sigma x_{(2)} & \Sigma xt & \Sigma t_{(2)} \\ \Sigma x & \Sigma x^2 & \Sigma xt & \Sigma xx_{(2)} & \Sigma x^2 t & \Sigma xt_{(2)} \\ \Sigma t & \Sigma xt & \Sigma t^2 & \Sigma x_{(2)} t & \Sigma xt^2 & \Sigma tt_{(2)} \\ \Sigma x_{(2)} & \Sigma xx_{(2)} & \Sigma x_{(2)} t & \Sigma (x_{(2)})^2 & \Sigma xx_{(2)} t & \Sigma x_{(2)} t_{(2)} \\ \Sigma xt & \Sigma x^2 t & \Sigma xt^2 & \Sigma xx_{(2)} t & \Sigma x^2 t^2 & \Sigma xtt_{(2)} \\ \Sigma t_{(2)} & \Sigma xt_{(2)} & \Sigma tt_{(2)} & \Sigma x_{(2)} t_{(2)} & \Sigma xtt_{(2)} & \Sigma (t_{(2)})^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} \Sigma V(x, t) \\ \Sigma xV(x, t) \\ \Sigma tV(x, t) \\ \Sigma x_{(2)} V(x, t) \\ \Sigma xtV(x, t) \\ \Sigma t_{(2)} V(x, t) \end{bmatrix}$$

The valuation of the liability is then

$$\Sigma S_{xt} V(x, t) = a\Sigma S_{xt} + b\Sigma x S_{xt} + c\Sigma t S_{xt} + d\Sigma x_{(2)} S_{xt} + e\Sigma xt S_{xt} + f\Sigma t_{(2)} S_{xt}.$$

In order to obtain accurate results it is necessary that, in the computation of a, b, c, d, e and f, the range over which the summations extend should follow fairly closely the distribution of S_{xt}. For the whole-life table S_{xt} will be comprised within a triangle bounded approximately by $x = 15$, $t = 0$, $x+t = 90$. If it is appropriate to use such a triangle the laborious solution of six linear equations can be simplified by the use of orthogonal polynomials. A development of orthogonal polynomials for two variables, where the range over which the summations extend is a triangle, is given in section 1 of the Appendix to this note. The mathematics necessary to understand the use of these polynomials is simple and is given briefly below.

If $\phi_1, \phi_2, \ldots, \phi_n$ are a series of polynomials in x and t, such that $\Sigma\phi_r\phi_s = 0$ for $r \neq s$, where the summations extend over a range which covers or nearly covers the distribution S_{xt}, then we can fit $V(x, t)$ to a polynomial of the form $a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n$ very easily.

For, if we multiply both $V(x, t)$ and $a_1\phi_1 + \ldots + a_n\phi_n$ by ϕ_1, sum over the range, and equate the two results together, we get

$$a_1\Sigma\phi_1^2 = \Sigma\phi_1 V(x, t),$$

so that
$$a_1 = \Sigma\phi_1 V(x, t)/\Sigma\phi_1^2.$$

Similarly
$$a_2 = \Sigma\phi_2 V(x, t)/\Sigma\phi_2^2, \quad \text{and so on.}$$

Hence, if the resulting fit of $a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n$ to $V(x, t)$ is reasonably good,

$$\begin{aligned} \Sigma S_{xt} V(x, t) &= \Sigma S_{xt}(a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n) \\ &= a_1\Sigma\phi_1 S_{xt} + a_2\Sigma\phi_2 S_{xt} + \ldots + a_n\Sigma\phi_n S_{xt} \\ &= \frac{\Sigma\phi_1 S_{xt}\Sigma\phi_1 V(x, t)}{\Sigma\phi_1^2} + \frac{\Sigma\phi_2 S_{xt}\Sigma\phi_2 V(x, t)}{\Sigma\phi_2^2} + \ldots + \frac{\Sigma\phi_n S_{xt}\Sigma\phi_n V(x, t)}{\Sigma\phi_n^2}. \end{aligned}$$

$\phi_1, \phi_2, \ldots, \phi_n$ are fairly simple polynomials in terms of x and t, and thus $\Sigma\phi_1 S_{xt}$, etc., are easily derived from the moments of S_{xt}. Likewise $\Sigma\phi_1 V(x, t)$, etc., can be obtained from moments of the valuation factors. Expressions for ϕ_1, etc., and $\Sigma\phi_1^2$, etc., are given on p. 510 under a different notation, viz. $\phi_1 = P_{00}(x, y)$, $\phi_2 = P_{01}(x, y)$, etc.

Better results may sometimes be obtained by using weights in computing the constants. Thus, if the weights are $g(x, t)$, the orthogonal polynomials are chosen such that $\Sigma g(x,t)\phi_r\phi_s = 0$ for $r \neq s$. We compute a_1 by multiplying both $V(x,t)$ and $a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n$ by $g(x,t)\phi_1$, summing over the range, and equating the two results together. We get

$$a_1\Sigma g(x,t)\phi_1^2 = \Sigma g(x,t)\phi_1 V(x,t).$$

Hence $$\Sigma S_{xt}V(x,t) = \frac{\Sigma\phi_1 S_{xt}\Sigma g(x,t)\phi_1 V(x,t)}{\Sigma g(x,t)\phi_1^2} + \ldots + \frac{\Sigma\phi_n S_{xt}\Sigma g(x,t)\phi_n V(x,t)}{\Sigma g(x,t)\phi_n^2}.$$

Expressions for ϕ_1, etc., and $\Sigma g(x,t)\phi_1^2$, etc., where $g(x,t) = 1/(x+t+1)$, are given on p. 510 under a different notation, viz. $\phi_1 = P_{00}(x,y)$, $\phi_2 = P_{01}(x,y)$, etc. This choice of $g(x,t)$ is sometimes useful in distributions where S_{xt} becomes somewhat sparse as $x+t$ increases, a feature which might be expected in the case of the whole-life assurances of a life office.

Test of the above methods

Method (D) was used to present the data of the whole-life without profit business of a life office as at 31 December 1944. In order to simplify comparison of the results with an accurate valuation, a few cases where special net premiums had been used were eliminated.

The three classifications used were: (1) year of entry, (2) an office year of birth such that subtraction of this year from the valuation year gave the nearest age at 31 December of the valuation year, and (3) the difference between the year of entry and the office year of birth. The average duration at 31 December 1944 is half a year more than the difference between 1944 and the year of entry, and the average age at entry is half a year less than the difference between the year of entry and the office year of birth. Both these half-years were omitted in the tabulations and the error was corrected by valuing by $V(x-\frac{1}{2}, t+\frac{1}{2})$.

The data extended over duration $t = 0$ to $t = 34$, the bulk of the business being of less than 15 years' duration. The youngest age at entry, x, was 14 and the oldest was 72. $x+t$, the nearest age at 31 December 1944, extended from 18 to 90. Office premiums were tabulated against valuation age $x+t$. There is nothing to be gained by using approximate methods for the valuation of the office premiums.

By repeated summations the following moments were obtained:

$\Sigma S_{xt} = 265{,}101{\cdot}6$, $\Sigma(x-40)S_{xt} = 1{,}016{,}492{\cdot}3$, $\Sigma(t-8)S_{xt} = 592{,}297{\cdot}35$, $\Sigma(x+t-54)S_{xt} = 18{,}180{\cdot}05$, $\Sigma(x-40)_{(2)}S_{xt} = 20{,}018{,}782$, $\Sigma(t-8)_{(2)}S_{xt} = 7{,}506{,}489{\cdot}1$, $\Sigma(x+t-54)_{(2)}S_{xt} = 24{,}875{,}442{\cdot}25$.

There was no particular virtue in the choice of 40, 8 and 54 as origins. They were rough guesses at the respective means.

It will be noted that we have the check

$$\Sigma(x+t-54)S_{xt} = \Sigma(x-40)S_{xt} + \Sigma(t-8)S_{xt} - 6\Sigma S_{xt}.$$

In actual fact this check enabled an error in the tabulations to be discovered.

For Perks's method it was necessary to calculate $\bar{x}$, $\bar{t}$, σ_x, σ_t and r_{xt}. For this purpose it was useful to note that $\sigma_x^2 = 2\dfrac{\Sigma(x-a)_{(2)}S_{xt}}{\Sigma S_{xt}} - 2(\bar{x}-a)_{(2)}$, where a is any origin. Similar formulae hold for σ_t^2 and $\Sigma(x+t-\bar{x}-\bar{t})^2 S_{xt}/\Sigma S_{xt}$, and $\Sigma(x-\bar{x})(t-\bar{t})S_{xt}/\Sigma S_{xt} = \frac{1}{2}\{\Sigma(x+t-\bar{x}-\bar{t})^2 S_{xt}/\Sigma S_{xt} - \sigma_x^2 - \sigma_t^2\}$. The statistical measures of the distribution were:

$$\bar{x} = 43{\cdot}834, \quad \bar{t} = 10{\cdot}234, \quad \sigma_x = 11{\cdot}839, \quad \sigma_t = 7{\cdot}340, \quad r_{xt} = -{\cdot}036260.$$

The results of the valuation by Perks's method using the A 1924–29 ultimate table at $2\frac{3}{4}$ % were as follows:

	True	Perks's formula (9)	Error %	5-point formula of Appendix, section 2	Error %	6-point formula of Appendix, section 2	Error %
	£	£		£		£	
Net premiums	7,540	7,453	−1·15	7,490	−·66	7,489	−·68
Value of sums assured	158,001	157,948	−·03	158,003	·00	158,075	·05
Value of net premiums	92,493	92,292	−·22	91,619	−·94	91,802	−·75
Net liability	65,508	65,656	·23	66,384	1·34	66,273	1·17

Two-variable Henry's method

It is too laborious to use each age in calculating the constants. It was decided to use every third age and to change the origin to $x = 20$, $t = 0$. The moments of the data to be valued corresponding to this change of origin and scale have to be obtained. If dashed symbols denote the original units and undashed the new units, we have

$$x = \frac{x' - 20}{3}, \quad t = \frac{t'}{3},$$

and the new moments may easily be obtained, e.g.

$$\Sigma t_{(2)} S_{xt} = \tfrac{1}{9}\Sigma\{(t'-8)_{(2)} + 7(t'-8) + 20\} S_{xt} = 1{,}883{,}845.$$

The new moments were:

$$\Sigma S_{xt} = 265{,}101{\cdot}6, \quad \Sigma x S_{xt} = 2{,}106{,}175, \quad \Sigma t S_{xt} = 904{,}370,$$
$$\Sigma x_{(2)} S_{xt} = 9{,}377{,}712, \quad \Sigma xt S_{xt} = 7{,}092{,}213, \quad \Sigma t_{(2)} S_{xt} = 1{,}883{,}845.$$

In the first place an attempt was made to use orthogonal polynomials over the range bounded by $x = 0$, $t = 0$, $x + t = 22$, ($x' = 20, t' = 0, x' + t' = 86$ under the original origin and scale). The various moments for the functions $A_{3x+3t+20}$ and $\pi_{3x+19\frac{1}{2}}(\frac{1}{2} + a_{3x+3t+20})$ over this range were found by summation and multiplication, and also the weighted moments for these functions using weights $1/(x+t+1)$. It was thought that, in view of the sparseness of the data at the older ages, weighted formulae might give better results. The formulae used were those given on p. 510, n being taken as 23. The results are given in the following table.

	True	Unweighted formula	Error %	Weighted formula	Error %
	£	£		£	
Value of sums assured	158,001	158,441	·28	157,826	− ·11
Value of net premiums	92,493	93,712	1·32	93,241	·81
Net liability	65,508	64,729	−1·19	64,585	−1·41

The disappointing nature of the results was not entirely unexpected because the assumption that the data were distributed over a triangle was rather wide of the mark. In fact there was practically no business for $t > 10$ ($t' > 30$) or for $x > 17$ ($x' > 71$), so that over one-third of the range over which $A_{3x+3t+20}$ and $\pi_{3x+19\frac{1}{2}}(\frac{1}{2} + a_{3x+3t+20})$ had been graduated contained no data. (Orthogonal formulae over a triangle are likely to be more successful for endowment assurances.)

It was decided, therefore, to graduate the valuation factors by the formula $a+bx+ct+dx_{(2)}+ext+ft_{(2)}$ over the triangle bounded by $x=0, t=0, x+t=22$, where durations greater than $t=10$ and ages at entry greater than $x=17$ were excluded. The following equations (in matrix notation) resulted:

$$\begin{bmatrix} 183 & 1{,}448 & 860 & 7{,}241 & 6{,}390 & 2{,}460 \\ 1{,}448 & 15{,}930 & 6{,}390 & 91{,}405 & 66{,}422 & 17{,}346 \\ 860 & 6{,}390 & 5{,}780 & 30{,}016 & 41{,}082 & 19{,}158 \\ 7{,}241 & 91{,}405 & 30{,}016 & 570{,}623 & 360{,}029 & 76{,}994 \\ 6{,}390 & 66{,}422 & 41{,}082 & 360{,}029 & 409{,}090 & 130{,}989 \\ 2{,}460 & 17{,}346 & 19{,}158 & 76{,}994 & 130{,}989 & 69{,}738 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{array}{cc} \text{Value of sums assured factor} & \text{Value of net premiums factor} \\ \left[\begin{array}{cc} 116{\cdot}1719 & 57{\cdot}0536 \\ 1{,}047{\cdot}074 & 509{\cdot}707 \\ 586{\cdot}470 & 211{\cdot}074 \\ 5{,}543{\cdot}145 & 2{,}737{\cdot}896 \\ 4{,}892{\cdot}090 & 1{,}696{\cdot}895 \\ 1{,}730{\cdot}379 & 525{\cdot}976 \end{array}\right] \end{array}$$

Hence the following were obtained:

	Value of sums assured factor (1)	Value of net premiums factor (2)	Sums assured moments (3)	Value of sums assured (1)×(3) (4)	Value of net premiums (2)×(3) (5)	Net liability (4)−(5) (6)
a	·2368518	·2632039	265,101·6	62,790	69,776	
b	·0329693	·0286646	2,106,175	69,439	60,373	
c	·0303466	−·0125962	904,370	27,445	−11,392	
d	−·0001362	−·0005867	9,377,712	−1,277	− 5,502	
e	−·0001118	−·0032885	7,092,213	− 793	−23,323	
f	·0002809	·0014128	1,883,845	529	2,661	
				158,133	92,593	65,540
			Error %	·08	·11	·03

The value of the net premiums was obtained from ΣS_{xt} and the two x-moments $\Sigma x S_{xt}$ and $\Sigma x_{(2)} S_{xt}$. Furthermore, it was appropriate to graduate $\pi_{3x+19\frac{1}{2}}$ over the restricted range $x=0$ to $x=16$. As we were only concerned with one variable, orthogonal polynomials of one variable could be used with advantage. For convenience of reference, expressions for the first four orthogonal polynomials and the sums of their squares, the general terms of which appear in *J.I.A.* Vol. LXV, p. 281 and *J.I.A.* Vol. LXIV, p. 336, are given below:

$$\begin{aligned} \phi_0 &= 1, & \Sigma\phi_0^2 &= n, \\ \phi_1 &= 2x-(n-1), & \Sigma\phi_1^2 &= (n+1)^{(3)}/3, \\ \phi_2 &= 6x_{(2)}-3(n-2)x+(n-1)_{(2)}, & \Sigma\phi_2^2 &= (n+2)^{(5)}/20, \\ \phi_3 &= 20x_{(3)}-10(n-3)x_{(2)}+4(n-2)_{(2)}x-(n-1)_{(3)}, & \Sigma\phi_3^2 &= (n+3)^{(7)}/252. \end{aligned}$$

By taking $n=17$ and computing the moments of $\pi_{3x+19\frac{1}{2}}$ from $x=0$ to $x=16$, the value of the net premiums was found to be 7534·39, the percentage error being −·07.

Conclusion

It is clear that with skill and care excellent results can be obtained, certainly within the limits of permissible error, which I would place at ·3%. Although only moments to the second order have been used in the test described in this note, I think that while the method is in its experimental stage it is advisable to classify the data according to method (C) so that third moments can be obtained. The application of the two-variable Henry's method to third moments involves the solution of simultaneous linear equations in ten variables if, as seems the case, the use of orthogonal polynomials must be ruled out because the data do not cover a triangle sufficiently well. This admittedly is troublesome although the work may be done between the valuations. It may be mentioned that Dr Aitken in a paper, *The evaluation with applications of a certain triple product matrix* (*Proc. Royal Soc. Edinburgh*, Vol. LVII, Part II, no. 12), gives, amongst other things, a straightforward process for solving simultaneous linear equations. It is, however, simpler, although I feel sure less accurate, to use Perks's methods if third moments are retained.

APPENDIX

1. *Orthogonal polynomials in two variables over a triangle*

The problem is to find a set of polynomials $P_{rs}(x,y)$ such that

$$\Sigma\Sigma P_{rs}(x,y)\,P_{r's'}(x,y) = 0$$

so long as $r, s \neq r', s'$, where the summation extends over the integral points of the triangle $0 \leqslant x \leqslant n-1$, $0 \leqslant y \leqslant n-1$, $0 \leqslant x+y \leqslant n-1$. There will be one P of zero order which we shall denote by $P_{00}(x,y)$, two P's of order 1 which we shall denote by $P_{01}(x,y)$ and $P_{11}(x,y)$, three P's of order 2 which we shall denote by $P_{02}(x,y)$, $P_{12}(x,y)$ and $P_{22}(x,y)$, and so on. The set is not unique, but the further conditions defining the particular set to be developed will emerge in the subsequent analysis.

Let $\mu = x+y$, $\nu = x-y$, and let $P_{rs}(x,y) = Q_{rs}(\mu,\nu)$;

then $$\Sigma\Sigma P_{rs}(x,y)\,P_{r's'}(x,y) = \sum_{\mu=0}^{n-1}\sum_{\nu=-\mu}^{\mu}{}' Q_{rs}(\mu,\nu)\,Q_{r's'}(\mu,\nu),$$

where $\sum\limits_{\nu=-\mu}^{\mu}{}'$ denotes summation over the points $\nu = -\mu, -\mu+2, -\mu+4, \ldots, \mu-2, \mu$.

The particular set of orthogonal polynomials which we shall consider will be further defined by $Q_{rs}(\mu,\nu) \equiv U_r(\mu,\nu)\; V_{rt}(\mu)$, $t = s-r$, where $U_r(\mu,\nu)$ is a polynomial of degree r in μ and ν, and $V_{rt}(\mu)$ is of degree t in μ, so that $Q_{rs}(\mu,\nu)$ is correctly of degree s in μ and ν.

We have $$\sum_{\mu=0}^{n-1} V_{rt}(\mu)\,V_{r't'}(\mu) \sum_{\nu=-\mu}^{\mu}{}' U_r(\mu,\nu)\,U_{r'}(\mu,\nu) = 0,$$

unless both $r = r'$ and $t = t'$.

Now Dr Aitken in his paper, *On the graduation of data by the orthogonal polynomials of least squares* (*Proc. Royal Soc. Edinburgh*, Vol. LIII), has investigated the polynomials orthogonal over the range $-(q-\frac{1}{2})$ to $(q-\frac{1}{2})$. In *J.I.A.* Vol. LXV, p. 284, I denoted these polynomials by $\hat{T}_r(\xi, q)$ and gave expressions for them in formulae (38) and (39) in terms of central factorials $\xi^{\{r\}} \equiv \left(\xi+\frac{r-1}{2}\right)\left(\xi+\frac{r-3}{2}\right)\cdots\left(\xi-\frac{r-1}{2}\right)$ and $q^{\{r\}}$. If we take $\xi = \frac{\nu}{2}$, $q = \frac{\mu+1}{2}$, the resulting polynomials $\hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right)$ are orthogonal over the range $\sum\limits_{\nu=-\mu}^{\mu}{}'$.

There is no difficulty in obtaining the general formula which results from this substitution. $U_r(\mu,\nu) = \hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right)$ can, however, be expressed in a more elegant and practical form, for

$$\hat{T}_r(\xi, q) = \Delta^r\left\{\left(\xi + \frac{2q-1}{2}\right)_{(r)}\left(\xi - \frac{2q+1}{2}\right)_{(r)}\right\}.$$

Hence $$\hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right) = \Delta^r\left\{\left(\xi+\frac{\mu}{2}\right)_{(r)}\left(\xi - \frac{\mu+2}{2}\right)_{(r)}\right\}, \qquad \xi = \frac{\nu}{2},$$

$$= \Delta^r\left\{\left(\xi + \frac{\nu}{2} + \frac{\mu}{2}\right)_{(r)}\left(\xi + \frac{\nu}{2} - \frac{\mu+2}{2}\right)_{(r)}\right\}, \quad \xi = 0,$$

$$= \Delta^r\{(\xi+x)_{(r)}(\xi-y-1)_{(r)}\}, \qquad \xi = 0.$$

Now $$\Delta^r\{u(\xi)v(\xi)\} = u(\xi)\Delta^r v(\xi) + r\Delta u(\xi)\Delta^{r-1}v(\xi+1)$$
$$+ r_{(2)}\Delta^2 u(\xi)\Delta^{r-2}v(\xi+2) + \ldots + \Delta^r u(\xi)v(\xi+r).$$

Hence

$$\hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right) = x_{(r)} + rx_{(r-1)}(-y)_{(1)} + r_{(2)}x_{(r-2)}(-y+1)_{(2)} + \ldots + (-y+r-1)_{(r)}$$
$$= x_{(r)} - rx_{(r-1)}y_{(1)} + r_{(2)}x_{(r-2)}y_{(2)} - \ldots + (-)^r y_{(r)}. \qquad (1)$$

The first four polynomials $U_r(\mu,\nu) = \hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right)$ are:

$$U_0(\mu,\nu) = 1,$$

$$U_1(\mu,\nu) = \nu = x - y,$$

$$U_2(\mu,\nu) = \tfrac{1}{4}\{3\nu^2 - \mu(\mu+2)\} = x_{(2)} - 2xy + y_{(2)},$$

$$U_3(\mu,\nu) = \frac{\nu}{12}\{5(\nu^2-1) - 3(\mu+3)(\mu-1)\} = x_{(3)} - 3x_{(2)}y + 3xy_{(2)} - y_{(3)}.$$

The adoption of $U_r(\mu,\nu) = \hat{T}_r\left(\frac{\nu}{2}, \frac{\mu+1}{2}\right)$ will solve the problem for $r \neq r'$ and will leave $V_{rt}(\mu)$ at our disposal to complete the determination of $P_{rs}(x,y)$ when $r = r'$.

Now $$\sum_{\nu=-\mu}^{\mu}{}' \{U_r(\mu,\nu)\}^2 = \sum_{\xi=-(q-\frac{1}{2})}^{(q-\frac{1}{2})} \{\hat{T}_r(\xi,q)\}^2 = 2^{2r+1}q^{\{r\}}q^{\{r+1\}}/(r!)^2(2r+1)$$

(see *J.I.A.* Vol. LXV, p. 284)

$$= (\mu+r+1)^{(2r+1)}/(r!)^2(2r+1).$$

Hence $$\sum_{\mu=0}^{n-1}(\mu+r+1)^{(2r+1)}V_{rt}(\mu)V_{rt'}(\mu) = 0, \quad t \neq t'.$$

Hence $$\sum_{\mu=0}^{n-1}(\mu+r+t'+1)^{(2r+t'+1)}V_{rt}(\mu) = 0, \quad t' = 0, 1, 2, \ldots, t-1. \qquad (2)$$

Let $$V_{rt}(\mu) \equiv \alpha_0 + \alpha_1\frac{(\mu-r)^{(1)}}{(n-r-1)^{(1)}} + \alpha_2\frac{(\mu-r)^{(2)}}{(n-r-1)^{(2)}} + \ldots + \alpha_t\frac{(\mu-r)^{(t)}}{(n-r-1)^{(t)}}.$$

Hence
$$\alpha_t = \mathrm{K}\frac{(2r+2t+1)!}{(2r+t+1)!},$$
$$\alpha_{t-1} = -\mathrm{K}\frac{t!}{1!(t-1)!}\frac{(2r+2t)!}{(2r+t)!},$$
$$\alpha_{t-2} = \mathrm{K}\frac{t!}{2!(t-2)!}\frac{(2r+2t-1)!}{(2r+t-1)!},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_0 = (-)^t\mathrm{K}\frac{t!}{t!}\frac{(2r+t+1)!}{(2r+1)!}.$$

We will standardize $\mathrm{V}_{rt}(\mu)$ by taking $\mathrm{K} = (n-r-1)^{(t)}/(t!)^2$, so that

$$\mathrm{V}_{rt}(\mu) = \frac{(2r+2t+1)!}{(2r+t+1)!\,t!}(\mu-r)_{(t)} - \frac{(2r+2t)!}{(2r+t)!\,t!}(n-r-t)_{(1)}(\mu-r)_{(t-1)}$$
$$+\frac{(2r+2t-1)!}{(2r+t-1)!\,t!}(n-r-t+1)_{(2)}(\mu-r)_{(t-2)}\cdots$$
$$+(-)^t\frac{(2r+t+1)!}{(2r+1)!\,t!}(n-r-1)_{(t)}. \qquad (3)$$

In multiplying $\mathrm{V}_{rt}(\mu)$ by $x_{(r)} - rx_{(r-1)}y_{(1)} + r_{(2)}x_{(r-2)}y_{(2)}\cdots + (-)^r y_{(r)}$ to form $\mathrm{Q}_{rs}(\mu,\nu)$ and hence $\mathrm{P}_{rs}(x,y)$, it is helpful to note that

$$x_{(r)}(\mu-r)_{(s)} = x_{(r)}(x-r+y)_{(s)} = x_{(r)}\{(x-r)_{(s)} + (x-r)_{(s-1)}y_{(1)}$$
$$+(x-r)_{(s-2)}y_{(2)} + \dots + y_{(s)}\}$$
$$= \frac{(r+s)!}{r!\,s!}x_{(r+s)} + \frac{(r+s-1)!}{r!\,(s-1)!}x_{(r+s-1)}y_{(1)} + \frac{(r+s-2)!}{r!\,(s-2)!}x_{(r+s-2)}y_{(2)} + \dots + x_{(r)}y_{(s)},$$

$$x_{(r-1)}y_{(1)}(\mu-r)_{(s)} = x_{(r-1)}y_{(1)}(x-r+1+y-1)_{(s)}$$
$$= x_{(r-1)}y_{(1)}\{(x-r+1)_{(s)} + (x-r+1)_{(s-1)}(y-1)_{(1)} + \dots + (y-1)_{(s)}\}$$
$$= \frac{(r+s-1)!}{(r-1)!\,s!}x_{(r+s-1)}y_{(1)} + \frac{(r+s-2)!}{(r-1)!\,(s-1)!}\frac{2!}{1!\,1!}x_{(r+s-2)}y_{(2)} + \dots$$
$$+\frac{(s+1)!}{1!\,s!}x_{(r-1)}y_{(s+1)},$$

and so on.

A general formula for $\mathrm{P}_{rs}(x,y)$ would be very cumbrous. The first ten P's are given on p. 510.

A more elegant though less practical expression can be found for $\mathrm{V}_{rt}(\mu)$ by writing (2) in the form $\sum\limits_{\mu=0}^{n-1}(\mu+r+1)_{(2r+1)}\mu_{(t')}\mathrm{V}_{rt}(\mu) = 0$, $t' = 0, 1, 2, \dots, t-1$, and using the relation $\sum\limits_{\mu=-r-1}^{n-1} u(\mu)v(\mu) = \mathrm{U}(n)v(n) - \sum\limits_{\mu=-r-1}^{n-1}\mathrm{U}(\mu+1)\Delta v(\mu)$, where $\mathrm{U}(\mu) = \sum\limits_{\nu=-r-1}^{\mu-1} u(\nu)$, to transform the functions $\psi_m(\mu)$ defined such that

$$\psi_m(\mu) = \sum_{\mu=-r-1}^{\mu-1}\psi_{m-1}(\mu),\ m = 1, 2, \dots, t;\ \psi_0(\mu) = (\mu+r+1)_{(2r+1)}\mathrm{V}_{rt}(\mu).$$

The argument follows almost exactly that in *J.I.A.* Vol. LXIV, pp. 337 and 338. $\psi_t(\mu)$ is shown to vanish when $\mu = -r-1, -r, \ldots, r+t-1$, and also when $\mu = n, n+1, \ldots, n+t-1$. The use of Newton's divided-difference theorem applied to $\psi_t(\mu)$ shows that $\psi_t(\mu) = \mathrm{K}(\mu+r+1)^{(2r+t+1)}(\mu-n)^{(t)}$; hence (after replacing K by an appropriate constant)

$$\mathrm{V}_{rt}(\mu) = \Delta^t\{(\mu+r+1)^{(2r+t+1)}(\mu-n)^{(t)}\}/(t!)^2(\mu+r+1)^{(2r+1)}$$

$$\Sigma\Sigma\mathrm{P}^2_{rs}(x,y) = \sum_{\mu=0}^{n-1}\mathrm{V}^2_{rt}(\mu)\sum_{\nu=-\mu}^{\mu}{}'\,\mathrm{U}^2_r(\mu,\nu)$$

$$= \sum_{\mu=0}^{n-1}(\mu+r+1)^{(2r+1)}\mathrm{V}^2_{rt}(\mu)/(r!)^2(2r+1)$$

$$= \sum_{\mu=0}^{n-1}(\mu+r+1)^{(2r+1)}\left[\alpha_0+\alpha_1\frac{(\mu-r)^{(1)}}{(n-r-1)^{(1)}}+\ldots+\alpha_t\frac{(\mu-r)^{(t)}}{(n-r-1)^{(t)}}\right]\mathrm{V}_{rt}(\mu)/(r!)^2(2r+1)$$

$$= \frac{\alpha_t}{(r!)^2(2r+1)(n-r-1)^{(t)}}\sum_{\mu=0}^{n-1}(\mu+r+1)^{(2r+1)}(\mu-r)^{(t)}\mathrm{V}_{rt}(\mu), \quad \text{from (2)},$$

$$= \frac{(2r+2t+1)!}{(2r+t+1)!\,(t!)^2(r!)^2(2r+1)}\sum_{\mu=0}^{n-1}(\mu+r+1)^{(2r+t+1)}\mathrm{V}_{rt}(\mu)$$

$$= \frac{(2r+2t+1)!}{(2r+t+1)!\,(t!)^2(r!)^2(2r+1)}\sum_{\mu=0}^{n-1}(\mu+r+t+1)^{(2r+t+1)}\mathrm{V}_{rt}(\mu),$$

since the only term that matters is $\Sigma\mu^t(\mu+r+1)^{(2r+1)}\mathrm{V}_{rt}(\mu)$,

$$= \frac{(2r+2t+1)!}{(2r+t+1)!\,(t!)^2(r!)^2(2r+1)}(n+r+t+1)^{(2r+t+2)} \times\left[\frac{\alpha_0}{2r+t+2}+\frac{\alpha_1}{2r+t+3}+\ldots+\frac{\alpha_t}{2r+2t+2}\right]$$

$$= \frac{(2r+2t+1)!}{(2r+t+1)!\,(t!)^2(r!)^2(2r+1)}(n+r+t+1)^{(2r+t+2)}\int_0^1\mu^{2r+t+1}\mathrm{W}_{rt}(\mu)\,d\mu.$$

Now $$\int_0^1\mu^{2r+t+1}\mathrm{W}_{rt}(\mu)\,d\mu = \int_0^1\mu^t\psi_0(\mu)\,d\mu$$

$$= -\int_0^1 t\mu^{t-1}\psi_1(\mu)\,d\mu = \int_0^1 t(t-1)\mu^{t-2}\psi_2(\mu)\,d\mu = \ldots$$

$$= (-)^t t!\int_0^1\psi_t(\mu)\,d\mu = (-)^t\mathrm{K}(t!)\int_0^1\mu^{2r+1+t}(\mu-1)^t\,d\mu$$

$$= \mathrm{K}(t!)\,\beta(2r+t+2,t+1),$$

where $\beta(2r+t+2,t+1)$ represents the first Eulerian integral,

$$= [(n-r-1)^{(t)}/t!]\,[(2r+t+1)!\,t!/(2r+2t+2)!].$$

Hence

$$\Sigma\Sigma\mathrm{P}^2_{rs}(x,y) = \frac{(n+r+t+1)^{(2r+2t+2)}}{(r!)^2(t!)^2(2r+2t+2)(2r+1)} = \frac{(n+s+1)^{(2s+2)}}{(r!)^2((s-r)!)^2(2s+2)(2r+1)}. \tag{4}$$

The methods of the preceding pages may be used in part to find polynomials orthogonal with regard to weights $1/(x+y+1)$, i.e., polynomials $\mathrm{P}_{rs}(x,y)$ such that $\Sigma\Sigma P_{rs}(x,y)\,P_{r's'}(x,y)/(x+y+1)=0$ so long as $r,s\neq r',s'$, where the summation extends over the triangle $0\leqslant x\leqslant n-1$, $0\leqslant y\leqslant n-1$, $0\leqslant x+y\leqslant n-1$.

As before let $\mu=x+y$, $\nu=x-y$, $P_{rs}(x,y)\equiv Q_{rs}(\mu,\nu)$, and limit $Q_{rs}(\mu,\nu)$ by the relation $Q_{rs}(\mu,\nu)\equiv U_r(\mu,\nu)\,V_{rt}(\mu)\,(t=s-r)$, where $U_r(\mu,\nu)$ is a polynomial of degree r in μ and ν and $V_{rt}(\mu)$ is of degree t in μ. It will be found that

$$U_r(\mu,\nu)=\hat{T}_r\left(\frac{\nu}{2},\frac{\mu+1}{2}\right)=x_{(r)}-rx_{(r-1)}y_{(1)}+r_{(2)}x_{(r-2)}y_{(2)}-\ldots+(-)^r y_{(r)},$$

but relation (2) defining $V_{rt}(\mu)$ becomes

$$\sum_{\mu=0}^{n-1}\frac{(\mu+r+t'+1)^{(2r+t'+1)}}{\mu+1}V_{rt}(\mu)=0,\quad t'=0,1,2,\ldots,t-1. \tag{5}$$

Also $$\Sigma\Sigma P_{rs}^2(x,y)/(x+y+1)=\sum_{\mu=0}^{n-1}\frac{(\mu+r+1)^{(2r+1)}}{\mu+1}V_{rt}^2(\mu)/(r!)^2(2r+1). \tag{6}$$

Unless $r=0$, the method used in the unweighted case to find $V_{rt}(\mu)$ breaks down when applied to equations (5) because of the missing factor $\mu+1$ in the middle of $(\mu+r+t'+1)^{(2r+t'+1)}$. It is possible, however, to solve (5) and (6) for small values of s by somewhat laborious methods which are not worth reproducing here. The first 10 P's and the corresponding weighted sums of the squares of P are given on p. 510.

Analogous methods may be used to find the polynomials $P_{rs}(x,y)$ such that $\iint P_{rs}(x,y)\,P_{r's'}(x,y)\,dx\,dy=0$ over the triangle $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant x+y\leqslant 1$, so long as $r,s\neq r',s'$. The solution is

$$P_{rs}(x,y)\equiv Q_{rs}(\mu,\nu)\equiv D^r\{(\lambda^2-1)^r\}\,D^{s-r}\{\mu^{r+s+1}(\mu-1)^{s-r}\}/r!\,(s-r)!\,\mu^{r+1}, \tag{7}$$

where $\lambda=\nu/\mu$,

and $$\iint P_{rs}^2(x,y)\,dx\,dy=2^{2r-1}/(2r+1)(s+1). \tag{8}$$

Likewise the polynomials $P_{rs}(x,y)$ such that

$$\iint\{P_{rs}(x,y)\,P_{r's'}(x,y)/(x+y)\}\,dx\,dy=0$$

over the triangle $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant x+y\leqslant 1$, so long as $r,s\neq r',s'$, are

$$P_{rs}(x,y)\equiv Q_{rs}(\mu,\nu)\equiv D^r\{(\lambda^2-1)^r\}\,D^{s-r}\{\mu^{r+s}(\mu-1)^{s-r}\}/r!\,(s-r)!\,\mu^r, \tag{9}$$

where $\lambda=\nu/\mu$,

and $$\iint\{P_{rs}^2(x,y)/(x+y)\}\,dx\,dy=2^{2r}/(2r+1)(2s+1). \tag{10}$$

2. *n-point formulae where the points lie on circles whose centre is the mean*

The interesting graphic idea by which Perks develops his formulae (5) to (11) may be extended. Perks takes the mean as origin and measures along the x-axis in the scale of $\sigma_x=1$ and along the t-axis in the scale of $\sigma_t=1$. His points lie on a circle with radius $\sqrt{2}$. Suppose that with this origin and scale we take n points equally spaced on a circle of radius k. Let the points subtend at the origin angles θ_1, $\theta_2=\theta_1+\frac{2\pi}{n}$, $\theta_3=\theta_1+\frac{4\pi}{n}$, ..., $\theta_n=\theta_1+\frac{2(n-1)\pi}{n}$ with the x-axis, so

Unweighted P's

r	s	$P_{rs}(x, y)$	$\Sigma\Sigma P^2_{rs}(x, y)$
0	0	1	$(n+1)^{(2)}/2$
0	1	$3(x+y)-2(n-1)$	$(n+2)^{(4)}/4$
1	1	$x-y$	$(n+2)^{(4)}/12$
0	2	$10(x_{(2)}+xy+y_{(2)})-6(n-2)(x+y)+3(n-1)_{(2)}$	$(n+3)^{(6)}/24$
1	2	$10(x_{(2)}-y_{(2)})-4(n-2)(x-y)$	$(n+3)^{(6)}/18$
2	2	$x_{(2)}-2xy+y_{(2)}$	$(n+3)^{(6)}/120$
0	3	$35(x_{(3)}+x_{(2)}y+xy_{(2)}+y_{(3)})-20(n-3)(x_{(2)}+xy+y_{(2)})+10(n-2)_{(2)}(x+y)-4(n-1)_{(3)}$	$(n+4)^{(8)}/288$
1	3	$21(3x_{(3)}+x_{(2)}y-xy_{(2)}-3y_{(3)})-30(n-3)(x_{(2)}-y_{(2)})+10(n-2)_{(2)}(x-y)$	$(n+4)^{(8)}/96$
2	3	$21(x_{(3)}-x_{(2)}y-xy_{(2)}+y_{(3)})-6(n-3)(x_{(2)}-2xy+y_{(2)})$	$(n+4)^{(8)}/160$
3	3	$x_{(3)}-3x_{(2)}y+3xy_{(2)}-y_{(3)}$	$(n+4)^{(8)}/2016$

Weighted P's. Weights $1/(x+y+1)$

r	s	$P_{rs}(x, y)$	$\Sigma\Sigma P^2_{rs}(x, y)/(x+y+1)$
0	0	1	n
0	1	$2(x+y)-(n-1)$	$(n+1)^{(3)}/3$
1	1	$x-y$	$(2n+5)\,n^{(2)}/18$
0	2	$6(x_{(2)}+xy+y_{(2)})-3(n-2)(x+y)+(n-1)_{(2)}$	$(n+2)^{(5)}/20$
1	2	$2(2n+5)(x_{(2)}-y_{(2)})-\frac{1}{2}(3n+7)(n-2)(x-y)$	$(n+5)(2n+5)(n+2)^{(5)}/120$
2	2	$x_{(2)}-2xy+y_{(2)}$	$(6n^2+33n+47)\,n^{(3)}/600$
0	3	$20(x_{(3)}+x_{(2)}y+xy_{(2)}+y_{(3)})-10(n-3)(x_{(2)}+xy+y_{(2)})+4(n-2)_{(2)}(x+y)-(n-1)_{(3)}$	$(n+3)^{(7)}/252$
1	3	$15(n+5)(3x_{(3)}+x_{(2)}y-xy_{(2)}-3y_{(3)})-5(4n+19)(n-3)(x_{(2)}-y_{(2)})+2(3n+13)(n-2)_{(2)}(x-y)$	$(n+5)(2n+17)(n+3)^{(7)}/168$
2	3	$3(6n^2+33n+47)(x_{(3)}-x_{(2)}y-xy_{(2)}+y_{(3)})-(5n^2+27n+37)(n-3)(x_{(2)}-2xy+y_{(2)})$	$(6n^2+33n+47)(n^2+9n+23)(n+3)^{(7)}/840$
3	3	$x_{(3)}-3x_{(2)}y+3xy_{(2)}-y_{(3)}$	$(10n^3+95n^2+299n+319)\,n^{(4)}/17640$

that $x_1 = k\cos\theta_1$, $t_1 = k\sin\theta_1$, $x_2 = k\cos\left(\theta_1+\frac{2\pi}{n}\right)$, $t_2 = k\sin\left(\theta_1+\frac{2\pi}{n}\right)$, etc. Suppose the valuation function is a second-degree curve

$$a+bx+ct+2dx^2+2ext+2ft^2.$$

If this curve passes through (x_1, t_1), we have

$$\begin{aligned} \mathrm{V}(x_1,t_1) = \mathrm{V}(\theta_1)\,(\text{say}) &= a+k(b\cos\theta_1+c\sin\theta_1)+k^2\{d(1+\cos 2\theta_1) \\ &\qquad +e\sin 2\theta_1+f(1-\cos 2\theta_1)\}, \\ \mathrm{V}(x_2,t_2) = \mathrm{V}(\theta_2) &= a+k(b\cos\theta_2+c\sin\theta_2)+k^2\{d(1+\cos 2\theta_2) \\ &\qquad +e\sin 2\theta_2+f(1-\cos 2\theta_2)\}, \end{aligned}$$

and so on.

We will use the well-known relations that

$$\sum_{r=1}^{n}\cos\theta_r = 0, \quad \sum_{r=1}^{n}\sin\theta_r = 0.$$

From these it is easy to deduce that

$$\left.\begin{aligned} &\sum_{r=1}^{n}\cos p\theta_r\cos q\theta_r = 0 \\ &\quad\text{or}\quad \frac{n}{2}\cos(p+q)\,\theta_1 \quad\text{or}\quad \frac{n}{2}\cos(p-q)\,\theta_1, \\ &\sum_{r=1}^{n}\sin p\theta_r\cos q\theta_r = 0 \\ &\quad\text{or}\quad \frac{n}{2}\sin(p+q)\,\theta_1 \quad\text{or}\quad \frac{n}{2}\sin(p-q)\,\theta_1, \\ &\sum_{r=1}^{n}\sin p\theta_r\sin q\theta_r = 0 \\ &\quad\text{or}\quad -\frac{n}{2}\cos(p+q)\,\theta_1 \quad\text{or}\quad \frac{n}{2}\cos(p-q)\,\theta_1, \end{aligned}\right\} \begin{array}{l}\text{according as both } p+q \text{ and} \\ p-q\neq mn \text{ or } p+q=mn \text{ or} \\ p-q=mn \text{ respectively, } m \\ \text{being an integer.}\end{array}$$

Hence $$\frac{1}{n}\sum_{r=1}^{n}\mathrm{V}(\theta_r) = a+k^2(d+f), \quad n>2,$$

and $$\frac{2}{n}\sum_{r=1}^{n}\sin 2\theta_r\,\mathrm{V}(\theta_r) = k^2e, \quad n>4.$$

Now $$\Sigma \mathrm{S}_{xt}\mathrm{V}(x,t) = (\Sigma \mathrm{S}_{xt})(a+2d+2r_{xt}e+2f).$$

Hence, if $k^2 = 2$,

$$\Sigma \mathrm{S}_{xt}\mathrm{V}(x,t) = (\Sigma \mathrm{S}_{xt})\left\{\frac{1}{n}\sum_{r=1}^{n}\mathrm{V}(\theta_r)+\frac{2r_{xt}}{n}\sum_{r=1}^{n}\sin 2\theta_r\,\mathrm{V}(\theta_r)\right\}.$$

(If $n=4$, $\theta_1=\frac{\pi}{4}$, we have $\frac{1}{4}\{\mathrm{V}(\theta_1)-\mathrm{V}(\theta_2)+\mathrm{V}(\theta_3)-\mathrm{V}(\theta_4)\} = k^2e$, from which Perks's formula (9) can be derived.)

Nothing seems to be gained by increasing the number of points indefinitely. Useful formulae are obtained when $n = 5$ or 6.

If $n = 5$, $\theta_1 = \dfrac{2\pi}{5}$,

$$(\theta_1) = \left(\bar{x}+\frac{\sqrt{10-\sqrt{2}}}{4}\sigma_x;\ \bar{t}+\frac{\sqrt{(20+4\sqrt{5})}}{4}\sigma_t\right) = (\bar{x}+\cdot 43702\sigma_x;\ \bar{t}+1\cdot 34500\sigma_t),$$

$$(\theta_2) = \left(\bar{x}-\frac{\sqrt{10+\sqrt{2}}}{4}\sigma_x;\ \bar{t}+\frac{\sqrt{(20-4\sqrt{5})}}{4}\sigma_t\right) = (\bar{x}-1\cdot 14412\sigma_x;\ \bar{t}+\cdot 83126\sigma_t),$$

$$(\theta_3) = \left(\bar{x}-\frac{\sqrt{10+\sqrt{2}}}{4}\sigma_x;\ \bar{t}-\frac{\sqrt{(20-4\sqrt{5})}}{4}\sigma_t\right) = (\bar{x}-1\cdot 14412\sigma_x;\ \bar{t}-\cdot 83126\sigma_t),$$

$$(\theta_4) = \left(\bar{x}+\frac{\sqrt{10-\sqrt{2}}}{4}\sigma_x;\ \bar{t}-\frac{\sqrt{(20+4\sqrt{5})}}{4}\sigma_t\right) = (\bar{x}+\cdot 43702\sigma_x;\ \bar{t}-1\cdot 34500\sigma_t),$$

$$(\theta_5) = (\bar{x}+\sqrt{2}\,\sigma_x;\ \bar{t}\qquad) = (\bar{x}+1\cdot 41421\sigma_x;\ \bar{t}\qquad),$$

and

$$\Sigma S_{xt} V(x,t) = (\Sigma S_{xt})\left\{\tfrac{1}{5}(V(\theta_1)+V(\theta_2)+V(\theta_3)+V(\theta_4)+V(\theta_5)) + \frac{2r_{xt}}{5}\left(\frac{\sqrt{(10-2\sqrt{5})}}{4}V(\theta_1)-\frac{\sqrt{(10+2\sqrt{5})}}{4}V(\theta_2) + \frac{\sqrt{(10+2\sqrt{5})}}{4}V(\theta_3)-\frac{\sqrt{(10-2\sqrt{5})}}{4}V(\theta_4)\right)\right\}$$

$$= (\Sigma S_{xt})\{\tfrac{1}{5}(V(\theta_1)+V(\theta_2)+V(\theta_3)+V(\theta_4)+V(\theta_5)) + r_{xt}(\cdot 23512\,V(\theta_1)-\cdot 38042\,V(\theta_2)+\cdot 38042\,V(\theta_3)-\cdot 23512\,V(\theta_4))\}.$$

If $n = 6$, $\theta_1 = \dfrac{2\pi}{6}$,

$$(\theta_1) = \left(\bar{x}+\frac{\sqrt{2}}{2}\sigma_x;\ \bar{t}+\sqrt{\tfrac{3}{2}}\,\sigma_t\right) = (\bar{x}+\cdot 70711\sigma_x;\ \bar{t}+1\cdot 22474\sigma_t),$$

$$(\theta_2) = \left(\bar{x}-\frac{\sqrt{2}}{2}\sigma_x;\ \bar{t}+\sqrt{\tfrac{3}{2}}\,\sigma_t\right) = (\bar{x}-\cdot 70711\sigma_x;\ \bar{t}+1\cdot 22474\sigma_t),$$

$$(\theta_3) = (\bar{x}-\sqrt{2}\sigma_x;\ \bar{t}\qquad) = (\bar{x}-1\cdot 41421\sigma_x;\ \bar{t}\qquad),$$

$$(\theta_4) = \left(\bar{x}-\frac{\sqrt{2}}{2}\sigma_x;\ \bar{t}-\sqrt{\tfrac{3}{2}}\,\sigma_t\right) = (\bar{x}-\cdot 70711\sigma_x;\ \bar{t}-1\cdot 22474\sigma_t),$$

$$(\theta_5) = \left(\bar{x}+\frac{\sqrt{2}}{2}\sigma_x;\ \bar{t}-\sqrt{\tfrac{3}{2}}\,\sigma_t\right) = (\bar{x}+\cdot 70711\sigma_x;\ \bar{t}-1\cdot 22474\sigma_t),$$

$$(\theta_6) = (\bar{x}+\sqrt{2}\,\sigma_x;\ \bar{t}\qquad) = (\bar{x}+1\cdot 41421\sigma_x;\ \bar{t}\qquad),$$

and

$$\Sigma S_{xt} V(x,t) = (\Sigma S_{xt})\left\{\tfrac{1}{6}(V(\theta_1)+V(\theta_2)+V(\theta_3)+V(\theta_4)+V(\theta_5)+V(\theta_6)) + \frac{\sqrt{3}}{6}r_{xt}(V(\theta_1)-V(\theta_2)+V(\theta_4)-V(\theta_5))\right\}$$

$$= (\Sigma S_{xt})\{\tfrac{1}{6}(V(\theta_1)+V(\theta_2)+V(\theta_3)+V(\theta_4)+V(\theta_5)+V(\theta_6)) + \cdot 28868 r_{xt}(V(\theta_1)-V(\theta_2)+V(\theta_4)-V(\theta_5))\}.$$

Now suppose the valuation function is a third-degree curve

$$a+bx+ct+2dx^2+2ext+2ft^2+4gx^3+4hx^2t+4ixt^2+4jt^3.$$

If this curve passes through $V(\theta)$,

$$\begin{aligned}V(\theta) &= a+k(b\cos\theta+c\sin\theta)+k^2\{d(1+\cos 2\theta)+e\sin 2\theta+f(1-\cos 2\theta)\}\\ &\quad +k^3\{g(3\cos\theta+\cos 3\theta)+h(\sin\theta+\sin 3\theta)+i(\cos\theta-\cos 3\theta)\\ &\quad +j(3\sin\theta-\sin 3\theta)\}.\end{aligned}$$

Hence
$$\frac{1}{n}\sum_{r=1}^{n} V(\theta_r) = a+k^2(d+f), \quad n>3,$$

$$\frac{2}{n}\sum_{r=1}^{n}\cos\theta_r V(\theta_r) = kb+k^3(3g+i), \quad n>4,$$

$$\frac{2}{n}\sum_{r=1}^{n}\sin\theta_r V(\theta_r) = kc+k^3(h+3j), \quad n>4,$$

$$\frac{2}{n}\sum_{r=1}^{n}\cos 2\theta_r V(\theta_r) = k^2(d-f), \quad n>5,$$

$$\frac{2}{n}\sum_{r=1}^{n}\sin 2\theta_r V(\theta_r) = k^2 e, \quad n>5,$$

$$\frac{2}{n}\sum_{r=1}^{n}\cos 3\theta_r V(\theta_r) = k^3(g-i), \quad n>6,$$

$$\frac{2}{n}\sum_{r=1}^{n}\sin 3\theta_r V(\theta_r) = k^3(h-j), \quad n>6.$$

If $n=6$, the two last equations may be replaced by the one equation

$$\begin{aligned}&\tfrac{1}{6}(V(\theta_1)-V(\theta_2)+V(\theta_3)-V(\theta_4)+V(\theta_5)-V(\theta_6))\\ &\qquad = k^3\{g\cos 3\theta_1+h\sin 3\theta_1-i\cos 3\theta_1-j\sin 3\theta_1\}.\end{aligned}$$

We wish to find $\Sigma S_{xt} V(x,t)$

$$= (\Sigma S_{xt})(a+2d+2r_{xt}e+2f+4gr_{x^3}+4hr_{x^2t}+4ir_{xt^2}+4jr_{t^3}).$$

It is clear that we can never separate g from i or h from j from the above equations. If, however, we take points (k_1,θ_1), (k_1,θ_2), ..., (k_1,θ_n) on a circle of radius k_1 and points (k_2,ϕ_1), (k_2,ϕ_2), ..., (k_2,ϕ_m) on another circle of radius k_2, we can obtain the extra equations we want because

$$\frac{2}{k_1 n}\sum_{r=1}^{n}\cos\theta_r V(k_1,\theta_r)-\frac{2}{k_2 m}\sum_{r=1}^{m}\cos\phi_r V(k_2,\phi_r) = (k_1^2-k_2^2)(3g+i)$$

and
$$\frac{2}{k_1 n}\sum_{r=1}^{n}\sin\theta_r V(k_1,\theta_r)-\frac{2}{k_2 m}\sum_{r=1}^{m}\sin\phi_r V(k_2,\phi_r) = (k_1^2-k_2^2)(h+3j).$$

It will be a convenience to take $k_1^2+k_2^2=4$, so that

$$\frac{1}{2n}\sum_{r=1}^{n} V(k_1,\theta_r)+\frac{1}{2m}\sum_{r=1}^{m} V(k_2,\phi_r) = a+2d+2f,$$

and
$$\frac{1}{n}\sum_{r=1}^{n}\sin 2\theta_r V(k_1,\theta_r)+\frac{1}{m}\sum_{r=1}^{m}\sin 2\phi_r V(k_2,\phi_r) = 2e.$$

We require at least six points on each circle. If $n>6$ and $m>6$, we obtain two equations each for $g-i$ and $h-j$. Taking $n=m=6$, we can obtain $h-j$ from

$$\tfrac{1}{6}\{V(k_1,\theta_1)-V(k_1,\theta_2)+V(k_1,\theta_3)-V(k_1,\theta_4)+V(k_1,\theta_5)-V(k_1,\theta_6)\}$$
$$=k_1^3(h-j),\quad \text{if } \theta_1=\frac{\pi}{6},$$

and $g-i$ from

$$\tfrac{1}{6}\{V(k_2,\phi_1)-V(k_2,\phi_2)+V(k_2,\phi_3)-V(k_2,\phi_4)+V(k_2,\phi_5)-V(k_2,\phi_6)\}$$
$$=-k_2^3(g-i),\quad \text{if } \phi_1=\frac{\pi}{3}.$$

The equations in $3g+i$ and $h+3j$ require that k_1 should be reasonably separated from k_2. A limit must, however, be placed on $|k_1-k_2|$ because, if either k_1 or k_2 is too large, one of the points may come outside the table of valuation factors. It will be found that the restraint is usually imposed by the size of σ_t relative to $\bar{t}$, because $\bar{x}$ is usually very much greater than σ_x. The points which need watching are (k_1,θ_5), (k_2,ϕ_4), and (k_2,ϕ_5), where the t co-ordinates are $\bar{t}-k_1\sigma_t$, $\bar{t}-k_2\frac{\sqrt{3}}{2}\sigma_t$, and $\bar{t}-k_2\frac{\sqrt{3}}{2}\sigma_t$ respectively. Satisfactory values of k_1 and k_2 are obtained by making these t co-ordinates equal to one another, so that $k_1=k_2\frac{\sqrt{3}}{2}$, and thus $k_1=\sqrt{\tfrac{12}{7}}$, $k_2=\sqrt{\tfrac{16}{7}}$. The following solution is obtained:

$$\Sigma S_{xt}V(x,t)=(\Sigma S_{xt})(a+2d+2r_{xt}e+2f+4gr_{x^3}+4hr_{x^2t}+4ir_{xt^2}+4jr_{t^3}),$$

where

$$\begin{bmatrix}(a+2d+2f)/\frac{1}{12}\\ 2e\Big/\frac{\sqrt{3}}{12}\\ 4g\Big/\frac{7\sqrt{7}}{384}\\ 4h\Big/\frac{7\sqrt{21}}{96}\\ 4c\Big/\frac{7\sqrt{7}}{384}\\ 4j\Big/\frac{7\sqrt{21}}{864}\end{bmatrix}=\begin{bmatrix}1&1&1&1&1&1\\ 1&0&-1&1&0&-1\\ -8&0&8&8&0&-8\\ 0&-2&0&0&2&0\\ -8&0&8&8&0&-8\\ -8&-10&-8&8&10&8\end{bmatrix}\begin{bmatrix}V(\theta_1)\\ V(\theta_2)\\ V(\theta_3)\\ V(\theta_4)\\ V(\theta_5)\\ V(\theta_6)\end{bmatrix}+\begin{bmatrix}1&1&1&1&1&1\\ 1&-1&0&1&-1&0\\ 3&-3&-9&-3&3&9\\ 1&1&0&-1&-1&0\\ 7&-7&-5&-7&7&5\\ 9&9&0&-9&-9&0\end{bmatrix}\begin{bmatrix}V(\phi_1)\\ V(\phi_2)\\ V(\phi_3)\\ V(\phi_4)\\ V(\phi_5)\\ V(\phi_6)\end{bmatrix}$$

$$\left(\frac{\sqrt{3}}{12}=\cdot 14434,\ \frac{7\sqrt{7}}{384}=\cdot 048230,\ \frac{7\sqrt{21}}{96}=\cdot 33415,\right.$$
$$\left.\frac{7\sqrt{7}}{384}=\cdot 048230,\ \frac{7\sqrt{21}}{864}=\cdot 037127\right),$$

and

$$(\theta_1)=\left(\bar{x}+\frac{3\sqrt{7}}{7}\sigma_x;\ \bar{t}+\frac{\sqrt{21}}{7}\sigma_t\right)=(\bar{x}+1\cdot 13389\sigma_x;\ \bar{t}+\cdot 65465\sigma_t),$$

$$(\theta_2)=\left(\bar{x}\qquad;\ \bar{t}+\frac{2\sqrt{21}}{7}\sigma_t\right)=(\bar{x}\qquad;\ \bar{t}+1\cdot 309310\sigma_t),$$

$$(\theta_3)=\left(\bar{x}-\frac{3\sqrt{7}}{7}\sigma_x;\ \bar{t}+\frac{\sqrt{21}}{7}\sigma_t\right)=(\bar{x}-1\cdot 13389\sigma_x;\ \bar{t}+\cdot 65465\sigma_t),$$

$$(\theta_4) = \left(\bar{x} - \frac{3\sqrt{7}}{7}\sigma_x;\ \bar{t} - \frac{\sqrt{21}}{7}\sigma_t\right) = (\bar{x} - 1\cdot13389\sigma_x;\ \bar{t} - \cdot65465\sigma_t),$$

$$(\theta_5) = \left(\bar{x} \qquad ;\ \bar{t} - \frac{2\sqrt{21}}{7}\sigma_t\right) = (\bar{x} \qquad ;\ \bar{t} - 1\cdot309310\sigma_t),$$

$$(\theta_6) = \left(\bar{x} + \frac{3\sqrt{7}}{7}\sigma_x;\ \bar{t} - \frac{\sqrt{21}}{7}\sigma_t\right) = (\bar{x} + 1\cdot13389\sigma_x;\ \bar{t} - \cdot65465\sigma_t),$$

$$(\phi_1) = \left(\bar{x} + \frac{2\sqrt{7}}{7}\sigma_x;\ \bar{t} + \frac{2\sqrt{21}}{7}\sigma_t\right) = (\bar{x} + \cdot75593\sigma_x;\ \bar{t} + 1\cdot309310\sigma_t),$$

$$(\phi_2) = \left(\bar{x} - \frac{2\sqrt{7}}{7}\sigma_x;\ \bar{t} + \frac{2\sqrt{21}}{7}\sigma_t\right) = (\bar{x} - \cdot75593\sigma_x;\ \bar{t} + 1\cdot309310\sigma_t),$$

$$(\phi_3) = \left(\bar{x} - \frac{4\sqrt{7}}{7}\sigma_x;\ \bar{t} \qquad \right) = (\bar{x} - 1\cdot51186\sigma_x;\ \bar{t} \qquad),$$

$$(\phi_4) = \left(\bar{x} - \frac{2\sqrt{7}}{7}\sigma_x;\ \bar{t} - \frac{2\sqrt{21}}{7}\sigma_t\right) = (\bar{x} - \cdot75593\sigma_x;\ \bar{t} - 1\cdot309310\sigma_t),$$

$$(\phi_5) = \left(\bar{x} + \frac{2\sqrt{7}}{7}\sigma_x;\ \bar{t} - \frac{2\sqrt{21}}{7}\sigma_t\right) = (\bar{x} + \cdot75593\sigma_x;\ \bar{t} - 1\cdot309310\sigma_t),$$

$$(\phi_6) = \left(\bar{x} + \frac{4\sqrt{7}}{7}\sigma_x;\ \bar{t} \qquad \right) = (\bar{x} + 1\cdot51186\sigma_x;\ \bar{t} \qquad).$$

Numerical results with this formula were disappointing. The valuation of King's 50-year model office by O^{M} 3 % gave a liability of £693,105 compared with £688,999, an error of ·60 %. The valuation of the distribution on p. 501 by A 1924–29 ultimate at $2\frac{3}{4}$ % was as follows:

	True	Approximate	Error %
	£	£	
Net premiums	7,540	7,495	−·60
Value of sums assured	158,001	158,168	·11
Value of net premiums	92,493	92,176	−·34
Net liability	65,508	65,992	·74